Journal of Mathematical Physics

Bloch waves and non-commutative tori of magnetic translations

Cite as: J. Math. Phys. **62**, 101701 (2021); https://doi.org/10.1063/5.0063174
Submitted: 12 July 2021 • Accepted: 14 September 2021 • Published Online: 06 October 2021

Tekin Dereli and Todor Popov

View Online Export Citation CrossMark

ARTICLES YOU MAY BE INTERESTED IN

Variational principle for stochastic mechanics based on information measures
Journal of Mathematical Physics **62**, 102104 (2021); https://doi.org/10.1063/5.0056779

Moyal equation–Wigner distribution functions for anharmonic oscillators
Journal of Mathematical Physics **62**, 102103 (2021); https://doi.org/10.1063/5.0021132

Generalized Casimir operators for Lie superalgebras
Journal of Mathematical Physics **62**, 101703 (2021); https://doi.org/10.1063/5.0056538



AIP Publishing

© 2021 Author(s).

Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

Bloch waves and non-commutative tori of magnetic translations

Cite as: J. Math. Phys. 62, 101701 (2021); doi: 10.1063/5.0063174
Submitted: 12 July 2021 • Accepted: 14 September 2021 •
Published Online: 6 October 2021

View Online Export Citation CrossMark

Tekin Dereli[1,a] and Todor Popov[2,3,b]

AFFILIATIONS

[1] Department of Physics, Koç University, Sariyer, İstanbul 34450, Turkey
[2] Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, bld. Tzarigradsko chaussée, 72, Sofia 1784, Bulgaria
[3] American University in Bulgaria, Svoboda Bachvarova Str. 8, Blagoevgrad 2700, Bulgaria

[a] E-mail: tdereli@ku.edu.tr
[b] Author to whom correspondence should be addressed: tpopov@aubg.edu

ABSTRACT

We review the Landau problem of an electron in a constant uniform magnetic field. The magnetic translations are the invariant transformations of the free Hamiltonian. A Kähler polarization of the plane has been used for the geometric quantization. Under the assumption of quasi-periodicity of the wavefunction, the Zak's magnetic translations in the Bravais lattice generate a non-commutative quantum torus. We concentrate on the case when the magnetic flux density is a rational number. The Bloch wavefunctions form a finite-dimensional module of the noncommutative torus of magnetic translations as well as of its commutant, which is the non-commutative torus of magnetic translations in the dual Bravais lattice. The bi-module structure of the Bloch waves is shown to be the connecting link between two Morita equivalent non-commutative tori. The main focus of our review is the Kähler structure on the Hilbert space of Bloch waves and its inherent quantum toric geometry. We reveal that the metaplectic group $Mp(2,\mathbb{R})$ of the automorphisms of magnetic translation algebras is represented by the quantum optics squeezing operators.

Published under an exclusive license by AIP Publishing. https://doi.org/10.1063/5.0063174

I. INTRODUCTION

A magnetic field modifies the geometry of space in the sense that the geodesic motion of a free electron in a constant uniform magnetic field is no longer a straight line but a helix. In the center of mass system, the trajectory is reduced to a circular orbit in the plane perpendicular to the magnetic field. A quantum mechanical description of the electron's geodesic motion in the plane is given by coherent states through geometric quantization, while their internal symmetry is captured by Zak's magnetic translations.[17] Kähler manifolds are especially suitable for geometric quantization, and their holomorphic sections provide the Bargmann–Fock Hilbert space. The different complex structures give different quantizations. However, on the plane of motion, these are all equivalent due to an intertwining action of the metaplectic group $Mp(2,\mathbb{R})$ that interpolates between Heisenberg representations. When imposing periodic boundary conditions on the plane, the electron motion (Bloch waves) is described by the holomorphic sections on the torus. The space of sections depends on the choice of the complex structure on the torus specified by a modular parameter τ. On the torus, different values of the modular parameter can give physically equivalent theories, which is an example of duality. The rational values of the magnetic flux density upon quantization lead to a Hilbert space with finitely degenerated Landau levels. The Landau problem on a torus is equivalent to the topological sector of the Maxwell–Chern–Simons theory coupled with matter describing the low energy effective theory of the Fractional Quantum Hall (FQH) effect.[6,8,9] The structure of the degeneracy of the ground states in FQH effect is a major problem in condensed matter physics[8] related to the modular transformations of the Hall states.

The main contribution of this review is to exploit the full power of the complex coordinates in the Landau problem; we start with the standard coordinates $z = x + iy$ on the plane, and then while accounting for the Bloch waves quasi-periodicity on a torus, we switch to the general complex structure $z = x + \tau y$ related to the Bravais lattice with a complex modulus τ. We find out that the squeezing operators

Published under an exclusive license by AIP Publishing

employed in quantum optics perform a deformation of the complex structure, these operators belong to a representation of the metaplectic group $Mp(2,\mathbb{R})$, and their action results into a deformation of the Bravais lattice via Bogoliubov transformation of the magnetic translation operators. Once the complex structure modulus τ has been fixed, we still have residual modular group action $SL(2,\mathbb{Z})$ on physically equivalent complex moduli. The physical T-duality relates the algebra of the magnetic translations in Bravais lattice and its commutant algebra with translations in the dual Bravais lattice. In the mathematical parlance, we are dealing with a pair of Morita equivalent non-commutative tori leading to equivalent physical theories.[15]

The remainder of this paper is organized as follows: In Sec. II, we review the Landau problem on the plane in terms of holomorphic z and anti-holomorphic $\bar{z}$ coordinates and derivatives. In Sec. III, we endow the phase space with a structure of a Kähler manifold and consider its quantization. The generic complex structure is parameterized by a modular parameter τ. In Sec. IV, we introduce the non-commutative algebra of Zak's magnetic translations.[17] In Sec. V, we introduce a periodic potential and compactify the configuration space to a torus: the complex plane modulo the Bravais lattice Λ. The magnetic translations along Λ define a non-commutative torus $\mathbb{T}_\kappa$, while its commutant algebra $(\mathbb{T}_\kappa)'$ spanned by the magnetic translations in the dual Bravais lattice Λ^* turns out to be isomorphic to a non-commutative torus $\mathbb{T}_{1/\kappa}$ for a different value of the magnetic flux parameter $\kappa' = 1/\kappa$. In Sec. VI, we consider the magnetic translation eigenvalue problem imposing the quasi-periodic conditions on the fundamental domain Λ when the magnetic flux is a rational number $\kappa = \frac{M}{N}$. We show that the degeneracy of any Landau level is MN and the energy eigenfunctions that are parameterized by the factor-lattice $\mathbb{Z}_M \times \mathbb{Z}_N$ are constructed in Sec. VII with the help of the Jacobi theta function and Dedekind eta function. In Sec. VIII, we reveal that the squeezing operators employed in quantum optics represent a metaplectic group acting on the Hilbert space. In fact, the squeezing operator acts as a Bogoliubov transform interpolating between isomorphic magnetic translation algebras and changes the complex structure of the Kähler space. In Sec. IX, we show that the Bloch waves in any Landau level form a matrix with a left action of the magnetic translation torus $\mathbb{T}_\kappa$ and a right action of the dual magnetic translation torus $\mathbb{T}_{1/\kappa}$. The $(\mathbb{T}_\kappa, \mathbb{T}_{1/\kappa})$-bimodule guarantees the Morita equivalence of the two non-commutative tori. Section X consists of a conclusion and perspectives for future work.

II. LANDAU LEVELS AND THEIR DEGENERACY

The classical motion of an electron of mass m and charge e in an uniform magnetic field $\vec{B}$ is the Larmour (cyclotron) motion with constant velocity along the magnetic field $\vec{B}$ and circular motion in the system of center of mass. We choose the $\hat{z}$-axis along the magnetic field, so $B_i = (0,0,B)$ and assume $B > 0$, i.e., the vector $\vec{B}$ is pointing upward. In this way, the dynamics is reduced to a rotation in a (x,y)-plane with the cyclotron frequency

$$\omega = \frac{|eB|}{mc},$$

where the direction of the rotation of a particle with negative charge $-e$ is anti-clockwise. The magnetic field introduces also a length scale, *the magnetic length*, given by

$$l_B = \sqrt{\frac{\hbar c}{eB}}.$$

The flux BA of the magnetic field B through an area A is quantified in units of elementary magnetic flux $\frac{hc}{e}$. We let

$$\frac{BA}{hc/e} = \frac{BA}{2\pi\hbar c/e} = \frac{A}{2\pi l_B^2} = \kappa.$$

The number κ is the density of the magnetic field lines (quanta of the flux).

The quantum mechanical description of the cyclotron motion of an electron is first studied by Landau.[18] The Hamiltonian of the Landau problem is defined by the minimal coupling of the electron to an external magnetic field described by the electromagnetic vector potential $\boldsymbol{A}$,

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 =: \frac{1}{2m}\boldsymbol{P}^2. \tag{1}$$

The constant uniform magnetic field $\boldsymbol{B} = B\hat{z}$ along the z-direction can be obtained from different potentials $\boldsymbol{A} = (A_x, A_y)$ in the plane. One particular choice is the symmetric gauge

$$\boldsymbol{A} = (A_x, A_y) = \frac{B}{2}(-y, x), \qquad A_i = -\frac{B}{2}\epsilon_{ij}x^j. \tag{2}$$

The coordinates on the plane (x,y) and the canonical momenta $\boldsymbol{p} = (p_x, p_y) = -i\hbar(\partial/\partial x, \partial/\partial y)$ provide the standard (so-called Schrödinger) representation of the Heisenberg commutation relations

$$[p_k, p_l] = 0, \qquad [x_k, x_l] = 0, \qquad [p_k, x_l] = -i\hbar\delta_{kl}.$$

Published under an exclusive license by AIP Publishing

However, other natural coordinates on the phase space (x, y, p_x, p_y) may better reflect the change of the geometry due to the magnetic field, namely, the kinetic momenta[19] $\boldsymbol{P} = (P_1, P_2) := (P_x, P_y)$ and the coordinates of the center of mass $\boldsymbol{X} = (X_1, X_2) := (X, Y)$ that are given by

$$P_i = m\dot{x}_i = m\frac{\partial H}{\partial p_i} = p_i - \frac{e}{c}A_i, \qquad X_i = x_i + \frac{1}{m\omega}\epsilon_{ij}P^j. \tag{3}$$

The center of mass coordinates $\boldsymbol{X} = (X, Y)$ are the coordinates of the center of the cyclotron motion. These are integrals of motion $\dot{X} = 0 = \dot{Y}$ and as such decouple from the system. Therefore, the observables on the phase space split into two mutually commuting algebras

$$[P^i, X_j] = 0,$$

$$[P_x, P_y] = i\frac{\hbar e}{c}B = i\hbar m\omega = \frac{i\hbar^2}{l_B^2}, \qquad [X, Y] = -i\frac{\hbar}{m\omega} = -il_B^2.$$

Since the "center of mass" coordinates (X, Y) commute[20] with the Hamiltonian (1), they are also referred to as "zero modes." The presence of the magnetic field $\pm B$ leads to a deformation of the commutators between the kinetic momenta P_x and P_y and similarly between the center of mass coordinates X and Y. Geometrically, the magnetic field $\pm B$ is the curvature of a connection $\boldsymbol{P}$.

A. Landau levels

In view of their noncommutativity, the operators P_x and P_y alone can be thought as canonically conjugated operators in a reduced phase space. Their subalgebra is quantized in terms of creation and annihilation operators as[21]

$$a^{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(-P_y \mp iP_x). \tag{4}$$

Indeed, it is straightforward to check that $[a^-, a^+] = 1$. The operators a^+ and a^- are the raising and lowering operators for the eigenstates of the Hamiltonian (1), which may also be given as

$$H = \hbar\omega\left(a^+a^- + \frac{1}{2}\right) = \hbar\omega\frac{1}{2}\{a^+, a^-\}, \qquad [H, a^{\pm}] = \pm a^{\pm}.$$

Hence, we conclude that the Landau problem is equivalent to a simple harmonic oscillator problem with the cyclotron frequency ω.

The Landau levels are the eigenspaces of the Hamiltonian H with energies

$$E = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots,$$

depending on the quantum number operator n that counts the quanta of the magnetic field B transversal to the plane of motion. The lowest Landau level is the ground state of the system, and it satisfies the equation

$$a^-\Phi_{(0)} = 0.$$

All other Landau levels are excited states over the ground state $\Phi_{(0)}$.

B. Complex coordinates and holomorphic sections

Let us introduce complex holomorphic and antiholomorphic coordinates on our (x, y)-plane,

$$z = (x + iy)/l_B, \qquad \bar{z} = (x - iy)/l_B.$$

The respective derivatives obeying $\partial z = 1 = \bar{\partial}\bar{z}$ and $\partial\bar{z} = 0 = \bar{\partial}z$ are explicitly given by

$$\partial = \frac{\partial}{\partial z} = \frac{l_B}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{l_B}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Published under an exclusive license by AIP Publishing

The energy creation and annihilation operators $a^{\pm}$ (4) after the substitution $P_i = p_i - \frac{e}{c}A_i$ (3) in the symmetric gauge $\boldsymbol{A} = (A_x, A_y) = \frac{B}{2}(-y, x)$ acquire the concise form in complex coordinates,

$$a^{-} = \frac{-P_y + iP_x}{\sqrt{2\hbar m\omega}} = \sqrt{2}\left(\bar{\partial} + \frac{z}{4}\right), \qquad a^{+} = \frac{-P_y - iP_x}{\sqrt{2\hbar m\omega}} = -\sqrt{2}\left(\partial - \frac{\bar{z}}{4}\right).$$

The differential equation for the ground state in complex coordinates reads

$$a^{-}\Phi_{(0)}(z,\bar{z}) = \sqrt{2}\left(\bar{\partial} + \frac{z}{4}\right)\Phi_{(0)}(z,\bar{z}) = 0,$$

and it is now easy to solve, providing the *holomorphic sections* of a line bundle with connection $\boldsymbol{P}$. Its solution is the Gaussian factor multiplying an *arbitrary* holomorphic function $f(z)$,

$$\Phi_{(0)}(z,\bar{z}) = f(z)\exp\left(-\frac{z\bar{z}}{4}\right). \tag{5}$$

The ground state becomes infinitely degenerate due to the residual symmetry of the "zero modes." A non-normalized basis in the space of different holomorphic sections (ground states) reads

$$\Phi_{(0),n}(z,\bar{z}) = N_n z^n \exp\left(-\frac{z\bar{z}}{4}\right), \qquad N_n^2 = 1/(\pi n! 2^{n+1}), \tag{6}$$

given by the monomial basis $\{z^n\}_{n\geq 0}$ of (polynomial) holomorphic functions $f(z)$. The space of holomorphic states has a positive definite norm

$$\langle \Phi_{(0)} | \Psi_{(0)} \rangle = \int (-i)dz \wedge d\bar{z} \exp\left(-\frac{z\bar{z}}{2}\right) f(\bar{z}) g(z)$$

where the non-holomorphic part yields a measure on the space of holomorphic functions $f(z)$.

C. Center of mass coordinates *X* and *Y*

The degeneracy of a Landau level can be removed if we consider the zero modes $\boldsymbol{X}$ as internal degrees of freedom. We quantize the center of mass coordinates X and Y by the creation b^{+} and annihilation b^{-} operators that are given by

$$b^{\pm} = \frac{1}{l_B\sqrt{2}}(X \pm iY), \qquad [b^{-}, b^{+}] = 1.$$

The operators $b^{\pm}$ commute with $a^{\pm}$ since these are made out of zero modes.

In complex coordinates, the "zero mode" operators are given by the concise expressions

$$b^{-} = \sqrt{2}\left(\partial + \frac{\bar{z}}{4}\right), \qquad b^{+} = -\sqrt{2}\left(\bar{\partial} - \frac{z}{4}\right).$$

It is worth noting that expressions for $b^{\pm}$ are mapped on to the ones for $a^{\pm}$ by complex conjugation $z \to \bar{z}$.

The operators b^{+} and b^{-} raise and lower the angular momentum such that the state $\Phi_{(0),0}(z,\bar{z})$ has minimal angular momentum,

$$b^{-}\Phi_{(0),0}(z,\bar{z}) = b^{-}N_0 \exp\left(-\frac{z\bar{z}}{4}\right) = N_0\sqrt{2}\left(\partial + \frac{\bar{z}}{4}\right)\exp\left(-\frac{z\bar{z}}{4}\right) = 0.$$

From this minimal state, we can obtain all other states in the lowest Landau level,

$$(b^{+})^n\Phi_{(0),0}(z,\bar{z}) = (b^{+})^n \exp\left(-\frac{z\bar{z}}{4}\right) = \sqrt{n!}N_n z^n \exp\left(-\frac{z\bar{z}}{4}\right) = \sqrt{n!}\Phi_{(0),n}(z,\bar{z}).$$

III. QUANTIZATION AND KÄHLER POLARIZATION

A Kähler manifold is a complex manifold equipped with a non-degenerate Hermitean form whose real part is a metric (symmetric) form and whose imaginary part is a symplectic (antisymmetric) form. A complex manifold can be seen as an even-dimensional real manifold

Published under an exclusive license by AIP Publishing

equipped with a complex structure J (i.e., a linear operator such that $\mathcal{J}^2 = -\mathbb{1}$). There are three mutually compatible structures on a Kähler manifold, namely, the metric, symplectic, and complex structures; it is sufficient to specify any two of these structures in order to obtain the third one.

The general holomorphic and antiholomorphic coordinates on the complex plane are introduced by the following mappings from $\mathbb{R}^2$ to $\mathbb{C}$:

$$z = (x + \tau y)/l_B, \qquad \bar{z} = (x + \bar{\tau} y)/l_B, \qquad Im\,\tau > 0 \tag{7}$$

where $\tau = \tau_x + i\tau_y$ is a complex number. The magnetic length l_B in the denominator fixes the unit of length. It makes the complex variable z dimensionless and simplifies some formulas. The condition Im $\tau > 0$ implies that the parallelogram is not degenerate and its area is oriented.

The choice of holomorphic z and antiholomorphic $\bar{z}$ coordinates given by Eq. (7) is equivalent to the choice of a complex structure $\mathcal{J} = \mathcal{J}(\tau)$ on the (x, y)-plane: the eigenspaces of the operator $\mathcal{J}$ being the holomorphic and antiholomorphic coordinates

$$\mathcal{J}z = -iz, \qquad \mathcal{J}\bar{z} = i\bar{z}.$$

The standard holomorphic $z = x + iy$ and antiholomorphic $\bar{z} = x - iy$ coordinates ($\tau = i$) correspond to the standard complex structure

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The general complex structure $\mathcal{J}$ is found from the linear system

$$\mathcal{J}(x + \tau y) = -i(x + \tau y), \qquad \mathcal{J}(x + \bar{\tau} y) = i(x + \bar{\tau} y). \tag{8}$$

Solving it for $\mathcal{J}(x)$ and $\mathcal{J}(y)$, we get the matrix of the operator $\mathcal{J}$,

$$\mathcal{J} = \frac{1}{Im\,\tau}\begin{pmatrix} Re\,\tau & |\tau|^2 \\ -1 & -Re\,\tau \end{pmatrix} \in SL(2, \mathbb{R}). \tag{9}$$

The metric on a Kähler space satisfies the compatibility condition $g(x, y) = \Omega(\mathcal{J}x, y)$. Hence, from the canonical symplectic form $\Omega = dx \wedge dy$ and the complex structure $\mathcal{J}$ given by Eq. (9), one gets the metric

$$g = \mathcal{J}^t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{pmatrix}.$$

The condition $Im\,\tau > 0$ guarantees the positivity of the metric.

In our Landau problem, we have tacitly introduced a Kähler manifold starting with a symplectic manifold $M \cong \mathbb{R}^4$ with coordinates (p_x, p_y, x, y) with the 2-form $\Omega = dp_x \wedge dx + dp_y \wedge dy$ and then choosing a standard complex structure given by the 4×4 matrix

$$\begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix}.$$

This is a Kähler polarization, which turns (M, Ω) into a Kähler manifold. When we consider the lowest Landau level, we reduce our phase space to a two dimensional one.

Having introduced general holomorphic and antiholomorphic coordinates [thus a complex structure $\mathcal{J}(\tau)$] we proceed by defining the holomorphic and antiholomorphic derivatives

$$\partial = \frac{\partial}{\partial z} = \frac{l_B}{\tau - \bar{\tau}}\left(-\bar{\tau}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{l_B}{\tau - \bar{\tau}}\left(\tau\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right),$$

such that $\partial z = 1 = \bar{\partial}\bar{z}$ and $\partial \bar{z} = 0 = \bar{\partial} z$ hold true [compatible with Eq. (7)].

Published under an exclusive license by AIP Publishing

The next step is the quantization of the Kähler manifold $(M,\Omega,\mathcal{J})$:[22] one defines the creation and annihilation operators depending on the complex structure $\mathcal{J}(\tau)$. Mimicking the standard case $\tau = i$, we define the algebra of raising and lowering operators $a_\tau^\pm$ changing the Landau level,

$$a_\tau^+ = \frac{(\tau P_x - P_y)l_B}{\sqrt{2\,\mathrm{Im}\,\tau}\sqrt{\hbar}}, \qquad a_\tau^- = \frac{(\bar\tau P_x - P_y)l_B}{\sqrt{2\,\mathrm{Im}\,\tau}\sqrt{\hbar}}, \qquad [a_\tau^-, a_\tau^+] = 1,$$

and its commutant algebra of "zero mode" operators $b_\tau^\pm$,

$$b_\tau^+ = \frac{X + \tau Y}{l_B\sqrt{2\,\mathrm{Im}\,\tau}}, \qquad b_\tau^- = \frac{X + \bar\tau Y}{l_B\sqrt{2\,\mathrm{Im}\,\tau}}, \qquad [b_\tau^-, b_\tau^+] = 1.$$

Translated into complex coordinates, these operators read

$$a_\tau^+ = -\sqrt{2\,Im\,\tau}\left(\frac{\partial}{\partial z} - \bar z/(4\,Im\,\tau)\right), \qquad a_\tau^- = \sqrt{2\,Im\,\tau}\left(\frac{\partial}{\partial \bar z} + z/(4\,Im\,\tau)\right), \tag{10}$$

$$b_\tau^+ = -\sqrt{2\,Im\,\tau}\left(\frac{\partial}{\partial \bar z} - z/(4\,Im\,\tau)\right), \qquad b_\tau^- = \sqrt{2\,Im\,\tau}\left(\frac{\partial}{\partial z} + \bar z/(4\,Im\,\tau)\right). \tag{11}$$

The ground states $\Phi_{0,\tau}$ (5) in the lowest Landau level are annihilated by the lowering operator a_τ^- ($\Phi_{0,\tau}$ are covariantly constant sections),

$$a_\tau^-\,\Phi_{0,\tau} = 0.$$

The solutions turn out to be Gaussians centered around the origin $z = 0$,

$$\Phi_{0,\tau} = C\,\exp\left(-\frac{z\bar z}{4\,Im\,\tau}\right)$$

where the spread of the "bell" depends on $\mathrm{Im}\,\tau$. The function $\Phi_{0,\tau}$ yields a *coherent state*, that is, a state that saturates the lower bound in the Heisenberg uncertainty inequalities.

More generally, a coherent state centered around the point $z = z_0$ satisfies

$$a_\tau^-\,\Phi_{z_0,\tau} = z_0\Phi_{z_0,\tau}, \qquad \Phi_{z_0,\tau} = C\,\exp\left(-\frac{|z - z_0|^2}{4\,\mathrm{Im}\,\tau}\right).$$

In fact, all the ground states Φ_{z_0} are *coherent states*. Indeed, the position operator expectation value yields

$$\langle\Phi_{z_0,\tau}|z|\Phi_{z_0,\tau}\rangle = z_0.$$

IV. MAGNETIC TRANSLATIONS

Each Landau level is infinitely degenerate in energy. The center of mass coordinates X and Y give rise to infinitesimal translation operators $\boldsymbol{T} = (T_x, T_y)$ that after rescaling

$$T_i = m\omega\epsilon_{ij}X^j, \qquad [T_x, T_y] = -im\omega\hbar.$$

The translation operators (T_x, T_y) commute with the momentum $\boldsymbol{P}$ and consequently with the creation and annihilation operators $a^\pm$ and the Hamiltonian H. These do not alter any Landau level, hence their name "zero modes." The generators T_i are similar to the momentum operators P_i,

$$T_i = -p_i - \frac{e}{c}A_i, \qquad P_i = p_i - \frac{e}{c}A_i, \qquad [T_i, P_j] = 0.$$

The operators T_i are the generators of the center of mass translations.

Published under an exclusive license by AIP Publishing

Definition 4.1. Magnetic translation operator $D(\boldsymbol{u})$ is a finite translation operator along a given displacement vector $\boldsymbol{u}$,

$$D(\boldsymbol{u}) = \exp\left(\frac{i}{\hbar}\boldsymbol{u}\cdot\boldsymbol{T}\right). \tag{12}$$

The unitary operator $D(\boldsymbol{u})$ is also referred to as the displacement operator.

The degeneracy of Landau levels reflects their invariance with respect to $D(\boldsymbol{u})$.

The magnetic translation along a vector $\boldsymbol{u} = (u_x, u_y)$ acts on states of the Hilbert space $f(x,y)$ as translation of coordinates and change of the phase (by the flux of the magnetic field through a unit area),

$$D(\boldsymbol{u})f(x,y) = e^{\frac{i}{\hbar}(u_xT_x+u_yT_y)}f(x,y) = e^{-i\frac{2\pi e}{hc}(A_xu_x+A_yu_y)}f(x-u_x,y-u_y).$$

Alternatively, we write the magnetic translations in terms of complex null variables u and $\bar{u}$ as

$$D(\boldsymbol{u}) = \exp\frac{(ub^+ - \bar{u}b^-)}{\sqrt{2}}, \qquad u = (u_x + iu_y)/l_B. \tag{13}$$

Then, the translations along the rays $u = u^+$ and $\bar{u} = u^-$ yield

$$D(\boldsymbol{u})f(z,\bar{z}) = \exp\left(-u\left(\partial + \frac{\bar{z}}{4}\right) + \bar{u}\left(\bar{\partial} + \frac{z}{4}\right)\right)f(z,\bar{z}) = e^{\frac{\bar{u}z-u\bar{z}}{4}}f(z-u,\bar{z}-\bar{u}). \tag{14}$$

More generally, a magnetic translation operator in terms of complex null holomorphic $u_\tau = (u_x + \tau u_y)/l_B$ and anti-holomorphic $\bar{u}_\tau = (u_x + \bar{\tau}u_y)/l_B$ coordinates[23] reads

$$D(\boldsymbol{u}) = \exp\left(\frac{i}{l_B^2}\Omega(\boldsymbol{u},\boldsymbol{X})\right) = \exp\frac{\bar{u}_\tau b_\tau^+ - u_\tau b_\tau^-}{\sqrt{2\,\mathrm{Im}\,\tau}}, \qquad \mathcal{J}u = iu, \quad \mathcal{J}\bar{u} = -i\bar{u}.$$

The magnetic translation action on states $f(z,\bar{z})$, in parallel with Eq. (14), is given by[24]

$$D(\boldsymbol{u})f(z,\bar{z}) = \exp\frac{\bar{u}z - u\bar{z}}{4\,\mathrm{Im}\,\tau}f(z-u,\bar{z}-\bar{u}), \qquad \mathcal{J}u = iu \quad \mathcal{J}\bar{u} = -i\bar{u}. \tag{15}$$

Applying the displacement operator $D(z_0)$ to the "centered" ground state $\Phi_{0,\tau}$ will translate it and multiply by a factor depending on the angular momentum,

$$D(z_0)\Phi_{0,\tau} = \exp\left(\frac{z_0\bar{z} - \bar{z}_0 z}{4\,\mathrm{Im}\,\tau}\right)\Phi_{z_0,\tau}.$$

Thus, the ground states $\Phi_{z_0,\tau}$ are obtained as solutions of the differential equation $(a_\tau^- - z_0)\Phi = 0$ in the symmetric gauge (2).

The symplectic form Ω is derivable from the so-called Kähler potential $h(z,\bar{z})$,

$$h(z,\bar{z}) = -\frac{z\bar{z}}{2\,\mathrm{Im}\,\tau}, \qquad \Omega = i\partial\bar{\partial}h.$$

The Hilbert space of holomorphic sections together with the metric will be given by

$$\Phi(z,\bar{z}) = f(z)\exp\left(-\frac{h}{2}\right), \qquad \|\Phi(z,\bar{z})\|^2 = \int |f(z)|^2\exp(-h)\Omega. \tag{16}$$

V. QUANTUM TORI

Introduction of a periodic potential adds to our system a new scale, namely, the crystal lattice scale l besides the magnetic length scale l_B. The structure of the Landau levels is intimately related to the interplay between l and l_B.

Published under an exclusive license by AIP Publishing

A. Periodicity conditions

The crystal periods span a Bravais lattice $\Lambda = \mathbb{Z}\boldsymbol{e}_1 + \mathbb{Z}\boldsymbol{e}_2$ with lattice vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. We add a periodic potential $V(x,y)$ to the free Hamiltonian,

$$H' = H + V(x,y), \qquad V(\boldsymbol{x}) = V(\boldsymbol{x} + \Lambda). \tag{17}$$

The crystal lattice periods are the vectors $\boldsymbol{e}_1 := (l, 0)$ and $\boldsymbol{e}_2 := (l\tau_x, l\tau_y)$ in $(x,y) \in \mathbb{R}^2$ such that

$$V(x,y) = V(x + l, y) = V(x + l\tau_x, y + l\tau_y).$$

We imbed the Bravais lattice Λ in $\mathbb{C}$ by $\boldsymbol{e}_1 = l$ and $\boldsymbol{e}_2 = l\tau$. The new periodic Hamiltonian H' becomes naturally defined on the quotient of the complex plane by the Bravais lattice

$$T_\tau \cong \mathbb{C}/\Lambda = \mathbb{C}/(l\mathbb{Z} \oplus l\tau\mathbb{Z}), \qquad \tau = \tau_x + i\tau_y,$$

which is a torus with a complex structure $\mathcal{J}$ of modulus[25] τ, $z = x + \tau y$.

The translation invariance is broken for generic $\boldsymbol{u}$,

$$[H + V(x,y), D(\boldsymbol{u})] \neq 0.$$

However, the periodic Hamiltonian H' commutes with a subset of the magnetic translation operators given by

$$D(\boldsymbol{m}) = \exp\left(\frac{i}{\hbar}\boldsymbol{m}\cdot\boldsymbol{T}\right), \qquad \boldsymbol{m} \in \Lambda := \left\{m^1\boldsymbol{e}_1 + m^2\boldsymbol{e}_2 | m^i \in \mathbb{Z}\right\}, \tag{18}$$

namely, the operators generating the Bravais lattice Λ. The magnetic translations $D(\boldsymbol{m})$, $\boldsymbol{m} \in \Lambda$ are the discrete symmetries of the potential $V(x,y)$.

B. Quantum torus

The elementary translations $D(\boldsymbol{e}_1)$ and $D(\boldsymbol{e}_2)$ along the lattice vectors $\boldsymbol{e}_i$ of the Bravais lattice commute with the Hamiltonian H', but, in general, they do not commute with each other. The lack of commutativity is due to the flux of the magnetic field $\boldsymbol{B}$ through one plaquette of the Bravais lattice. The magnetic flux $BA = \boldsymbol{B}\cdot(\boldsymbol{e}_1 \times \boldsymbol{e}_2)$ through a cell with area $A = |\boldsymbol{e}_1 \times \boldsymbol{e}_2| = l^2\tau_y$ in units of elementary magnetic flux $\frac{hc}{e}$ is the dimensionless quantity κ given by

$$\kappa = \frac{BA}{hc/e} = \frac{A}{2\pi l_B^2} = \frac{\mathrm{Im}\,\tau}{2\pi}\frac{l^2}{l_B^2}. \tag{19}$$

The holonomy of the magnetic translation operators $D(\boldsymbol{u})$ [Eq. (18)] around the Bravais plaquette

$$D(\boldsymbol{e}_1)D(\boldsymbol{e}_2)D(\boldsymbol{e}_1)^{-1}D(\boldsymbol{e}_2)^{-1} = e^{\frac{ie}{\hbar c}\oint \boldsymbol{A}\cdot d\boldsymbol{r}} = q$$

is measured by the dimensionless parameter depending on the magnetic flux,

$$q = \exp\left(2\pi i\frac{e}{hc}\boldsymbol{B}\cdot(\boldsymbol{e}_1 \times \boldsymbol{e}_2)\right) = \exp\ 2\pi i\kappa.$$

In other words, the transversal (background) magnetic field B induces a curvature form Ω on the lattice Λ,

$$\Omega(\boldsymbol{m},\boldsymbol{n}) := \kappa\epsilon_{ij}m^i n^j = \kappa(\boldsymbol{m}\times\boldsymbol{n})\cdot\boldsymbol{e}_3 \qquad \text{where} \qquad (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = \boldsymbol{e}_3.$$

So far, we have obtained a symplectic redefinition to the magnetic translation (18) in the spirit of Stokes's theorem. It relates the circulation of the vector potential around a loop (i.e., the holonomy of $\boldsymbol{A}$) to the flux of the magnetic field $\boldsymbol{B}$ through a surface spanned on that loop. In coordinate-free notation, one has

Published under an exclusive license by AIP Publishing

$$D(\boldsymbol{m}) = \exp\left(\frac{i}{\hbar}\boldsymbol{m}\cdot\boldsymbol{T}\right) = \exp\ \frac{i}{l_B^2}dx\wedge dy(\boldsymbol{m},\boldsymbol{X}) = \exp\ i2\pi\Omega(\boldsymbol{m},\boldsymbol{X}), \tag{20}$$

where Ω is the symplectic form $\Omega = \kappa\, dx\wedge dy$ of the magnetic flux, which is the product of the density of the magnetic lines κ and the area of an elementary cell. Here, we have bind together the symplectic structure Ω with the metric (i.e., scalar product $\cdot$) as usual for the Kähler spaces.

In the compact picture, the magnetic translations along the periods $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ become the Wilson line operators around the nontrivial loops of the torus T_τ. Any open path on the lattice gives a non-trivial loop in T_τ. By construction, the magnetic flux through a non-contractible loop is a non-observable quantity, being a phase $i2\pi\mathbb{Z}$. Thus, $D(\boldsymbol{e}_i)$ are global gauge transformations.

The magnetic translation operators $D(\boldsymbol{u})$ generate a *quantum torus* $\mathbb{T}_\kappa$ such that

$$D(\boldsymbol{m})D(\boldsymbol{n}) = e^{i\pi\Omega(\boldsymbol{m},\boldsymbol{n})}D(\boldsymbol{m}+\boldsymbol{n})$$

or simply in the form of Heisenberg group relation

$$D(\boldsymbol{m})D(\boldsymbol{n}) = e^{i2\pi\kappa(\boldsymbol{m}\times\boldsymbol{n})}D(\boldsymbol{n})D(\boldsymbol{m}).$$

The phase factor measures the flux κ of the magnetic field through the parallelogram spanned by $\boldsymbol{m}$ and $\boldsymbol{n}$. When the flux κ is an integer, we get a commutative torus $\mathbb{C}/\Lambda$.

C. Dual quantum torus

We consider also the translations in the dual lattice Λ^*,

$$\widetilde{D}(\boldsymbol{n}^*) = \exp\left(\frac{i}{\hbar}\boldsymbol{n}^*\cdot\boldsymbol{T}\right), \qquad \boldsymbol{n}^*\in\Lambda^* := \left\{n_1\boldsymbol{e}^{*1}+n_2\boldsymbol{e}^{*2}|n_i\in\mathbb{Z}\right\}$$

with basis $\boldsymbol{e}^{*i}(\boldsymbol{e}_j) = \delta^i_j$. The basis $\boldsymbol{e}^{*i}$ span the dual lattice Λ^* of the Bravais lattice Λ, and the translations $\widetilde{D}(\boldsymbol{n}^*)$ are referred to as *dual magnetic translations*.

By construction, every magnetic translation $D(\boldsymbol{m})$ commutes with every dual magnetic translation $\widetilde{D}(\boldsymbol{n}^*)$,

$$\left[D(\boldsymbol{m}),\widetilde{D}(\boldsymbol{n}^*)\right] = 0.$$

The converse is also true; every magnetic translation [Eq. (18)] commuting with $D(\boldsymbol{m})$ is a dual magnetic translation $\widetilde{D}(\boldsymbol{n}^*)$ for some $n^*\in\Lambda^*$.

Dual magnetic translations $\widetilde{D}(\boldsymbol{n}^*)$ generate a dual quantum torus $\mathbb{T}'_\kappa\cong\mathbb{T}_{1/\kappa}$ such that

$$\widetilde{D}(\boldsymbol{m}^*)\widetilde{D}(\boldsymbol{n}^*) = e^{i\pi\widetilde{\Omega}(\boldsymbol{m}^*,\boldsymbol{n}^*)}\widetilde{D}(\boldsymbol{m}^*+\boldsymbol{n}^*), \qquad \widetilde{\Omega}(\boldsymbol{m}^*,\boldsymbol{n}^*) = -\frac{1}{\kappa}\epsilon^{ij}m_in_j.$$

The curvature $\widetilde{\Omega} = -\Omega^{-1}$ measures the holonomy $\tilde{q} = \exp(\frac{2\pi i}{\kappa})$ around the plaquette of the dual lattice Λ^*. The dual algebra is therefore isomorphic to a quantum torus with reciprocal value of the parameter, $(\mathbb{T}_\kappa)'\cong\mathbb{T}_{1/\kappa}$.

It is now clear that the algebra $\mathbb{T}_\kappa$ of finite translations $D(\boldsymbol{m})$ in momentum space and the algebra $(\mathbb{T}_\kappa)' = \mathbb{T}_{1/\kappa}$ of dual magnetic translations $\widetilde{D}(\boldsymbol{m}^*)$ (global gauge transformation and winding operators) are commutants of each other in a bigger algebra $\mathbb{T}_\kappa\otimes\mathbb{T}_{1/\kappa}$,

$$D(\boldsymbol{e}_1)D(\boldsymbol{e}_2) = e^{2\pi i\kappa}D(\boldsymbol{e}_2)D(\boldsymbol{e}_1), \tag{21}$$

$$\widetilde{D}(\boldsymbol{e}_1^*)\widetilde{D}(\boldsymbol{e}_2^*) = e^{\frac{2\pi i}{\kappa}}\widetilde{D}(\boldsymbol{e}_2^*)\widetilde{D}(\boldsymbol{e}_1^*), \tag{22}$$

$$D(\boldsymbol{e}_i)\widetilde{D}(\boldsymbol{e}_j^*) = \widetilde{D}(\boldsymbol{e}_j^*)D(\boldsymbol{e}_i). \tag{23}$$

The double commutant property implies Morita equivalence as we shall show in Sec. IX.

Alternatively, the quantum tori relations $\mathbb{T}_\kappa$ and $(\mathbb{T}_\kappa)'$ can be written as two commuting copies of the W_∞ algebra (see Ref. 4),

$$[D(\boldsymbol{m}),D(\boldsymbol{n})] = 2i\,\sin(\pi\kappa(\boldsymbol{m}\times\boldsymbol{n}))D(\boldsymbol{m}+n), \tag{24}$$

$$\left[\widetilde{D}(\boldsymbol{m}^*),\widetilde{D}(\boldsymbol{n}^*)\right] = 2i\,\sin\left(\frac{\pi}{\kappa}(\boldsymbol{m}\times\boldsymbol{n})\right)\widetilde{D}(\boldsymbol{m}^*+\boldsymbol{n}^*). \tag{25}$$

Published under an exclusive license by AIP Publishing

We end up with a splitting $W_\infty \otimes \widetilde{W}_\infty$ of the algebra of "zero modes" of the Hamiltonian H' with a periodic potential into two commuting algebras depending on the Bravais lattice of periods and its dual, respectively.

A word of caution: the algebras W_∞ and $\widetilde{W}_\infty$ both commute with the algebra of $a^\pm$ and the free Hamiltonian. Hence, they do not mix different Landau levels. The operators of the algebra $\mathbb{T}_\kappa$ are exponentials of the "zero mode" operators $b^\pm$; they create coherent states that are superposition of all states with different angular momenta that live in the lowest Landau level. One can also consider the algebra of (momentum space) displacement operators $W_{k,\bar{k}} = \exp(\frac{ka^+ - \bar{k}a^-}{\sqrt{2}})$, which will give rise to another couple of W_∞ algebras interpolating between different Landau levels.

VI. MAGNETIC TRANSLATION EIGENFUNCTIONS

Here, we get the wavefunctions of Landau levels with quasi-periodic conditions.[6] The elements of the quantum tori $\mathbb{T}_\kappa$ and $(\mathbb{T}_\kappa)'$ are expressed via the elementary magnetic translations

$$\begin{aligned} D(\boldsymbol{m}) &= q^{-\frac{1}{2}(m^1 m^2)} \quad D(\boldsymbol{e}_1)^{m^1} D(\boldsymbol{e}_2)^{m^2}, \quad \boldsymbol{m} = m^1 e_1 + m^2 e_2, \\ \widetilde{D}(\boldsymbol{n}^*) &= \tilde{q}^{-\frac{1}{2}(n_1 n_2)} \quad \widetilde{D}(\boldsymbol{e}_1^*)^{n_1} \widetilde{D}(\boldsymbol{e}_2^*)^{n_2}, \quad \boldsymbol{n} = n_1 e^{*1} + n_2 e^{*2}, \end{aligned} \tag{26}$$

where the exponents $m^i, n_i \in \mathbb{Z}$ are the winding numbers around the periods of the quantum torus $\mathbb{T}_\kappa$ and its dual, respectively. Every element in $\mathbb{T}_\kappa$ commutes with every element in $\mathbb{T}_{1/\kappa}$ since the magnetic translations commute with the dual magnetic translations.

When the flux density κ is rational, meaning that some rational number of quantized fluxes passes through a plaquette of the lattice Λ, the algebras $\mathbb{T}_\kappa$ and $\mathbb{T}_{1/\kappa}$ have a huge center, and the algebras are finite over their center.

When the flux takes a rational value $\kappa = N/M$, the operators $D(\boldsymbol{e_1})$ and $D(\boldsymbol{e_2})$ do not commute. However, $D(\boldsymbol{e_1})^M$ and $D(\boldsymbol{e_2})^M$ do commute. Moreover, one has $D(\boldsymbol{e_1})^M = \widetilde{D}(\boldsymbol{e_1^*})^N$ and $D(\boldsymbol{e_2})^M = \widetilde{D}(\boldsymbol{e_2^*})^N$; hence, we can choose a basis of common eigenfunctions ψ of (dual) magnetic translations

$$D(\boldsymbol{e_1})^M \psi = e^{i\alpha_1} \psi \big(= \psi \widetilde{D}(\boldsymbol{e_1^*})^N\big), \quad D(\boldsymbol{e_2})^M \psi = e^{i\alpha_2} \psi \big(= \psi \widetilde{D}(\boldsymbol{e_2^*})^N\big), \tag{27}$$

where the eigenvalues α_i will be referred to as *vacuum angles*. The common eigenfunctions ψ transform as a left module of $\mathbb{T}_\kappa$ and as a right module of $\mathbb{T}_{\kappa'}$. The translations $D(\boldsymbol{e_1})^M$ and $D(\boldsymbol{e_2})^M$ as well as the operators $\widetilde{D}(\boldsymbol{e_2^*})^N$ and $\widetilde{D}(\boldsymbol{e_1^*})^N$ shift on the bigger lattice

$$\Lambda_{MN} = \mathbb{Z}Me_1 \oplus \mathbb{Z}Me_2 = \mathbb{Z}Ne_1^* \oplus \mathbb{Z}Ne_2^*. \tag{28}$$

The lattice Λ_{MN} generates the center of both the algebra $\mathbb{T}_\kappa$ and its dual $\mathbb{T}_{1/\kappa}$. The flux through an elementary plaquette of Λ_{MN} is given by

$$M^2 l^2 \operatorname{Im} \tau / (2\pi l_B^2) = M^2 \kappa = MN. \tag{29}$$

With our assumption, MN is an integer that equals the co-dimension of the center.

We will give explicitly the eigenstates (27) of the (center of) the magnetic translations and with their help construct the modules of the quantum tori $\mathbb{T}_\kappa$ and $\mathbb{T}_{1/\kappa}$.

Magnetic translation of a wavefunction $\psi(z,\bar{z}) = \langle z, \bar{z} | \psi \rangle$ [Eq. (14)] on the Bravais lattice for the complex coordinates $z = (x+iy)/l_B$ (i.e., for the standard complex structure) looks like

$$\begin{aligned} D(\boldsymbol{e}_1)\psi(z,\bar{z}) &= e^{\frac{(z-\bar{z})l_0}{4}} \psi(z - l_0, \bar{z} - l_0), \\ D(\boldsymbol{e}_2)\psi(z,\bar{z}) &= e^{\frac{(\tilde{\tau}z - \tau\bar{z})l_0}{4}} \psi(z - \tau l_0, \bar{z} - \tilde{\tau} l_0), \end{aligned} \tag{30}$$

where we use the dimensionless spacing $l_0 = l/l_B$.

The center of $\mathbb{T}_\kappa$ (and $\mathbb{T}_{1/\kappa}$) is generated by the magnetic translations $D(\boldsymbol{m}')$ in the bigger lattice Λ_{MN}, and $D(\boldsymbol{m}')$ have the eigenfunctions ψ according to Eq. (30),

$$e^{-M(z-\bar{z})l_0/4} \psi(z + Ml_0, \bar{z} + Ml_0) = e^{i\alpha_1} \psi(z, \bar{z}), \tag{31}$$

$$e^{-M(\tilde{\tau}z - \tau\bar{z})l_0/4} \psi(z + \tau Ml_0, \bar{z} + \tilde{\tau} Ml_0) = e^{i\alpha_2} \psi(z, \bar{z}). \tag{32}$$

Published under an exclusive license by AIP Publishing

We now introduce a new variable $\mathfrak{z} = z/Ml_0$. The wavefunction renamed after rescaling $\Psi(\mathfrak{z}) = \psi(z/Ml_0)$ is quasi-periodic with conditions on periods 1 and τ,

$$\Psi(\mathfrak{z}+1,\bar{\mathfrak{z}}+1) = e^{i\alpha_1+\pi MN(\mathfrak{z}-\bar{\mathfrak{z}})/2\,\mathrm{Im}\,\tau}\Psi(\mathfrak{z},\bar{\mathfrak{z}}), \tag{33}$$

$$\Psi(\mathfrak{z}+\tau,\bar{\mathfrak{z}}+\bar{\tau}) = e^{i\alpha_2+\pi MN(\bar{\tau}\mathfrak{z}-\tau\bar{\mathfrak{z}})/2\mathrm{Im}\tau}\Psi(\mathfrak{z},\bar{\mathfrak{z}}) \tag{34}$$

where we have used[26] Eq. (29) so that $M^2 l_0^2\, Im\,\tau = M^2 2\pi\kappa = 2\pi MN$.

The function Ψ with the latter boundary conditions is determined by a holomorphic function $F(\mathfrak{z})$ times an appropriate factor $\eta(\mathfrak{z},\bar{\mathfrak{z}})$ [such that we retrieve a periodic condition of $F(\mathfrak{z}+1) = F(\mathfrak{z})$],

$$\psi(\mathfrak{z},\bar{\mathfrak{z}}) = \eta(\mathfrak{z},\bar{\mathfrak{z}})F(\mathfrak{z}).$$

In fact, the scale factor $\eta(\mathfrak{z},\bar{\mathfrak{z}})$ that does the job reads

$$\eta(\mathfrak{z},\bar{\mathfrak{z}}) = \exp\left(\frac{\pi MN\mathfrak{z}(\mathfrak{z}-\bar{\mathfrak{z}})}{2\,\mathrm{Im}\,\tau} + i\alpha_1\mathfrak{z}\right).$$

The boundary conditions on the holomorphic part $F(\mathfrak{z})$ are periodic for the shift $\mathfrak{z}\to\mathfrak{z}+1$ and quasi-periodic for $\mathfrak{z}\to\mathfrak{z}+\tau$ with period τ,

$$F(\mathfrak{z}+1,\tau) = F(\mathfrak{z},\tau), \qquad F(\mathfrak{z}+\tau,\tau) = e^{-i\pi MN\tau}e^{-i2\pi MN(\mathfrak{z}+\gamma)}F(\mathfrak{z},\tau),$$

for $\gamma = \frac{\tau\alpha_1-\alpha_2}{2\pi MN}$ depending on the vacuum angles α_1 and α_2.

The quasi-periodic boundary conditions for $F(z)$ are to be compared to the functional equations of the theta function $\vartheta(z,\tau)$, which is a quasi-periodic function (holomorphic section) on the torus $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$.

Definition 6.1. The theta function $\vartheta_r^K(z,\tau)$ of level $K\in\mathbb{Z}$ with modular parameter τ reads

$$\vartheta_r^K(z,\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau K\left(n+\frac{r}{K}\right)^2} e^{i2\pi Kz\left(n+\frac{r}{K}\right)} \qquad r\in\mathbb{Z}/K\mathbb{Z} := \mathbb{Z}_K.$$

We compare the boundary conditions of our holomorphic function $F(\mathfrak{z},\tau)$ with the one of the theta function $\vartheta_r^K(z,\tau)$ of level $K = MN$,

$$\vartheta_r^K(z+1,\tau) = \vartheta_r^K(z,\tau), \qquad \vartheta_r^K(z+\tau,\tau) = e^{-i\pi K\tau}e^{-i2\pi Kz}\vartheta_r^K(z,\tau). \tag{35}$$

Hence, we can identify[27] the function $F(\mathfrak{z})$ with the theta function $\vartheta_r^{MN}(\mathfrak{z}+\gamma,\tau)$ of level MN. To summarize, we have just proven the following proposition:

Proposition 6.2. Satisfying the quasi-periodic boundary conditions on the Bravais lattice $\Lambda = \mathbb{Z}e_1\oplus\mathbb{Z}e_2$ (18),

$$\langle z,\bar{z}|jk\rangle = \Psi_{jk}(\mathfrak{z},\bar{\mathfrak{z}}), \qquad \mathfrak{z} = \frac{zl_B}{Ml}$$

are eigenfunctions of the center of the magnetic translation algebra generated in $\Lambda_{MN} = \mathbb{Z}Me_1\oplus\mathbb{Z}Me_2 = \mathbb{Z}Ne_1^*\oplus\mathbb{Z}Ne_2^*$ (28),

$$D(\boldsymbol{e_1})^M\psi = e^{i\alpha_1}\psi, \quad D(\boldsymbol{e_2})^M\psi = e^{i\alpha_2}\psi,$$

where the eigenvalues depend on the vacuum angles α_1 and α_2. Then, the basis of the lowest Landau level is spanned by the ϑ-functions[28] of level MN,

$$\Psi_{jk}(\mathfrak{z},\bar{\mathfrak{z}}) := \exp\left(\pi MN\frac{\mathfrak{z}(\mathfrak{z}-\bar{\mathfrak{z}})}{2\,\mathrm{Im}\,\tau} + i\alpha_1\mathfrak{z}\right)\vartheta_{r_{jk}}^{MN}(\mathfrak{z}+\gamma,\tau). \tag{36}$$

Published under an exclusive license by AIP Publishing

Here, the shift angle is $\gamma = \frac{\tau\alpha_1 - \alpha_2}{2\pi MN}$ *and the parameters* r_{jk} *belong to the factor-lattice* $\Lambda^*_{MN}/\Lambda_{MN}$,

$$r_{jk} = jN + kM \in \Lambda^*_{MN}/\Lambda_{MN} \cong \mathbb{Z}_M \times \mathbb{Z}_N.$$

Remark. The degree of degeneracy MN is equal to the dimension of the translation algebra $\mathbb{T}_\kappa \times \mathbb{T}_{1/\kappa}$, quotiented by its center generated in the lattice Λ_{MN}. The algebra $\mathbb{T}_\kappa \times \mathbb{T}_{1/\kappa}$ (considered as a discrete subgroup of all translations) is finite over its center and its finite part is isomorphic to $\Lambda^*_{MN}/\Lambda_{MN} \cong \mathbb{Z}_N \times \mathbb{Z}_M$.

Corollary 6.3. Let the parameter κ *be a rational number* $\kappa = \frac{N}{M}$ *with* M *and* N *coprimes,* $gcd(M, N) = 1$. *Then, the degree of degeneracy of any Landau level under the assumption of the quasi-periodicity of the wavefunctions is* MN.

Proof. In fact, the degeneracy is not a privilege of the lowest Landau level. Similarly, at the nth Landau level, the eigenstates of energy $\hbar\omega(n + \frac{1}{2})$ are obtained by the action of the raising operator a^+,

$$\Psi^{(n)}_{jk} = a^{+n}\Psi_{jk}.$$

These functions span a basis of the nth Landau level. Thus, the degeneracy on any Landau level is again MN. □

Remark. In a recent paper,[5] Haldane stressed that the origin of holomorphic structures is not a landmark to the "lowest-Landau-level wavefunction," but it is a manifestation of a more-fundamental "quantum geometry" permeating the whole Hilbert space.

The states in any Landau level are expressed in terms of classical theta functions. The operators behind this construction are the magnetic translation operators. The state-operator correspondence has been used by Manin in order to introduce the quantum theta-functions,[11] these are operator-valued functions acting on the states-wavefunctions (spanning a module of the non-commutative torus).[29]

VII. MODULAR INVARIANCE AND COHERENT STATES

In the Landau problem on toroidal geometry or rather in the problem of Bloch electrons in a periodic potential, we encounter two different types of tori: the configuration space torus T_τ is a one-dimensional complex curve, whereas the magnetic translation operators close a non-commutative algebra, which is the quantum torus $\mathbb{T}_\kappa$ [Eq. (21)].

The magnetic translation operators (26) with loops wrapped around the periods of T_τ are the generators of $\mathbb{T}_\kappa$. These are the topologically nontrivial global gauge transformations. The magnetic flux through a minimal loop, i.e., through the lattice facette, is κ depending on the ratio l/l_B and Im τ [Eq. (19)]. The parameter τ of T_τ encodes the ratio of the periods of the crystal lattice (and thus the complex structure on T_τ, which is scale invariant).

The norm of a state in a Landau level is a complex function of the modular parameter τ. For the states in the lowest Landau level [Eq. (36)], the norm reads

$$\langle\Psi_{jk}|\Psi_{jk}\rangle(\tau,\bar\tau) = \int \Omega_\tau e^{-\frac{\pi MN}{2\,\mathrm{Im}\,\tau}|\mathfrak{z}-\bar{\mathfrak{z}}|^2} e^{i\alpha_1(\mathfrak{z}-\bar{\mathfrak{z}})}\vartheta^{MN}_{r_{jk}}(\mathfrak{z}+\gamma,\tau)\vartheta^{MN}_{r_{jk}}(\bar{\mathfrak{z}}+\bar\gamma,\bar\tau), \tag{37}$$

where the volume form is $\Omega_\tau = \pi MN d\mathfrak{z}\wedge d\bar{\mathfrak{z}}/(\tau-\bar\tau)$.

We introduce a new shifted complex coordinate on the torus,[30] $z = \mathfrak{z} + \gamma$, and obtain after completing the square[31] a Gaussian measure on the space of holomorphic and anti-holomorphic functions,

$$\langle\Psi_{jk}|\Psi_{jk}\rangle(\tau,\bar\tau) = \int \frac{dz\wedge d\bar z}{\tau-\bar\tau}\ \exp\left(-\frac{\pi MN}{2\,\mathrm{Im}\,\tau}|z-\bar z|^2\right)\vartheta^{MN}_{r_{jk}}(z,\tau)\vartheta^{MN}_{r_{jk}}(\bar z,\bar\tau). \tag{38}$$

A. Double (quasi)periodicity of the norm

The above integrand is invariant under a translation of the z-variable with the periods of the torus. Indeed, the translation invariance is obvious for the period $z \to z+1$. To show the covariance for the other period $z \to z+\tau$, we note

$$\vartheta^{MN}_{r_{jk}}(z+\tau,\tau)\vartheta^{MN}_{r_{jk}}(\bar z+\bar\tau,\bar\tau)\exp\left(\frac{i\pi MN}{\tau-\bar\tau}(z-\bar z+\tau-\bar\tau)^2\right) \tag{39}$$

$$= \vartheta^{MN}_{r_{jk}}(z,\tau)\vartheta^{MN}_{r_{jk}}(\bar z,\bar\tau)\exp\left(\frac{i\pi MN}{\tau-\bar\tau}(z-\bar z)^2\right), \tag{40}$$

where one has to make use of the quasi-periodicity of the $\vartheta^K(z,\tau)$ [Eq. (35)]. The choice of the non-holomorphic prefactor is crucial for the periodicity. This is a translationally invariant gauge.

Hence, the states are indeed well defined on the factor space of the complex plane,

$$T_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}).$$

Published under an exclusive license by AIP Publishing

B. Modular transformations

There are different values of τ for which the tori T_τ are isomorphic as complex manifolds. These are precisely the tori whose parameters are related by the modular transformation

$$T_\tau \cong T_{\tau'} \qquad \Leftrightarrow \qquad \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

A modular transformation changes the basis of the lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, thus keeping the quotient invariant, $T_\tau = \mathbb{C}/\Lambda$. However, a choice of the basis of Λ fixes the periods and hence the holonomy loops for the magnetic translation operators, providing the basis of the non-commutative torus $\mathbb{T}_\kappa$. Therefore, the moduli space of the tori is the space of parameters τ modulo the action of the modular group $SL(2,\mathbb{Z})$. The transformations $T : \tau \to \tau' = \tau + 1$ and $S : \tau \to \tau' = -1/\tau$ are in $SL(2,\mathbb{Z})$ and correspond to matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is not difficult to see that S and T generate the modular group $SL(2,\mathbb{Z})$. The fundamental domain for the $SL(2,\mathbb{Z})$-orbits can be chosen to be the domain $\tau \in \mathbb{C}$,

$$\{\operatorname{Im}\tau > 0\} \qquad \text{and} \qquad \begin{cases} |\tau| > 1 & -1/2 < \operatorname{Re}\tau < 0, \\ |\tau| \geq 1 & 0 \leq \operatorname{Re}\tau \leq 1/2. \end{cases}$$

Every $SL(2,\mathbb{Z})$-orbit has one point in the fundamental domain. This is the moduli space of the complex structures on the torus. More on the modular invariance in the context of conformal field theory can be found in the lectures.[14]

C. Partition function

In conformal field theory, we have path-integral formalism for any Riemann surface. In particular, for the Riemann surface of genus 1 and modulus τ, that is, the torus T_τ, the *partition function* $Z(\tau)$ is the vacuum functional,

$$Z(\tau) = \operatorname{Tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right), \qquad q = e^{i2\pi\tau},$$

where L_0 and $(\bar{L}_0)$ are the holomorphic and antiholomorphic conformal energy operators. The constant c is the central charge of the conformal theory. The characters of the chiral affine algebra are

$$\chi_\mu(\tau) = \operatorname{Tr}\left(q^{L_0 - \frac{c}{24}}\right), \qquad q = e^{i2\pi\tau},$$

and they carry representation of the modular group $SL(2,\mathbb{Z})$. The partition function $Z(\tau)$ is a sesquilinear pairing of characters with non-negative integer coefficients $Z_{\mu\nu}$,

$$Z(\tau) = \sum_\mu \chi_\mu^* Z_{\mu\nu} \chi_\nu, \quad Z_{\mu\nu} \in \mathbb{Z}_{\geq 0}.$$

The integer matrices $Z_{\mu\nu}$ satisfying the $SL(2,\mathbb{Z})$-invariance conditions

$$SZ = ZS, \qquad TZ = ZT$$

give rise to modular invariant partition functions $Z(\tau)$.

A free boson on the torus has the partition function

$$Z_{boson}(\tau) = \frac{1}{\sqrt{\operatorname{Im}\tau}} \frac{1}{|\eta(\tau)|^2},$$

where the Dedekind function $\eta(\tau)$ is given as the infinite product,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi\tau}.$$

Published under an exclusive license by AIP Publishing

One can check its modular properties under T and S transformations,

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau).$$

The latter transformations exactly compensate the transformations of $\operatorname{Im}\tau$ and thus guarantee that the partition function of the free boson $Z_{boson}(\tau)$ is modular invariant.

The affine lattice characters are written as the ratio of theta functions of level K and the Dedekind function,

$$\chi_\mu(z,\tau) = \vartheta^K_\mu(z,\tau)/\eta(\tau), \quad \mu \in \Lambda^*_K/\Lambda_K.$$

The role of $\eta(\tau)$ is again to compensate for the nontrivial modular transformations of the theta functions, which are modular half-forms, just as the Dedekind function.

The modular transformations of the affine characters are remarkably beautiful,[7]

$$\chi_\mu(z,\tau+1) = e^{2\pi i\left(\frac{|\mu|^2}{2}-\frac{1}{24}\right)}\chi_\mu(z,\tau), \tag{41}$$

$$e^{-i\pi Kz^2/\tau}\chi_\mu\left(\frac{z}{\tau},-\frac{1}{\tau}\right) = |\Lambda^*_K/\Lambda_K|^{-\frac{1}{2}}\sum_{\mu'\in\Lambda^*_K/\Lambda_K} e^{-i2\pi(\mu|\mu')}\chi_{\mu'}(z,\tau). \tag{42}$$

Thus, the partition function is a sum over the Hilbert space of states with weights depending on the energy. We sum first on states of a given Landau level and then sum with weights over all levels. As the degeneracy of each Landau level is the same, we expect to have factorization of the partition function. All the physically meaningful quantities in a theory defined on a torus should be modular invariant.

Lemma 7.1. The sum of the norms of all states in the lowest Landau level normalized by the Dedekind function $\eta(\tau)$ is a modular invariant,

$$\tilde{Z}(\tau) = \sum_{r_{jk}\in\mathbb{Z}_M\times\mathbb{Z}_N}\frac{\langle\Psi_{jk}|\Psi_{jk}\rangle}{|\eta(\tau)|^2}. \tag{43}$$

Proof. The summation on the weights $\mu = r_{jk} \in \Lambda^*_{MN}/\Lambda_{MN} \cong \mathbb{Z}_M \times \mathbb{Z}_N$ yields

$$\tilde{Z}(\tau) = \int\frac{dz\wedge d\bar{z}}{\tau-\bar{\tau}}\exp\left(-\frac{\pi MN}{2\operatorname{Im}\tau}|z-\bar{z}|^2\right)\sum_{\mu\in\Lambda^*_{MN}/\Lambda_{MN}}\chi^*_\mu(z,\tau)\chi_\mu(z,\tau). \tag{44}$$

The modular invariance $\tilde{Z}(\tau+1) = \tilde{Z}(\tau)$ is straightforward due to the compensation of the phase factor in Eq. (41). The invariance $\tilde{Z}(-1/\tau) = \tilde{Z}(\tau)$ follows from Eq. (42) and the orthogonality relations

$$\delta_{\mu'\mu''} = (MN)^{-1}\sum_{\mu\in\Lambda^*_{MN}/\Lambda_{MN}} e^{-2\pi i(\mu'|\mu)}e^{2\pi i(\mu|\mu'')}.$$

We conclude that $Z(\tau)$ is a good candidate for the partition function on the torus.

A structure similar to the modular invariant partition function for the quantum Hall effect in the annulus geometry has been introduced by Cappelli and Zemba.[1] Indeed, the Gaussian factor in Eq. (43) together with the theta-function normalized by the Dedekind function $\eta(\tau)$ mimics the definition of the affine U(1)-characters [cf. (2.11) in Ref. 1].

VIII. METAPLECTIC REPRESENTATIONS AND SQUEEZED STATES

The Heisenberg algebra $\mathfrak{h}_3$ is realized as the algebra of creation and annihilation operators $b^\pm$ with the oscillator relation $[b^-,b^+] = 1$. The three quadratic symmetric polynomials in b^+ and b^-,

$$J_\pm = \frac{1}{4}\{b^\pm,b^\pm\}, \qquad J_0 = \frac{1}{4}\{b^+,b^-\}, \tag{45}$$

close an algebra $\mathfrak{sp}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$, known as the metaplectic representation of $\mathfrak{sl}(2,\mathbb{R})$,

$$[J_0,J_\pm] = \pm J_\pm, \qquad [J_+,J_-] = -2J_0. \tag{46}$$

Published under an exclusive license by AIP Publishing

The action of quadratic generators $J_\pm$ and J_0 on the generators $b^\pm$,

$$[J_0, b^\pm] = \pm\frac{1}{2}b^\pm, \qquad [J_\pm, b^\mp] = \mp b^\pm, \qquad [J_\pm, b^\pm] = 0, \tag{47}$$

gives the algebra of automorphisms of $\mathfrak{h}_3$. Hence, the quadratic and the linear polynomials in b^+, b^- together with the identity operator 1 close an algebra, which is a semidirect product $\mathfrak{h}_3 \rtimes \mathfrak{sl}(2,\mathbb{R})$.

When exponentiated, the metaplectic representation (47) is double valued, and it gives rise to a double cover of the group $SL(2,\mathbb{R})$, which is referred to as the *metaplectic representation* $Mp(2,\mathbb{R})$. The automorphisms of the Heisenberg group H_3 are given by the metaplectic group $Mp(2,\mathbb{R})$. The quantum torus or equivalently the algebra of the displacement operators $D(\boldsymbol{u})$ provides a representation of the Heisenberg group H_3, which also carries an action of $Mp(2,\mathbb{R})$.

We now describe the semi-direct group product $H_3 \rtimes Mp(2,\mathbb{R})$. We can either work with the double cover of $SL(2,\mathbb{R})$ or with the double cover of $SU(1,1)$. The Cartan operator $2J_0 = \frac{1}{2}\{b^-, b^+\} \in \mathfrak{su}(1,1)$. The operator $2J_0 = (N + \frac{1}{2})$ with the integer spectrum is exponentiated to the unitary operator $U_\phi = \exp(i\phi 2J_0)$ so that

$$U_\phi \begin{pmatrix} b^+ \\ b^- \end{pmatrix} U_\phi^{-1} = e^{\pm i\phi} \begin{pmatrix} b^+ \\ b^- \end{pmatrix}.$$

It might come as a surprise that the *squeezing operators* that are massively used in quantum optics,

$$S(w) = \exp\frac{1}{2}\left(w(b^+)^2 - \bar{w}(b^-)^2\right), \quad w = re^{i\varphi}, \tag{48}$$

belong to the metaplectic representation of $SU(1,1)$.

The operator $S(w)$ acting on the creation and annihilation operators $b^\pm$ maps them to another pair $b_\tau^\pm$ given by

$$\begin{pmatrix} b_\tau^+ \\ b_\tau^- \end{pmatrix} = S(w) \begin{pmatrix} b^+ \\ b^- \end{pmatrix} S^{-1}(w) = \begin{pmatrix} \cosh r & e^{-i\varphi}\sinh r \\ e^{i\varphi}\sinh r & \cosh r \end{pmatrix} \begin{pmatrix} b^+ \\ b^- \end{pmatrix}. \tag{49}$$

Thus, the squeezing operator $S(w)$ is intertwining different representations of $\mathfrak{h}_3$. Such intertwining of oscillator representations is also known as Bogoliubov transformations in, e.g., superfluidity theory. The $SU(1,1)$-symmetry applies to the geometry of the Laughlin fractional quantum Hall states in the framework of the Chern–Simons matrix model (see Ref. 10). In other words, the squeezing operator $S(w)$ interpolates between different complex structures, thus perturbing the standard complex structure $\mathcal{J}_0$ to a new one $\mathcal{J}(\tau)$,

$$S(w)\mathcal{J}_0 S^{-1}(w) = i\begin{pmatrix} \cosh 2r & -e^{i\varphi}\sinh 2r \\ e^{-i\varphi}\sinh 2r & -\cosh 2r \end{pmatrix} = \mathcal{J}(\tau),$$

$$= \begin{pmatrix} \cosh r & e^{-i\varphi}\sinh r \\ e^{i\varphi}\sinh r & \cosh r \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cosh r & -e^{-i\varphi}\sinh r \\ -e^{i\varphi}\sinh r & \cosh r \end{pmatrix}.$$

The "perturbed" complex structure is to be compared to the general complex structure $\mathcal{J}$ [given by Eq. (9) but in the (x,y)-basis] dependent on the modulus τ,

$$\mathcal{J}(\tau) = \frac{i}{2\,\mathrm{Im}\,\tau}\begin{pmatrix} 1+|\tau|^2 & -2i\,\mathrm{Re} + (1-|\tau|^2) \\ -2i\,\mathrm{Re} - (1-|\tau|^2) & -1-|\tau|^2 \end{pmatrix}. \tag{50}$$

Comparing the matrices above, one determines the ("squeezing") parameters $w = w(\tau) = r(\tau)e^{i\phi(\tau)}$ as functions of the complex structure modulus τ,

$$\cosh 2r = \frac{1+|\tau|^2}{2\,\mathrm{Im}\tau}, \qquad \sinh 2r\cos\varphi = \frac{1-|\tau|^2}{2\,\mathrm{Im}\,\tau}, \qquad \sinh 2r\sin\varphi = -\frac{\mathrm{Re}\,\tau}{\mathrm{Im}\,\tau}.$$

Published under an exclusive license by AIP Publishing

Thus, we get the dependence

$$\tan\,\varphi = \frac{\mathrm{Re}\,\tau}{|\tau|^2-1}, \qquad \sinh\,2r = \frac{1-|\tau|^2}{2\,\mathrm{Im}\,\tau}\sqrt{1+\tan^2\varphi}.$$

In particular, when w is real, we have $\mathrm{Re}\,\tau = 0$ and "squeezing" of the $Im\,\tau$,

$$\varphi = 0, \qquad Re\,\tau = 0, \qquad e^{-2r} = Im\,\tau.$$

Roughly speaking, in the plane geometry the operator $S(w)$ acting on the complex structure $\mathcal{J}_0$ is deforming it to another complex structure $\mathcal{J}$. For the compactified theory on the torus, different complex structures are related by a metaplectic representation of the modular group $SL(2,\mathbb{Z})$, the group of invariance of the Bravais lattice Λ.

The argument $z = x + \tau y$ of the theta function carries a representation of the Heisenberg group H_3 of the magnetic translations, while the argument τ is transforming in the metaplectic group $Mp(2,\mathbb{R})$. It was André Weil who revealed that the theta function $\vartheta(z,\tau)$ is a matrix representation of the cross-product $H_3 \ltimes Mp(2,\mathbb{Z})$.[12]

IX. MATRIX ALGEBRAS AND MORITA EQUIVALENCE

Magnetic translations commute with the Hamiltonian of the Landau problem and hence do not alter the energy. The lowest Landau level consisting of the ground states in Hilbert space is a representation of both the magnetic algebra $\mathbb{T}_\kappa$ and the dual magnetic algebra $\mathbb{T}_{\kappa'}$. In fact, any Landau level carries one and the same structure of a module of two non-commutative tori whose magnetic flux parameters are related by $\kappa' = 1/\kappa$.

Lemma 9.1. Let us consider a rational magnetic flux $\kappa = N/M$. The wave functions $\Psi_{jk}(z,\bar z) = \langle z,\bar z|jk\rangle$, $j \in \mathbb{Z}_N$ and $k \in \mathbb{Z}_M$, span a left module of the quantum torus $\mathbb{T}_\kappa$ and a right module of the dual quantum torus $\mathbb{T}_{\kappa'}$,

$$D(\boldsymbol{e_1})\Psi_{jk} = e^{i(\alpha_1+2\pi jN)/M}\Psi_{jk}, \quad \Psi_{jk}\widetilde{D}(\boldsymbol{e_1^*}) = e^{i\alpha_2/N}\Psi_{j\,k-1},$$

$$D(\boldsymbol{e_2})\Psi_{jk} = e^{i\alpha_2/M}\Psi_{j-1\,k}, \qquad \Psi_{jk}\widetilde{D}(\boldsymbol{e_2^*}) = e^{i(\alpha_1+2\pi kM)/N}\Psi_{jk}.$$

The operator $D(\boldsymbol{e}_1)$ acts on the left module by the $M\times M$-matrix

$$D(\boldsymbol{e}_1) = e^{i\alpha_1/M}\begin{pmatrix} 1 & & & \\ & e^{i2\pi N/M} & & \\ & & \ddots & \\ & & & e^{i2\pi(M-1)N/M}\end{pmatrix},$$

where we have written only the nonzero entries. Similarly, $D(\boldsymbol{e}_2)$ acts by

$$D(\boldsymbol{e}_2) = e^{i\alpha_2/M}\begin{pmatrix} 0 & & & 1\\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{pmatrix}.$$

Note that both matrices are of trace zero (on the diagonal of the first matrix, we have the sum of roots of unity, $\sum_{j=0}^{M-1}\omega^j = 0$ when $\omega^M = 1$).

Definition 9.2. Let E be a (A,B)-bimodule. The algebra A is Morita equivalent to the algebra B if there exists a (B,A)-bimodule F such that the following isomorphisms hold true:

$$E\otimes_B F \cong A \qquad as \qquad A-bimodule,$$

$$F\otimes_A E \cong B \qquad as \qquad B-bimodule.$$

Published under an exclusive license by AIP Publishing

In fact, here we encounter an example of T-duality, which is the physical jargon for what the mathematicians call Morita equivalence.[15]

Proposition 9.3. Let the flux κ be a rational number $\kappa = N/M$. Then, any MN-fold degenerate Landau level is spanned by the wavefunctions Ψ_{jk} (36). The $M \times N$ matrix

$$\Psi = \left\{\Psi_{jk}, j \in \mathbb{Z}_M, k \in \mathbb{Z}_N\right\} \qquad is\ a \quad (\mathbb{T}_\kappa, \mathbb{T}_{\kappa'}) - bimodule,$$

whereas the $N \times M$ matrix

$$\bar{\Psi} = \left\{\bar{\Psi}_{kj}, k \in \mathbb{Z}_N, j \in \mathbb{Z}_M\right\} \qquad is\ a \quad (\mathbb{T}_{\kappa'}, \mathbb{T}_\kappa) - bimodule.$$

The bimodules Ψ and $\bar{\Psi}$ provide the Morita equivalence of the quantum torus T_κ and its dual $\mathbb{T}_{\kappa'}$ with $\kappa' = 1/\kappa$.

To summarize, the theta functions exhibit a bimodule structure that guarantees the T-duality of the algebra of magnetic translations $\mathbb{T}_\kappa$ and its dual $\mathbb{T}_{\kappa'}$.

Finally, we consider the universal quantum enveloping algebra $U_q(sl_2)$ generated by

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}}, \tag{51}$$

that can be built from the magnetic translation operators[9,6] along the light-cone directions on the lattice. Indeed, it is straightforward to check that the following combinations realize the algebra $U_q(sl_2)$:

$$J_\pm := \frac{D(\pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2) - D(\mp\boldsymbol{e}_1 \pm \boldsymbol{e}_2)}{q - q^{-1}}, \qquad q^{\pm J_3} := D(\pm\boldsymbol{e}_1) \quad \text{with} \quad q = e^{i2\pi\kappa}.$$

On the other hand, another "dual" copy of $U_{\widetilde{q}}(sl_2)$ algebra for the dual parameter $\widetilde{q} = e^{i2\pi/\kappa}$ is built through the dual magnetic translations

$$\widetilde{J}_\pm := \frac{\widetilde{D}(\pm\boldsymbol{e}_1^* \pm \boldsymbol{e}_2^*) - \widetilde{D}(\mp\boldsymbol{e}_1^* \pm \boldsymbol{e}_2^*)}{\widetilde{q} - \widetilde{q}^{-1}}, \qquad \widetilde{q}^{\pm J_3} := \widetilde{D}(\pm\boldsymbol{e}_1^*) \quad \text{with} \quad \widetilde{q} = e^{i2\pi/\kappa}.$$

The Bloch waves $\Psi_{jk}^{(n)} = a^{+n}\Psi_{jk}$ given by Eq. (36), living at the nth Landau level, are entries of a $M \times N$ matrix carrying a left M-dimensional $U_q(sl_2)$-module and right N-dimensional $U_{\tilde{q}}(sl_2)$-module.[6]

X. CONCLUSION AND OUTLOOK

From the geometric perspective, theta-functions arise as sections of a bundle over the torus of the compactified coordinate space in the presence of a periodic potential. Applying the non-commutative geometry paradigm, we replace vector bundles with modules of non-commutative algebras; hence, the Bloch waves for electrons without spin are treated as modules of Heisenberg–Weyl algebras of magnetic translations. These modules provide a textbook example of Morita equivalence.

The next natural step is to consider Bloch waves for the electron with spin. The geometric point of view has the merit that the spin interaction of the electron with the magnetic field is incorporated into fermionic degrees of freedom living on spinorial bundles over the torus. The super-symmetric counterparts of the magnetic translations operators will satisfy the super-sine algebra,[3,4] that is, the super-symmetric version of W_∞ (24), which we can refer to as a non-commutative super-torus. We expect that Bloch waves with spin can be expressed as appropriate supersymmetric theta-functions, the super-holomorphic sections transforming into a module of pair of Morita equivalent non-commutative super-tori.

The maximal symmetry of the Landau problem is given by the symplectic morphisms $Sp(4,\mathbb{R})$ of the phase space $(\mathbb{R}^4, \Omega)$. The isomorphism $sp(4,\mathbb{R}) \cong so(2,3)$ relates the 2 + 1 Minkowski space conformal group $SO(2,3)$ to its double cover $Sp(4,\mathbb{R})$. The conformal symmetry $SO(2,3)$ has a natural description in terms of the Jordan algebra of 2×2 symmetric real matrices, and hence, we have prepared the ground for treating the symmetries of the Landau problem in the framework of Jordan algebras.

ACKNOWLEDGMENTS

T.P. was supported by TUBITAK through a BIDEP-2230 Visiting Scholar Grant and by the Bulgarian National Science Fund Research Grant (No. DN 18/3). We also thank TUBA for partial financial support. We are acknowledge Koç University for hospitality, especially the Office of VPRD (Vice President for Research and Development) for support that came at the right times.

Published under an exclusive license by AIP Publishing

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

[1] A. Cappelli and G. R. Zemba, "Modular invariant partition functions in the quantum Hall effect," Nucl. Phys. B **490**, 595 (1997).
[2] E. Chang-Young and H. Kim, "Theta vectors and quantum theta functions," J. Phys. A: Math. Gen. **38**(19), 4255 (2005).
[3] T. Dereli and A. Verçin, "A physical realization of the super-sine algebra," Phys. Lett. B **288**(1), 109–112 (1992).
[4] T. Dereli and A. Vercin, "Lie algebra deformation of $Dif f_A \mathbb{T}^2$ and Bloch electrons in a constant uniform magnetic field," J. Phys. A: Math. Gen. **26**, 6961–6971 (1993).
[5] F. D. M. Haldane, "The origin of holomorphic states in Landau levels from non-commutative geometry and a new formula for their overlaps on the torus," J. Math. Phys. **59**(8), 081901 (2018).
[6] C.-L. Ho, "W_∞ and $SL_q(2)$ algebras in the Landau problem and Chern-Simons theory on a torus," Mod. Phys. Lett. A **10**, 2665–2673 (1995).
[7] V. G. Kac, *Infinite-Dimensional Lie Algebras* (Cambridge University Press, 1994), Vol. 44.
[8] E. Keski-Vakkuri and X.-G. Wen, "The ground state structure and modular transformations of fractional quantum Hall states on a torus," Int. J. Mod. Phys. B **7**(25), 4227–4259 (1993).
[9] I. I. Kogan, "Area-preserving diffeomorphisms, W_∞ and U_q(sl2) in the Chern-Simons theory and the quantum Hall system," Int. J. Mod. Phys. A **9**, 3887–3911 (1994).
[10] M. F. Lapa and T. L. Hughes, "Hall viscosity and geometric response in the Chern-Simons matrix model of the Laughlin states," Phys. Rev. B **97**(20), 205122 (2018).
[11] Y. I. Manin, "Functional equations for quantum theta functions," Publ. Res. Inst. Math. Sci. **40**(3), 605–624 (2004); arXiv:math/0307393 (2003).
[12] D. Mumford, M. Nori, and P. Norman, *Tata Lectures on Theta III* (Birkhäuser, Boston, MA, 2007), Vol. 43.
[13] H. Murayama, Lectures available at http://hitoshi.berkeley.edu/221A/landau.pdf.
[14] N. M. Nikolov and I. T. Todorov, "Lectures on elliptic functions and modular forms in conformal field theory," in *Proceedings of 3rd Summer School in Modern Mathematical Physics (MPHYS3)* (Institute of Physics, 2005), pp. 1–87; arXiv:math-ph/0412039 (2004).
[15] A. Schwarz, "Morita equivalence and duality," Nucl. Phys. B **534**(3), 720–738 (1998).
[16] I. Todorov, "Quantization is a mystery," Bulg. J. Phys. **39**, 107–149 (2012); arXiv:1206.3116.
[17] J. Zak, "Magnetic translation group," Phys. Rev. **134**(6A), A1602 (1964).
[18] For an concise introduction to Landau problem, we send the reader to the inspiring lectures.[13]
[19] We suppose that we have a positive metric $g_{ij} = \boldsymbol{\delta}_{ij}$, and hence, $P_i = P^i$.
[20] This commutation is specific for the symmetric gauge.
[21] Our conventions slightly differ from the standard $a^\pm = \frac{1}{\sqrt{2\hbar m\omega}}(P_x \mp iP_y)$.
[22] For expository notes on quantization on Kähler manifolds, see Ref. 16.
[23] It is worth noting that our displacement vector $\boldsymbol{u}$ have dimension of length, while the (anti)holomorphic coordinates u and $\bar{u}$ are chosen to be dimensionless.
[24] By abuse of notation, we write $u(\bar{u})$ for the (anti)holomorphic coordinates $u_\tau(\bar{u}_\tau)$ whenever the complex structure is clear from the context.
[25] The dependence on the scale l cancels in the ratio of the periods $\tau = \frac{l\tau}{l}$ of $T_{\boldsymbol{\tau}}$.
[26] We could have used the complex structure $z = (x + \boldsymbol{\tau}y)/l_B$ to get the same result through magnetic translations, Eq. (15). However, we have $M^2 l_0^2 = 2\pi MN$ due to $\mathrm{Im}_{\mathcal{J}}\, \tau = 1$ with respect to the complex structure $\mathcal{J}$, i.e., the Bravais plaquette is the image of a square in the (x, y)-plane.
[27] The solution has an additional freedom of multiplication by a function $g(\boldsymbol{\tau})$, which is not function of z, $\partial g/\partial\mathfrak{z} = 0$, that is, $g(\tau)F(\mathfrak{z})$ will again satisfy the same functional equations as $F(\mathfrak{z})$.
[28] These are coherent ground states.
[29] For nice and friendly physicist introduction to Manin's quantum theta-functions based on kq-representation, see Ref. 2.
[30] By some abuse, we denote it again by z. The old $z = (x + \boldsymbol{\tau}y)/l_B$ and the new z are related by a shift and scaling $\frac{z}{Ml} + \gamma \to \mathfrak{z} + \gamma \to z$, and thus, the complex structure is not altered.
[31] Since the vacuum angle $\alpha_1 = i\pi MN(\gamma - \bar{\gamma})/\mathrm{Im}\,\tau$, we can perform a global gauge transformation of the wavefunction $\Psi_{jk}(\mathfrak{z}, \bar{\mathfrak{z}}) \to \Psi_{jk}(\mathfrak{z}, \bar{\mathfrak{z}}) e^{i\pi MN \frac{\gamma(\gamma-\bar{\gamma})}{(\tau-\bar{\tau})}}$.

Published under an exclusive license by AIP Publishing