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On the $SU(3)$ Parametrization of Qutrits

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Abstract. Parametrization of qutrits on the complex projective plane $\mathcal{C}P^2 = SU(3)/U(2)$ is given explicitly. A set of constraints that characterize mixed state density matrices is found.

Many recent ideas of quantum information theory are based on the notion of qubits. A qubit may be represented by a point on the Poincaré sphere S^2 that is homeomorhic to the complex projective line $\mathcal{H}^{(2)} = \mathcal{C}P^1 = SU(2)/U(1)$. A similar parametrization in the case of higher dimensional quantum systems is desirable both from theoretical [1] and technical points of view [2], [3]. A qutrit may be represented by a point on the complex projective plane $\mathcal{H}^{(3)} = \mathcal{C}P^2 = SU(3)/U(2)$. Such a representation is given explicitly in terms of Gell-Mann matrices [4]. We determine a set of constraints that characterize mixed states of qutrits below.

A qutrit $|\psi>= \alpha_0|0> +\alpha_1|1> +\alpha_2|2>$where $\alpha_0, \alpha_1, \alpha_2 \in \mathbf{C}$, $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$, is a state vector in the Hilbert space of states $\mathcal{H}^{(3)}$ of a 3-level system. It is spanned by an orthonormal basis $\{|0>, |1>, |2>\}$ which in matrix notation reads

$$|0>\rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1>\rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |2>\rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore

$$|\psi>\rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathbf{C}^3 \simeq \mathbf{R}^6.$$

Since $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$ and since $|\psi>$ is determined up to a multiplicative phase factor, $\dim\mathcal{H}^{(3)} = 4$.

Any 3×3 density matrix can be written as

$$\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$$

where $\vec{n}$ is a real 8-vector, and components of $\vec{\lambda}$ are the (Hermitian, traceless) Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

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$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The product of two Gell-Mann matrices is given by

$$\lambda_j \lambda_k = \frac{2}{3}\delta_{jk} + \sum_l d_{jkl}\lambda_l + i\sum_l f_{jkl}\lambda_l$$

where $j, k = 1, 2, \ldots, 8$. The f-symbols (structure constants of the Lie algebra $\mathsf{su}(3)$) are totally anti-symmetric :

$$\begin{aligned} f_{123} &= 1, f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}, \\ f_{147} &= f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \tfrac{1}{2}, \end{aligned}$$

and the d-symbols are totally symmetric:

$$\begin{aligned} d_{118} &= d_{228} = d_{338} = -d_{888} = \tfrac{1}{\sqrt{3}}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\tfrac{1}{2\sqrt{3}}, \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \tfrac{1}{2}. \end{aligned}$$

Given two real 8-vectors $\vec{a}$ and $\vec{b}$, we define their Euclidean inner product

$$\vec{a} \cdot \vec{b} = \sum_k a_k b_k \quad ,$$

skew-symmetric vector $\wedge$-product

$$(\vec{a} \wedge \vec{b})_j = \sqrt{3} \sum_{k,l} f_{jkl}\ a_k b_l \quad ,$$

and symmetric vector $\star$-product

$$(\vec{a} \star \vec{b})_j = \sqrt{3} \sum_{k,l} d_{jkl}\ a_k b_l \quad .$$

The pure states that satisfy $\rho^2 = \rho$ are therefore characterized by

$$|\vec{n}|^2 = 1 \quad \text{and} \quad \vec{n} \star \vec{n} = \vec{n}.$$

Suppose that $\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$ is the density matrix of a mixed state. It is Hermitian, positive with trace equal to 1. Therefore all the eigenvalues x_1, x_2, x_3 are positive and add to one: $x_1 + x_2 + x_3 = 1$. The Cayley-Hamilton equation satisfied by ρ reads

$$\rho^3 - \rho^2 + (x_1x_2 + x_2x_3 + x_1x_3)\rho - x_1x_2x_3 I = 0 \quad .$$

The following inequalities hold:

$$\frac{1}{3} \geq x_1x_2 + x_2x_3 + x_1x_3 \geq 0 \quad , \quad \frac{1}{27} \geq x_1x_2x_3 \geq 0 \quad .$$

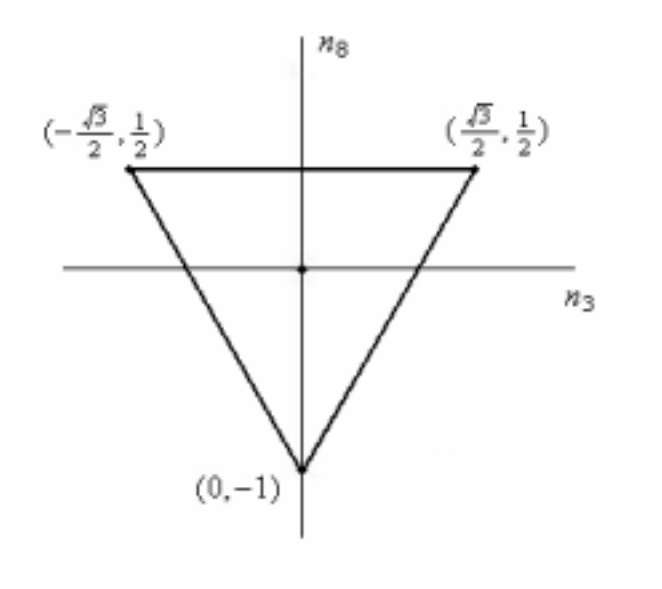


Figure 1.

Starting from these, a straightforward computation shows that the necessary and sufficient conditions for $\rho = \frac{1}{3}(I + \sqrt{3}n \cdot \lambda)$ to be a density matrix of a mixed state are given by

$$1 \geq |\vec{n}|^2 \geq 0 \quad \text{and} \quad 1 \geq 3|\vec{n}|^2 - 2\vec{n} \cdot (\vec{n} \star \vec{n}) \geq 0.$$

An arbitrary diagonal density matrix of a 3-level system will be

$$\rho = \frac{1}{3}(I + \sqrt{3}(n_3\lambda_3 + n_8\lambda_8)).$$

In this case, the mixed-state density matrix constraints reduce to

$$0 \leq n_3^2 + n_8^2 \leq 1 \quad \text{and} \quad 0 \leq 2n_8^3 - 6n_3^2 n_8 + 3n_3^2 + 3n_8^2 \leq 1.$$

The region in the $n_3 n_8$-plane where both the constraints are satisfied is bound by an equilateral triangle with vertices at the points

$$(n_3, n_8)_R = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (n_3, n_8)_B = (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(n_3, n_8)_G = (0, -1) \leftrightarrow . \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Vertices of the above triangle correspond to three mutually orthogonal pure-states. We labelled them Red, Blue, Green in analogy with colored quarks [2]. In fact two pure-state vectors $|\psi>$ and $|\psi'>$ are orthogonal if and only if $<\psi|\psi'>= 0$, so that $Tr(\rho\rho') = 0$. This implies $\vec{n} \cdot \vec{n}' = -\frac{1}{2}$. Then $\arccos(\vec{n} \cdot \vec{n}') = \pm\frac{2\pi}{3}$. This is equal to the geodesic distance between two orthogonal pure-states as measured by the standard Fubini-Study metric on $\mathcal{C}P^2$.

Points on the edges of the triangle correspond to mixing of two orthogonal pure-states of qutrits. In particular, at mid-points where bi-sectors intersect with the edges we have

$$(n_3,n_8)_C=(0,\frac{1}{2})\leftrightarrow\frac{1}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},(n_3,n_8)_B=(\frac{\sqrt{3}}{4},-\frac{1}{4})\leftrightarrow\frac{1}{2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$(n_3,n_8)_A=(-\frac{\sqrt{3}}{2},-\frac{1}{4})\leftrightarrow\frac{1}{2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Triply mixed-states correspond to points inside the triangle. In particular the origin

$$(n_3,n_8)_O=(0,0)\leftrightarrow\frac{1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

corresponds to the maximally mixed state.

For 2-level systems, the orbit of any diagonal 2×2 density matrix of qubits

$$\rho=\frac{1}{2}\begin{pmatrix}1+n_3&0\\0&1-n_3\end{pmatrix}\leftrightarrow(0,0,n_3)$$

under the action of the unitary group $\mathsf{SU(2)}$,

$$\rho\rightarrow U\rho U^\dagger$$

where $U\in\mathsf{SU(2)}$ (i.e. adjoint representation of $\mathsf{SU(2)}$, which is $\mathsf{SO(3)}$ applied on n_3) sweeps the whole Poincaré sphere S^2. Adjoint representation Ad of a given $\mathsf{U}\in\mathsf{SU(2)}$ is explicitly

$$\mathsf{Ad(U)}_{ij}=\frac{1}{2}\mathsf{Tr}(\sigma_i\mathsf{U}\sigma_j\mathsf{U}^\dagger)\in\mathsf{SO(3)},$$

so that $n_j\rightarrow\mathsf{Ad(U)}_{jk}n_k$.

In a similar way, for 3-level systems the orbit of each point (n_3,n_8) of the above triangle under the unitary action of $\mathsf{SU(3)}$ (i.e. adjoint representation of $\mathsf{SU(3)}$) will provide a generalization of the Poincaré sphere to 3-level systems. Adjoint representation Ad of a given $\mathsf{U}\in\mathsf{SU(3)}$ is found as follows:

$$\mathsf{Ad(U)}_{ij}=\frac{1}{2}\mathsf{Tr}(\lambda_i\mathsf{U}\lambda_j\mathsf{U}^\dagger)\in\mathsf{SO(8)}$$

so that

$$n_j\rightarrow\mathsf{Ad(U)}_{jk}n_k.$$

In fact $\mathsf{Ad(SU(3))}$ is an 8-parameter subgroup of the 28-parameter rotation group $\mathsf{SO(8)}$.

We also consider the *entropy of mixing* of ρ defined as

$$E(\rho)=-x_1\log_3(x_1)-x_2\log_3(x_2)-x_3\log_3(x_3)\quad.$$

Since in diagonal form

$$\rho=\frac{1}{3}\begin{pmatrix}1+\sqrt{3}n_3+n_8&&\\&1-\sqrt{3}n_3+n_8&\\&&1-2n_8\end{pmatrix},$$

the entropy of mixing of ρ becomes

$$E(\rho)=-(\frac{1+\sqrt{3}n_3+n_8}{3})\log_3(\frac{1+\sqrt{3}n_3+n_8}{3})$$
$$-(\frac{1-\sqrt{3}n_3+n_8}{3})\log_3(\frac{1-\sqrt{3}n_3+n_8}{3})-(\frac{1-2n_8}{3})\log_3(\frac{1-2n_8}{3})\quad.$$

The equi-mixing curves in the n_3n_8-plane are shown on the Figure 2.

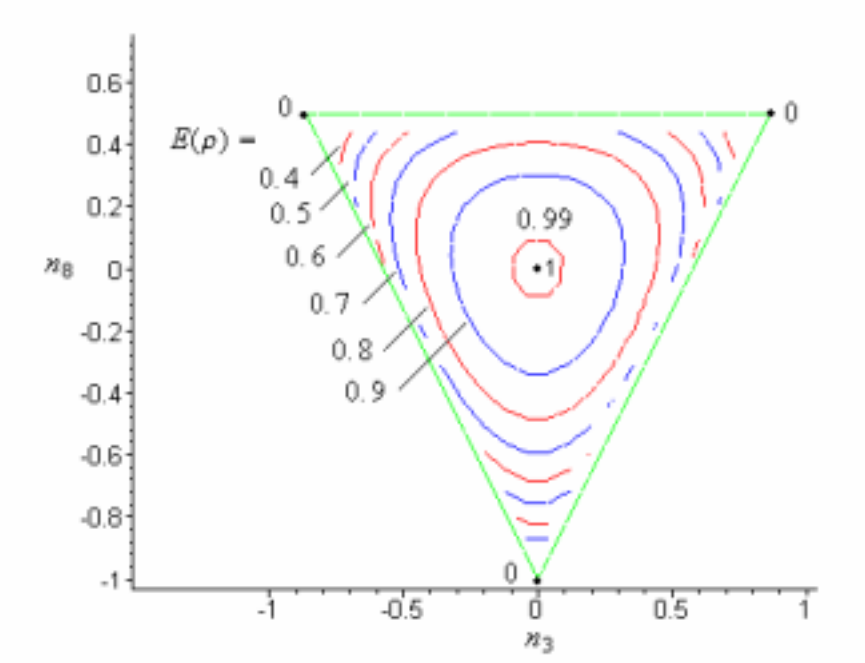


Figure 2.

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