Stark units and the main conjectures for totally real fields

Kâzım Büyükboduk

Abstract

The main theorem of the author's thesis suggests that it should be possible to lift the Kolyvagin systems of Stark units, constructed in an earlier paper also by the author, to a Kolyvagin system over the cyclotomic Iwasawa algebra. In this paper, we verify that this indeed is true. This construction of Kolyvagin systems over the cyclotomic Iwasawa algebra from Stark units gives the first example towards a more systematic study of Kolyvagin system theory over an Iwasawa algebra when the core Selmer rank (in the sense of Mazur and Rubin) is greater than one. As a result of this construction, we reduce the main conjectures of Iwasawa theory for totally real fields to a statement of local Iwasawa theory, assuming the truth of the Rubin-Stark conjecture and Leopoldt's conjecture. This statement of local Iwasawa theory turns out to be interesting in its own right as it suggests a relation between solutions to p-adic and complex Stark conjectures.

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1. Introduction

This paper is an attempt to understand the Kolyvagin system theory (over an Iwasawa algebra) when the core Selmer rank (in the sense of [MR04, Definition 4.1.8 and 4.1.11]) is greater than one,

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and the work here should be considered as a continuation of our earlier paper [Büy07a].

The Kolyvagin system machinery is designed to bound the size of a Selmer group. In all well-known cases the bounds obtained relate to L-values, and thus provide a link between arithmetic and analytic data. Well-known prototypes for such a relation between arithmetic and analytic data are the Birch and Swinnerton-Dyer conjecture (more generally, Bloch-Kato conjectures) and the main conjectures of Iwasawa theory. The Kolyvagin system machinery has been successfully applied by many authors to obtain deep results towards proving these conjectures.

In this paper, we construct and study Kolyvagin systems over the cyclotomic Iwasawa algebra (henceforth denoted by Λ). In [Büy07b], Kolyvagin systems over Λ are proved to exist in a wide variety of cases provided that the core Selmer rank is one. Further, Mazur and Rubin [MR04, §5.3] show that these Kolyvagin systems can be used to compute the correct size of an appropriately defined Selmer group. However, when the core Selmer rank is greater than one, not much is known.

The most basic example of a situation where the core Selmer rank is greater than one arises if one attempts to utilize the Euler system that would come from the Rubin-Stark elements introduced in [Rub96]. Rubin was the first to study the Euler system of Stark units in [Rub92], where he proved a Gras-type formula for the χ-isotypic component of a certain ideal class group under certain assumptions on the character χ (which essentially ensured that the core Selmer rank of the Galois representation $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, in the sense of [MR04, Definition 4.1.8 and 4.1.11], is one). These assumptions have been removed and a more general Gras-type conjecture was proved in [Büy07a]. The proof of the main results of loc.cit. relies on the introduction of an auxiliary Selmer structure in a systematic way so as to cut down the core Selmer rank to one. One then obtains a useful collection of Kolyvagin systems for this auxiliary Selmer structure, and reduce to the treatment of [MR04].

The principal objective of this article is to generalize the methods of [Büy07a] to an Iwasawa theoretic setting. We first show how to lift the Kolyvagin systems for the auxiliary Selmer structure constructed in [Büy07a] to Kolyvagin systems over the cyclotomic Iwasawa algebra. To achieve this, we modify (in §2) the *classical* Selmer structure along the cyclotomic tower. The main theorem of [Büy07b] implies that there are Kolyvagin systems over Λ for the modified Selmer structure (see also §2.5 below). In §3, we show how to obtain these Kolyvagin systems from the *Euler system of Stark units* (which were introduced in [Rub96, §6]).

This approach to construct Kolyvagin systems from Euler systems is of course standard. Kolyvagin's original descent argument has been systematized by Mazur and Rubin, who construct what they call the *Euler system to Kolyvagin system map* ([MR04, Theorem 5.3.3]). The problem one faces is that when the Euler system to Kolyvagin system map is directly applied on the Euler system of Stark units, one does not in general obtain a Kolyvagin system for the modified Selmer structure but rather for a much coarser Selmer structure. This issue is tackled in §3.3. Once we have the Kolyvagin system for the modified Selmer structure, we apply the Kolyvagin system machinery of [MR04] to deduce the main results of this paper.

Before stating our main results, we introduce some notation and set the hypotheses that will be in effect throughout this paper.

1.1 Notation and Hypotheses

For any field F and a fixed separable closure $\overline{F}$ of F, we write $G_F := \mathrm{Gal}(\overline{F}/F)$ for the Galois group of $\overline{F}/F$.

Fix a totally real number field k and let $r = [k : \mathbb{Q}]$; fix also an algebraic closure $\overline{k}$ of k, and a rational odd prime p. Let k_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of k. Let k_n be the unique sub-extension of k_∞/k such that $[k_n : k] = p^n$. Let μ_{p^n} denote the p^n-th roots of unity, and let $\mu_{p^\infty} = \varinjlim_n \mu_{p^n}$.

Let $\chi : G_k \to \mathbb{Z}_p^\times$ be a non-trivial *totally even* character of G_k (i.e., it is trivial on all complex conjugations inside G_k) that has finite order, and let L be the fixed field of $\ker(\chi)$ inside $\overline{k}$. Denote the conductor of χ by f_χ, and let Δ be the Galois group $\mathrm{Gal}(L/k)$.

Let S be a finite set of places of k, which contains the infinite places, but does not contain any prime of k lying above p. Suppose also that $|S| \geqslant r+1$.

For any abelian group A, let $A^\wedge$ denote its p-adic completion. Define also A^χ to be the χ-isotypic component of $A^\wedge$.

Throughout this paper, T will always stand for the G_k-representation $\mathbb{Z}_p(1) \otimes \chi^{-1}$ (except in the Appendix, where T will be an arbitrary $\mathbb{Z}_p[[G_k]]$-module which is free of finite rank over $\mathbb{Z}_p$ and unramified outside a finite number of places of k).

The following hypotheses will be occasionally assumed (and all of them are required for our main results):

(A1) $p \nmid f_\chi$ (i.e., L/k is unramified at all primes of k above p),

(A2) k is unramified at all primes above p,

(A3) $\chi(\mathrm{Frob}_\wp) \neq 1$ for any prime $\wp$ of k above p, where $\mathrm{Frob}_\wp$ denotes a Frobenius element at $\wp$ inside $\mathrm{Gal}(\overline{k}/k)$.

The hypothesis (A1) is already assumed in [Büy07a], although we think that it may be possible to remove this hypothesis (both in [Büy07a] and in this paper) using an argument similar to the one used in the proof of Proposition 3.9.

We are almost certain that this paper could have been written without (A2) as well. However, this would have at least necessitated fixing a prime $\wp \subset k$ above p to define our auxiliary local conditions (§2.2). We decided to avoid making this choice of a prime $\wp$ for the sake of consistency with [Büy07a]. In fact, one of the major points of this paper is to initiate a study of Λ-adic Kolyvagin systems (in the sense of [Büy07b]) when the core Selmer rank is strictly larger than *one*. From this viewpoint, we decided that we would rather assume (A2) than losing our liberty in modifying the local conditions at p.

The hypothesis (A3), however, is a more serious assumption. This is the assumption $\mathbb{H}$.**sEZ** of [Büy07b, §2.2], translated to our setting; and it appears here for exactly the same reason why it appears in [Büy07b]: To avoid the interference of the *exceptional zeros* (in the sense of [Gre94, MTT86]) of the relevant p-adic L-function.

1.2 Statement of the main results

Suppose that the finite set of places S does not contain any non-archimedean prime which splits completely in L/k. Let $\mathbf{c}_{k_\infty}^{\mathrm{stark}} := \{\varepsilon_{k_n}^\chi\}_n$, where $\varepsilon_{k_n}^\chi$ denotes an appropriate twist of the Rubin-Stark element[1] (see also §3.1, §3.2, §4 and Remark 4.5 below for details). Let M_∞ be the maximal abelian p-extension of L_∞ unramified outside the set of primes lying above p, and set $\mathbb{T} = T \otimes \Lambda$. Let $H^1(k_p, \mathbb{T})$ denote the semi-local cohomology group[2] at p. Finally, for a torsion Λ-module $\mathbb{A}$, let $\mathrm{char}(\mathbb{A})$ denote its characteristic ideal.

THEOREM A. *Assume the hypotheses* (A1)-(A3) *hold. Assume also that Leopoldt's conjecture and*

[1] The definition of $\varepsilon_{k_n}^\chi$ actually depends on the choice of the set S; however, as long as we assume that S contains no non-archimedean prime which splits completely in L/k, this choice does not effect our main results. That is why we feel free to drop S from our notation.

[2] Let k_n be the unique sub-extension of k_∞/k such that $[k_n : k] = p^n$. Set $L_n = L \cdot k_n$, and $\mathcal{U}_n^\chi$ denote the local units inside $(L_n \otimes \mathbb{Q}_p)^\chi$. By Kummer theory (*c.f.*, [Rub00, §1.6.C and Proposition 3.2.6] and Lemma 3.4 below) $H^1(k_p, \mathbb{T})$ may be identified with $\varprojlim_n \mathcal{U}_n^\chi$.

the Rubin-Stark conjecture [Rub96, Conjecture B′*] are true. Then*

$$\mathrm{char}\left(\mathrm{Gal}(M_\infty/L_\infty)^\chi\right) = \mathrm{char}\left(\bigwedge^r H^1(k_p,\mathbb{T})/\Lambda\cdot\mathbf{c}_{k_\infty}^{\mathrm{stark}}\right). \tag{1}$$

We remark that when $k=\mathbb{Q}$ (i.e., when $r=1$), the element $\mathbf{c}_{k_\infty}^{\mathrm{stark}}$ can be obtained from the cyclotomic units. Furthermore, in this case, the ideal on the right hand side of (1) is generated by a certain Kubota-Leopoldt p-adic L-function. This fact goes back to Iwasawa [Iwa64]. Therefore, when $k=\mathbb{Q}$, Theorem A is equivalent to the main conjecture of Iwasawa theory in the most classical setting.

Let $\mathcal{L}_k^\chi$ denote the Deligne-Ribet p-adic L-function attached to the character χ (see [DR80] for the construction of this p-adic L-function). In view of the main conjectures for totally real fields (proved by Wiles [Wil90]), Theorem A yields the following:

Theorem B. *The ideal* $\mathrm{char}\left(\bigwedge^r H^1(k_p,\mathbb{T})/\Lambda\cdot\mathbf{c}_{k_\infty}^{\mathrm{stark}}\right)\subset\Lambda$ *is generated by* $\mathcal{L}_k^\chi$.

It would be more reasonable to expect that a relation such as the one given in Theorem B between the Rubin-Stark elements and the p-adic L-function should rather exist between the Deligne-Ribet p-adic L-function and the "p-adic" Stark elements (which would be solutions to a p-adic Stark conjecture, *c.f.* [Sol02, Sol04], rather than the "complex" Rubin-Stark conjecture). This suggests a link between solutions to the p-adic Stark conjectures and complex Stark conjectures at $s=0$. In a future paper, we hope to formulate this relation more precisely. Motivated by the example above in the case $k=\mathbb{Q}$ and by the work of Perrin-Riou [PR94a, PR95], one may hope to deduce Theorem B directly (i.e., without appealing to the truth of the main conjectures) and deduce the main conjecture for the totally real field k itself (of course, modulo the hypotheses of Theorem A).

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2. Modified Selmer structures

2.1 Selmer groups for $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$

Throughout this section, we use the notation that was set in §1.1. In addition, set $\Gamma := \mathrm{Gal}(k_\infty/k)$; and let $\Lambda := \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra.

We first recall Mazur and Rubin's definition of a *Selmer structure*, in particular, the *canonical Selmer structure* on $T \otimes \Lambda$.

2.1.1 *Local conditions.* Let R be a complete local noetherian ring, and let M be a $R[[G_k]]$-module which is free of finite rank over R. In this paper we will be interested in the case when $R = \Lambda$ or its certain quotients, and M is $T \otimes \Lambda$ or its relevant quotients by an ideal of Λ.

For each prime λ of k, a *local condition* $\mathcal{F}$ (at the prime λ) on M is a choice of an R-submodule $H^1_{\mathcal{F}}(k_\lambda, M)$ of $H^1(k_\lambda, M)$. For the prime p, a local condition $\mathcal{F}$ at p will be a choice of an R-submodule $H^1_{\mathcal{F}}(k_p, M)$ of the semi-local cohomology group

$$H^1(k_p, M) := \bigoplus_{\wp | p} H^1(k_\wp, M),$$

where the direct sum is over all the primes of k which lie above p.

For examples of local conditions see [MR04, Definition 1.1.6 and 3.2.1].

Suppose that $\mathcal{F}$ is a local condition (at a prime λ of k) on M. If M' is a submodule of M (resp., M'' is a quotient module), then $\mathcal{F}$ induces local conditions (which we still denote by $\mathcal{F}$) on M' (resp., on M''), by taking $H^1_{\mathcal{F}}(k_\lambda, M')$ (resp., $H^1_{\mathcal{F}}(k_\lambda, M'')$) to be the inverse image (resp., the image) of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the natural maps induced from

$$M' \hookrightarrow M, \qquad M \twoheadrightarrow M''.$$

Definition 2.1. *Propagation* of a local condition $\mathcal{F}$ on M to a submodule M' (and a quotient M'') of M is the local condition $\mathcal{F}$ on M' (and on M'') obtained following the procedure in the paragraph above.

For example, if I is an ideal of R, then a local condition on M induces local conditions on M/IM and $M[I]$, by *propagation*.

Definition 2.2. Define the *Cartier dual* of M to be the $R[[G_k]]$-module

$$M^* := \mathrm{Hom}(M, \mu_{p^\infty}),$$

where μ_{p^∞} stands for the p-power roots of unity.

Let λ be a prime of k. There is the perfect local Tate pairing

$$< , >_\lambda : H^1(k_\lambda, M) \times H^1(k_\lambda, M^*) \longrightarrow H^2(k_\lambda, \mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p.$$

2.1.2 *Selmer structures and Selmer groups.* Notation from §2.1.1 is in effect throughout this section. We will also write $\mathcal{D}_\lambda$ for $G_{k_\lambda} := \mathrm{Gal}(\overline{k_\lambda}/k_\lambda)$, whenever we wish to identify the group G_{k_λ} with a closed subgroup of G_k; namely with a particular decomposition group $\mathcal{D}_\lambda \subset G_k$ at λ. We further define $\mathcal{I}_\lambda \subset \mathcal{D}_\lambda$ to be the inertia group and $\mathrm{Frob}_\lambda \in \mathcal{D}_\lambda/\mathcal{I}_\lambda$ to be the arithmetic Frobenius element at λ.

Definition 2.3. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

– A finite set $\Sigma(\mathcal{F})$ of places of k, including all infinite places and primes above p, and all primes where M is ramified.

– For every $\lambda \in \Sigma(\mathcal{F})$, a local condition (in the sense of §2.1.1) on M (which we view now as a $R[[\mathcal{D}_\lambda]]$-module), i.e., a choice of an R-submodule

$$H^1_{\mathcal{F}}(k_\lambda, M) \subset H^1(k_\lambda, M).$$

If $\lambda \notin \Sigma(\mathcal{F})$, we will also write $H^1_{\mathcal{F}}(k_\lambda, M) = H^1_{\mathrm{f}}(k_\lambda, M)$, where the module $H^1_{\mathrm{f}}(k_\lambda, M)$ is the *finite* part of $H^1(k_\lambda, M)$, defined as in [MR04, Definition 1.1.6].

Definition 2.4. If $\mathcal{F}$ is a Selmer structure on M, we define the *Selmer module* $H^1_{\mathcal{F}}(k, M)$ to be the kernel of the sum of the restriction maps

$$H^1(\mathrm{Gal}(k_{\Sigma(\mathcal{F})}/k), M) \longrightarrow \bigoplus_{\lambda \in \Sigma(\mathcal{F})} H^1(k_\lambda, M)/H^1_{\mathcal{F}}(k_\lambda, M),$$

where $k_{\Sigma(\mathcal{F})}$ is the maximal extension of k which is unramified outside $\Sigma(\mathcal{F})$.

Example 2.5. In this example we recall [MR04, Definition 5.3.2]. Let $R = \Lambda$ be the cyclotomic Iwasawa algebra, and let $\mathbb{M}$ be a free R-module endowed with a continuous action of G_k, which is unramified outside a finite set of places of k. Fix a finite set of places Σ containing all the primes of k which lie above p, all the places at infinity, and all the primes of k where $\mathbb{M}$ is ramified. We define a Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{M}$ by setting $\Sigma(\mathcal{F}_\Lambda) = \Sigma$ and $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}) = H^1(k_\lambda, \mathbb{M})$ for $\lambda \in \Sigma(\mathcal{F}_\Lambda)$. This is what we call the *canonical Selmer structure* on $\mathbb{M}$. As remarked in [MR04, Definition 5.3.2], this definition is independent of the choice of Σ.

As in Definition 2.1, the induced Selmer structure on the quotients $\mathbb{M}/I\mathbb{M}$ will still be denoted by $\mathcal{F}_\Lambda$. Note that $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}/I\mathbb{M})$ will not usually be the same as $H^1(k_\lambda, \mathbb{M}/I\mathbb{M})$.

Remark 2.6. When $R = \Lambda$ and $\mathbb{M} = T \otimes \Lambda$ with $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, we will see in §2.5 that the Selmer structure $\mathcal{F}_{\mathrm{can}}$ of [Büy07b, §2.1] on the quotients $T \otimes \Lambda/(f)$ may be identified, under the hypothesis (A3) on χ, with the propagation of $\mathcal{F}_\Lambda$ to the quotients $T \otimes \Lambda/(f)$, for every distinguished polynomial f inside Λ.

Definition 2.7. A *Selmer triple* is a triple $(M, \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ is a Selmer structure on M and $\mathcal{P}$ is a set of rational primes, disjoint from $\Sigma(\mathcal{F})$.

Remark 2.8. Although one might identify the cohomology groups in our setting (when the Galois module in question is $T \otimes \Lambda$ with $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, or its quotients by certain ideals of Λ) with certain groups of units, thanks to Kummer theory; we will insist on using the cohomological language for the sake of notational consistency with [MR04] from which we borrow the main technical results. This way, we also hope that it will be easier to hypothesize our approach for potential generalizations to other settings.

2.2 Modifying the local conditions at p

In [Büy07a, §1], the classical local conditions at the primes above p are modified to obtain a Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T. The objective of this section is to lift the Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T to a Selmer structure on $T \otimes \Lambda$.

In this section, we will make use of the results from the Appendix to determine the structure of the semi-local cohomology group $H^1(k_p, T \otimes \Lambda)$ as a Λ-module. Although there might be a more direct way (in this particular setting where $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$) to obtain these results on the structure of $H^1(k_p, T \otimes \Lambda)$, without appealing to the description of the Galois cohomology groups in terms of

(φ,Γ)-modules; we believe the more general approach via Fontaine's theory of (φ,Γ)-modules might allow our strategy to apply in many other settings.

Recall that k_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of k, and $\Gamma = \mathrm{Gal}(k_\infty/k)$. Assume that the hypotheses (A1)-(A3) hold until the end of §2.2. The hypothesis (A2) implies that the extension k_∞/k is totally ramified at all primes $\wp \subset k$ over p. Let $k_\wp$ denote the completion of k at $\wp$, and let $k_{\wp,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $k_\wp$. We may therefore identify $\mathrm{Gal}(k_{\wp,\infty}/k_\wp)$ by Γ for all $\wp|p$ and henceforth Γ will stand for any of these Galois groups. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra, as usual. We also fix a topological generator γ of Γ, and set $\mathbf{X} = \gamma - 1$ (and we occasionally identify Λ by the power series ring $\mathbb{Z}_p[[\mathbf{X}]]$).

Following the notation of Appendix A, write

$$H^1_{\mathrm{Iw}}(k_\wp, T) := \varprojlim_n H^1(k_{\wp,n}, T),$$

where $k_{\wp,n}$ denotes the unique subfield of $k_{\wp,\infty}$, which has degree p^n over $k_\wp$. By Shapiro's Lemma, one may canonically identify $H^i_{\mathrm{Iw}}(k_\wp, T)$ by $H^i(k_\wp, T\otimes\Lambda)$ (c.f., [Col98a, Proposition II.1.1]), for all $i \in \mathbb{Z}^+$. Define

$$H^i_{\mathrm{Iw}}(k_p, T) := \bigoplus_{\wp|p} H^i_{\mathrm{Iw}}(k_\wp, T) \text{ and } H^i(k_p, T\otimes\Lambda) := \bigoplus_{\wp|p} H^i(k_\wp, T\otimes\Lambda)$$

(and these two Λ-modules are canonically isomorphic by the argument above).

Set $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ for any local field K. We also define $T_m := T\otimes\Lambda/(\mathbf{X}^m)$ and $T_{s,m} := T\otimes\Lambda/(p^s,\mathbf{X}^m)$ for $s,m\in\mathbb{Z}^+$, following the notation of [Büy07b, §2.3.2].

Proposition 2.9.

(i) *$H^1(k_p, T\otimes\Lambda) = H^1_{\mathrm{Iw}}(k_p, T)$ is a free Λ-module of rank r.*

(ii) *The map $H^1(k_p, T\otimes\Lambda) \to H^1(k_p, T_m)$ is surjective for all $m\in\mathbb{Z}^+$.*

Proof. For every $\wp\subset k$ above p, it follows from our hypothesis (A3) that $T^{H_{k_\wp}} = 0$, and thus (i) is immediate from Theorem A.8. It also follows that $(T^*)^{G_{k_\wp}} = 0$, again thanks to (A3); hence the proof of [Büy07b, Lemma 2.11] shows that $H^2(k_\wp, T\otimes\Lambda) = 0$. But then

$$\mathrm{coker}\{H^1(k_p, T\otimes\Lambda) \longrightarrow H^1(k_p, T_m)\} = \bigoplus_{\wp|p} H^2(k_\wp, T\otimes\Lambda)[\mathbf{X}^m] = 0,$$

and (ii) follows. □

Fix a free Λ-direct summand $\mathbb{L}_\infty$ inside of $H^1(k_p, T\otimes\Lambda)$, which is free of rank one as a Λ-module. By Proposition 2.9, this also fixes a free $\Lambda/(\mathbf{X}^m)$-direct summand $\mathbb{L}_m$ of $H^1(k_p, T_m)$ which is free of rank one (as a $\Lambda/(\mathbf{X}^m)$-module). When $m=1$, we denote $\mathbb{L}_1$ by $\mathcal{L}$.

Definition 2.10. Let $\mathbb{L}_\infty$ be as above. We define the $\mathbb{L}_\infty$-*modified Selmer structure* $\mathcal{F}_{\mathbb{L}_\infty}$ on $T\otimes\Lambda$ as follows:

- $\Sigma(\mathcal{F}_{\mathbb{L}_\infty}) = \Sigma(\mathcal{F}_\Lambda)$,
- $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T\otimes\Lambda) = \mathbb{L}_\infty \subset H^1(k_p, T\otimes\mathbb{L}_\infty)$,
- $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_\lambda, T\otimes\Lambda) = H^1_{\mathcal{F}_\Lambda}(k_\lambda, T\otimes\Lambda)$, for $\lambda\nmid p$.

The induced Selmer structure on the collection of quotients $\mathcal{T}_0 := \{T_{s,m}\}$ will also be denoted by $\mathcal{F}_{\mathbb{L}_\infty}$ (except for the induced Selmer structure on $T_1 = T\otimes\Lambda/(\mathbf{X}) = T$ and its quotients $T_{s,1} = T\otimes\Lambda/(p^s,\mathbf{X}) = T/p^sT$, which will be denoted by $\mathcal{F}_\mathcal{L}$ for notational consistency with [Büy07a]).

2.3 Local duality and the dual Selmer structure

We will discuss local duality in great generality. Let R be a complete local noetherian ring and let M be a free R-module of finite rank which is endowed with a continuous action of G_k. Let $M^* = \mathrm{Hom}(M, \mu_{p^\infty})$ be the Cartier dual of M. For each prime λ of k, there is a perfect pairing, called the local Tate pairing

$$<,>_\lambda \colon H^1(k_\lambda, M) \times H^1(k_\lambda, M^*) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Let $\mathcal{F}$ be a Selmer structure on M. For each prime λ of k, define $H^1_{\mathcal{F}^*}(k_\lambda, M^*) := H^1_{\mathcal{F}}(k_\lambda, M)^\perp$ as the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the local Tate pairing. The Selmer structure $\mathcal{F}^*$ on M^* defined in this way will be called the *dual Selmer structure*.

As in Definition 2.4, the dual Selmer structure gives rise to the *dual Selmer module*

$$H^1_{\mathcal{F}^*}(k, M^*) := \ker \left\{ H^1(\mathrm{Gal}(k_{\Sigma(\mathcal{F})}/k), M^*) \longrightarrow \bigoplus_{\lambda \in \Sigma(\mathcal{F})} \frac{H^1(k_\lambda, M^*)}{H^1_{\mathcal{F}^*}(k_\lambda, M^*)} \right\}.$$

2.4 Comparison of Selmer modules

As our sights are set on Iwasawa's main conjecture over totally real number fields, we now construct the *correct* Iwasawa module: A Selmer module which should relate to the appropriate p-adic L-function (which is the Deligne-Ribet p-adic L-function in our particular setting, c.f., [DR80] for its construction).

Once this Selmer module is defined, we will use Poitou-Tate global duality to compare it to $H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k, (T \otimes \Lambda)^*)$, the dual Selmer module attached to the dual $\mathbb{L}_\infty$-modified Selmer structure.

Definition 2.11. The *p-strict Selmer structure* $\mathcal{F}_{\mathrm{str}}$ on $T \otimes \Lambda$ is defined by the following data:

- $\Sigma(\mathcal{F}_{\mathrm{str}}) = \Sigma(\mathcal{F}_\Lambda)$,
- $H^1_{\mathcal{F}_{\mathrm{str}}}(k_p, T \otimes \Lambda) = 0$,
- $H^1_{\mathcal{F}_{\mathrm{str}}}(k_\lambda, T \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(k_\lambda, T \otimes \Lambda) = H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_\lambda, T \otimes \Lambda)$, for $\lambda \nmid p$.

Hence, for the dual Selmer structure $\mathcal{F}^*_{\mathrm{str}}$ (in the sense of §2.3), we have:

- $H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_p, (T \otimes \Lambda)^*) = H^1(k_p, (T \otimes \Lambda)^*)$,
- $H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_\lambda, (T \otimes \Lambda)^*) = H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k_\lambda, (T \otimes \Lambda)^*)$, for $\lambda \nmid p$.

For any $\mathbb{Z}_p$-module A, let $A^\vee := \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ denote its Pontryagin dual.

Later in §4, we will explain why $H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_p, (T \otimes \Lambda)^*)^\vee$ is the *correct* Iwasawa module in this setting which relates[3] to the Deligne-Ribet p-adic L-function.

Proposition 2.12. *Assume Leopoldt's conjecture. Then there is an exact sequence*

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, T \otimes \Lambda) \xrightarrow{\mathrm{loc}_p} \mathbb{L}_\infty \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{str}}}(k, (T \otimes \Lambda)^*)^\vee \longrightarrow H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k, (T \otimes \Lambda)^*)^\vee \longrightarrow 0.$$

Proof. We allow ourselves to be sketchy in this proof, since similar versions of this Proposition can already be found in the literature (c.f., [Rub00, Theorem I.7.3 and III.2.10], [dS87, §III.1.7]).

[3] Wiles [Wil90] has already proved this relation using different techniques than ours. He systematically uses Hida's theory of Λ-adic Hilbert modular forms to construct certain unramified extensions from which he deduces the main conjectures.

Set $\gamma_n = \gamma^{p^n}$ for $n \in \mathbb{Z}_{\geqslant 0}$, and let $\mathcal{L}_n \subset H^1(k_p, T \otimes \Lambda/(\gamma_n - 1))$ be the image of $\mathbb{L}_\infty$ under the surjective map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T \otimes \Lambda/(\gamma_n - 1)).$$

Let $\mathcal{F}_{\mathcal{L}_n}$ denote the Selmer structure on $T \otimes \Lambda/(\gamma_n - 1)$, which is obtained by propagating the Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ on $T \otimes \Lambda$ to its quotient $T \otimes \Lambda/(\gamma_n - 1)$. The propagated Selmer structure from $\mathcal{F}_{\mathrm{str}}$ on $T \otimes \Lambda$ (defined in Definition 2.11) to the quotient $T \otimes \Lambda/(\gamma_n - 1)$ will still be denoted by $\mathcal{F}_{\mathrm{str}}$.

By Shapiro's lemma, one may canonically identify $H^1(k, T \otimes \Lambda/(\gamma_n - 1))$ by $H^1(k_n, T)$; and for every prime $\lambda \subset k$, $H^1(k_\lambda, T \otimes \Lambda/(\gamma_n - 1))$ by $H^1((k_n)_\lambda, T)$; c.f., [Rub00, §B.4 and §B.5]. This way, we may view $\mathcal{F}_{\mathcal{L}_n}$ and $\mathcal{F}_{\mathrm{str}}$ as Selmer structures on the G_{k_n}-representation T.

It is easy to see that one then has the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathcal{F}_{\mathrm{str}}}(k_n, T) & \longrightarrow & H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) & \xrightarrow{\mathrm{loc}_p} & \mathcal{L}_n \\ 0 & \longrightarrow & H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*) & \longrightarrow & H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_n, T^*) & \xrightarrow{\mathrm{loc}^*_p} & \frac{H^1_{\mathcal{F}^*_{\mathrm{str}}}((k_n)_p, T^*)}{H^1_{\mathcal{F}^*_{\mathcal{L}_n}}((k_n)_p, T^*)} \end{array} \tag{2}$$

Since we assume Leopoldt's conjecture, [Büy07a, Proposition 1.4] (applied with the totally real field k_n instead of k) and Kummer theory give:

i) $H^1(k_n, T) = (L_n^\times)^\chi$, and $H^1((k_n)_p, T) = (L_n \otimes \mathbb{Q}_p)^{\times,\chi} \supset \mathcal{L}_n$;

ii) $H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) = (\mathcal{O}_{L_n}^\times)^\chi \cap \mathcal{L}_n \subset \mathcal{L}_n$;

iii) $H^1_{\mathcal{F}_{\mathrm{str}}}(k_n, T) = \ker\left(H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) \to \mathcal{L}_n\right) = 0$.

Here, $\mathcal{O}_{L_n}^\times$ stands for the ring of integers of the the number field L_n. Thanks to Leopoldt's conjecture, we may identify $(\mathcal{O}_{L_n}^\times)^\chi$ with its isomorphic image under the localization map at p, and the intersection in (ii) above is taken inside $(L_n \otimes \mathbb{Q}_p)^{\times,\chi}$ after this identification.

The exact sequences of the diagram (2) can therefore be rewritten as

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) & \xrightarrow{\mathrm{loc}_p} & \mathcal{L}_n \\ 0 & \longrightarrow & H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*) & \longrightarrow & H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_n, T^*) & \xrightarrow{\mathrm{loc}^*_p} & \frac{H^1_{\mathcal{F}^*_{\mathrm{str}}}((k_n)_p, T^*)}{H^1_{\mathcal{F}^*_{\mathcal{L}_n}}((k_n)_p, T^*)} \end{array}$$

Further, the Poitou-Tate global duality theorem (c.f., [Mil86, I.4.10]) says that $\mathrm{im}(\mathrm{loc}_p) = \mathrm{im}(\mathrm{loc}^*_p)^\perp$ with respect to the local Tate pairing. This in return translates the diagram above into the following exact sequence, c.f., the proof of [Rub00, Theorem I.7.3] for details:

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) \xrightarrow{\mathrm{loc}_p} \mathcal{L}_n \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{str}}}(k_n, T^*)^\vee \longrightarrow H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*)^\vee \longrightarrow 0. \tag{3}$$

Now passing to the inverse limit with respect to n in (3) (which we may without harming the exactness thanks to [Rub00, Proposition B.1.1]) we obtain the exact sequence which we seek for. □

Suppose $c \in H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, T \otimes \Lambda)$ is any class.

Corollary 2.13. *Under the hypotheses of Proposition 2.12, the following sequence is exact:*

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, T \otimes \Lambda)/\Lambda \cdot c \xrightarrow{\mathrm{loc}_p} \mathbb{L}_\infty/\Lambda \cdot c \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{str}}}(k, (T \otimes \Lambda)^*)^\vee \longrightarrow H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k, (T \otimes \Lambda)^*)^\vee \longrightarrow 0.$$

2.5 Kolyvagin systems for the $\mathbb{L}_\infty$-modified Selmer triple - I

From this section on, we concentrate on the particular G_k-representations $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$ and $\mathbb{T} = T \otimes \Lambda$, with χ as in §1.1. Throughout §2.5, we assume (A1)-(A3).

Remark 2.14. Consider the following properties which play a role also in [MR04, Büy07b]:

(**H**.**1**) The residual $\mathbb{F}_p[[G_k]]$-representation T/pT is absolutely irreducible.

(**H**.**2**) There is a $\tau \in G_k$ such that $\tau = 1$ on μ_{p^∞} and $T/(\tau - 1)T$ is free of rank one over $\mathbb{Z}_p$.

(**H**.**3**) $H^0(k, T/pT) = H^0(k, T^*[p]) = 0$.

(**H**.**4**) Either
(**H**.**4a**) $\mathrm{Hom}_{\mathbb{F}_p[[G_k]]}(T/pT, T^*[p]) = 0$, or
(**H**.**4b**) $p > 4$.

Before we explain why $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$ has these properties, let us point out that the hypothesis (**H**.**3**) above is implied by what Mazur and Rubin [MR04, §3.5] call (**H**.**3**) (c.f., [MR04, Lemma 3.5.2]). The hypothesis (**H**.**3**) above is sufficient for our purposes.

The hypothesis (**H**.**1**) holds because T/pT is one dimensional (as an $\mathbb{F}_p$-vector space). The hypothesis (**H**.**2**) holds with $\tau = \mathrm{id} \in G_k$, and (**H**.**3**) holds since we assumed χ is non-trivial. Finally, it is easy to see that (**H**.**4a**) holds.

Observe further that the following versions of the hypotheses $\mathbb{H}$.**T** and $\mathbb{H}$.**sEZ** of [Büy07b, §2.2] hold for T as well:

($\mathbb{H}$.$\mathbf{T}_{/k}$) $(T \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{I}_\lambda}$ is divisible for every prime $\lambda \nmid p$, $\lambda \subset k$.

($\mathbb{H}$.$\mathbf{sEZ}_{/k}$) $(T^*)^{G_{k_\wp}} = 0$ for primes $\wp | p$.

Note that ($\mathbb{H}$.$\mathbf{sEZ}_{/k}$) is implied by our assumption (A3).

Next, we define the Selmer structure $\mathcal{F}_{\mathrm{can}}$ on certain quotients of $T \otimes \Lambda$, following [Büy07b, Definition 2.2]:

Definition 2.15. Suppose $f \in \Lambda$ is any *distinguished* polynomial, in the sense that the quotient $\Lambda/(f)$ is a free $\mathbb{Z}_p$-module of finite rank. Let $\mathcal{F}_{\mathrm{can}}$ be the following Selmer structure on $T_f := T \otimes \Lambda/(f)$:

- $\Sigma(\mathcal{F}_{\mathrm{can}}) = \Sigma(\mathcal{F}_\Lambda)$,
- The local conditions are given by

$$H^1_{\mathcal{F}_{\mathrm{can}}}(k_\lambda, T_f) = \begin{cases} H^1(k_\lambda, T_f) & , \quad \text{if } \lambda | p \\ \\ H^1_{\mathrm{f}}(k_\lambda, T_f) & , \quad \text{if } \lambda \in \Sigma(\mathcal{F}_{\mathrm{can}}) \text{ and } \lambda \nmid p \end{cases}$$

with

$$H^1_{\mathrm{f}}(k_\lambda, T_f) = \ker\{H^1(k_\lambda, T_f) \longrightarrow H^1(k_\lambda^{\mathrm{unr}}, T_f \otimes \mathbb{Q}_p)\},$$

where k_λ^{unr} is the maximal unramified extension of k_λ.

The induced Selmer structure on the quotients $T \otimes \Lambda/(p^s, f)$, which is obtained by *propagating* $\mathcal{F}_{\mathrm{can}}$ (in the sense of Definition 2.1) will also be denoted by $\mathcal{F}_{\mathrm{can}}$.

Recall the definition of the collection $\mathcal{T}_0 = \{T_{s,m}\}_{s,m}$.

Remark 2.16. By the definition of $\mathcal{F}_{\mathbb{L}_\infty}$, the local conditions on $T_{s,m}$ at primes $\lambda \nmid p$ determined by $\mathcal{F}_{\mathbb{L}_\infty}$ coincide with the local conditions determined by $\mathcal{F}_\Lambda$; and thanks to [Büy07b, Corollary 2.8

and 2.9], they also coincide with the local conditions determined by $\mathcal{F}_{\mathrm{can}}$, since $\mathbb{H}.\mathbf{T}_{/k}$ holds true. Indeed, it is proved in loc.cit. that all these local conditions coincide with

$$H^1_{\mathrm{unr}}(k_\lambda, T_{s,m}) := \ker\left\{H^1(k_\lambda, T_{s,m}) \longrightarrow H^1(k_\lambda^{\mathrm{unr}}, T_{s,m})\right\},$$

as long as the hypothesis $\mathbb{H}.\mathbf{T}_{/k}$ holds true.

Recall that the rank-one $\mathbb{Z}_p$-direct summand $\mathcal{L}$ of $H^1(k_p, T)$ is defined as the image of $\mathbb{L}_\infty$ under the canonical (surjective) map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T).$$

Let $\mathcal{F}_{\mathcal{L}}$ denote the Selmer structure on T, which is obtained by propagating the Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$. This agrees with the definition of $\mathcal{F}_{\mathcal{L}}$ in [Büy07a, §1.1].

Proposition 2.17.

(i) *The Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ is cartesian on $\mathcal{T}_0$, in the sense of [Büy07b, Definition 2.4].*

(ii) *The core Selmer rank $\mathcal{X}(T, \mathcal{F}_{\mathcal{L}})$ (in the sense of [MR04, Definition 4.1.11]) of the Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T is one.*

Proof. (ii) is [Büy07a, Proposition 1.8].

As pointed out in Remark 2.16, the Selmer structures $\mathcal{F}_{\mathrm{can}}$ and $\mathcal{F}_{\mathbb{L}_\infty}$ determine the same local conditions on the quotients $T_{s,m}$, at every place $v \nmid p$. Hence, the local conditions determined by $\mathcal{F}_{\mathbb{L}_\infty}$ are cartesian at $v \nmid p$, by [Büy07b, Proposition 2.10]. It therefore suffices to check that $\mathcal{F}_{\mathbb{L}_\infty}$ is cartesian on $\mathcal{T}_0$ at p, i.e., we need to verify the following properties **C.1**-**C.3** of [Büy07b, Definition 2.4] for the local conditions at p, determined by $\mathcal{F}_{\mathbb{L}_\infty}$ on the collection $\mathcal{T}_0 = \{T_{s,m}\}$:

(C1) For positive integers $s \leqslant s'$ and $m \leqslant m'$, the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s,m})$ is the image of the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s',m'})$ under the canonical map

$$H^1(k_p, T_{s',m'}) \longrightarrow H^1(k_p, T_{s,m}).$$

(C2) For positive integers s, m and α, the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s,m})$ is the inverse image of the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s,m+\alpha})$ under the natural map

$$H^1(k_p, T_{s,m}) \longrightarrow H^1(k_p, T_{s,m+\alpha}),$$

which is induced from the injection $T_{s,m} \xrightarrow{[\mathbf{X}^\alpha]} T_{s,m+\alpha}$, where $[\mathbf{X}^\alpha]$ stands for the multiplication-by-$\mathbf{X}^\alpha$ map.

(C3) For positive integers s, m and α, the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s,m})$ is the inverse image of the module $H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s+\alpha,m})$ under the natural map

$$H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s,m}) \longrightarrow H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p, T_{s+\alpha,m}),$$

which is induced from the injection $T_{s,m} \xrightarrow{[p^\alpha]} T_{s+\alpha,f}$, where $[p^\alpha]$ is the multiplication-by-p^α map.

The property **C.1** holds by the definition of $\mathcal{F}_{\mathbb{L}_\infty}$ on $\mathcal{T}_0$, and the property **C.3** follows easily from [MR04, Lemma 3.7.1] (which applies since the line $\mathbb{L}_m \subset H^1(k_p, T_m)$ is a direct summand of $H^1(k_p, T_m)$, i.e., the $\Lambda/(\mathbf{X}^m)$-module $H^1(k_p, T_m)/\mathbb{L}_m$ is free, for any $m \in \mathbb{Z}^+$).

We now verify **C.2**. For $s \in \mathbb{Z}^+$, let $\mathbb{L}_{s,m}$ be the image of $\mathbb{L}_m$ under the reduction map

$$H^1(k_p, T_m) \longrightarrow H^1(k_p, T_{s,m}).$$

It is easy to see that $\mathbb{L}_{s,m}$ (resp., $H^1(k_p, T_{s,m})/\mathbb{L}_{s,m}$) is a free $\mathbb{L}/(p^s, \mathbf{X}^m)$-module of rank one (resp., of rank $[k:\mathbb{Q}]-1$). To complete the proof, we need to check that the map

$$H^1(k_p, T_{s,m})/\mathbb{L}_{s,m} \overset{[\mathbf{X}^{M-m}]}{\longrightarrow} H^1(k_p, T_{s,M})/\mathbb{L}_{s,M}$$

induced from the map

$$[\mathbf{X}^{M-m}] : \Lambda/(\mathbf{X}^m) \longrightarrow \Lambda/(\mathbf{X}^M)$$

is injective for all $M \geqslant m$. But this is evident, since $H^1(k_p, T_{s,m})/\mathbb{L}_{s,m}$ (resp., $H^1(k_p, T_{s,M})/\mathbb{L}_{s,M}$) is a free $\Lambda/(p^s, \mathbf{X}^m)$-module (resp., a free $\Lambda/(p^s, \mathbf{X}^M)$-module) of rank $[k:\mathbb{Q}]-1$. □

Let $\mathcal{P}$ denote the set primes of k, whose elements do not divide pf_χ. For positive integers s and m, define

$$\mathcal{P}_{s+m} := \{\mathfrak{q} \in \mathcal{P} : \mathfrak{q} \text{ splits completely in } L(\mu_{p^{s+m+1}})/k\}.$$

Note that the set $\mathcal{P}_j$ is exactly the set of primes which are determined by [MR04, Definition 3.1.6] or [Rub00, Definition IV.1.1], in the particular case $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$. Let $\mathcal{N}$ (resp., $\mathcal{N}_j$) denote the set of square free products of primes $\mathfrak{q}$ in $\mathcal{P}$ (resp., in $\mathcal{P}_j$), with the convention that $1 \in \mathcal{N}_j \subset \mathcal{N}$.

Definition 2.18.(Compare to [MR04, Definition 3.1.6].)

(i) Define the (generalized) module of Kolyvagin systems

$$\overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P}) := \varprojlim_s \varinjlim_j \mathbf{KS}(T/p^sT, \mathcal{F}_\mathcal{L}, \mathcal{P}_j),$$

where $\mathbf{KS}(T/p^sT, \mathcal{F}_\mathcal{L}, \mathcal{P}_j)$ is the module of Kolyvagin systems for the Selmer structure $\mathcal{F}_\mathcal{L}$ on the representation T/p^sT, as in [MR04, Definition 3.1.3].

(ii) Define the module of Λ-*adic Kolyvagin systems*

$$\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}) := \varprojlim_{s,m} \varinjlim_j \mathbf{KS}(T_{s,m}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}_j),$$

where $\mathbf{KS}(T_{s,m}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}_j)$ is the module of Kolyvagin systems for the Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ on the representation $T_{s,m}$.

Theorem 2.19.

(i) *The $\mathbb{Z}_p$-module $\overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P})$ is free of rank one.*

(ii) *The Λ-module $\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$ is free of rank one, and the canonical map*

$$\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}) \longrightarrow \overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P})$$

is surjective.

Any generator of these cyclic modules will be called a *primitive* Kolyvagin system.

Proof. Part (i) follows is [MR04, Theorem 5.2.10(ii)]. We briefly go over the hypotheses which are needed to apply [MR04, Theorem 5.2.10(ii)], and explain why they hold in our case.

To apply Theorem 5.2.10(ii) of [MR04] with the Selmer triple $(T, \mathcal{F}_\mathcal{L}, \mathcal{P})$, we first need to know that $\mathcal{X}(T, \mathcal{F}_\mathcal{L}) = 1$, and this is true thanks to Proposition 2.17(ii). Also, one needs to check that hypotheses H.1 through H.6 of [MR04, §3.5] hold. We already verified H.1-4 above; hypothesis H.5 holds with $\mathcal{P}$ given as above, and H.6 holds for $\mathcal{F}_\mathcal{L}$ thanks to Proposition 2.17(i) (in fact, we only need to know that the condition (**C.3**) holds with $m = 1$).

The result which is analogous to [MR04, Theorem 5.2.10(ii)] and which enables us to conclude (ii) has been previously proved by the author ([Büy07b, Theorem 3.23]). The statement of Theorem

3.23 of loc.cit. is almost identical to the statement of (ii), only the base field $\mathbb{Q}$ should be replaced by k, and the Selmer structure $\mathcal{F}_{\mathrm{can}}$ should be replaced by $\mathcal{F}_{\mathbb{L}_\infty}$. The proof of [Büy07b, Theorem 3.23] works *verbatim* after these changes as well: The technical points which need to be verified are:

(1) The G_k-representation T satisfies (**H**.**1**)-(**H**.**4**) as well as ($\mathbb{H}$.$\mathbf{T}_{/k}$) and ($\mathbb{H}$.$\mathbf{sEZ}_{/k}$).
(2) We have $\mathcal{X}(T, \mathcal{F}_{\mathcal{L}}) = 1$ for the core Selmer rank.
(3) Properties **C.1**-**C.3** hold for $\mathcal{F}_{\mathbb{L}_\infty}$.

Proposition 2.17 shows that (2) and (3) hold, and (1) is checked in Remark 2.14. □

In the next section, we will obtain a generator of the cyclic Λ-module $\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$ using the (conjectural) Rubin-Stark elements. Note, however, that the existence of Λ-adic Kolyvagin systems is unconditional: It does not rely on Rubin's conjecture [Rub96, Conjecture B′].

For the convenience of the reader, we record the main application of a Λ-adic Kolyvagin system $\boldsymbol{\kappa} = \left\{\{\kappa_\tau(s,m)\}_{\tau \in \mathcal{N}_{s+m}}\right\}_{s,m}$, which is a generator of the module $\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$. See [MR04, §3] for an explanation of our notation. We loosely say here that $\kappa_\tau(s,m) \in H^1(k, T_{s,m})$, and by definition, there is a well defined element

$$\kappa_1 = \{\kappa_1(s,m)\}_{s,m} \in \varprojlim_{s,m} H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, T_{s,m}) = H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, T \otimes \Lambda).$$

For notational simplicity, we write $\mathbb{T} = T \otimes \Lambda$. Recall that $\mathrm{char}(\mathbb{A})$ denotes the characteristic ideal of a finitely generated Λ-module $\mathbb{A}$, with the convention that $\mathrm{char}(\mathbb{A}) = 0$ unless $\mathbb{A}$ is Λ-torsion.

Theorem 2.20. *Under the assumptions* (A1)-(A3)*,*

$$\mathrm{char}(H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k, \mathbb{T}^*)^\vee) = \mathrm{char}(H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k, \mathbb{T})/\Lambda \cdot \kappa_1).$$

Proof. This is [MR04, Theorem 5.3.10(iii)] applied in our setting. We remark that all the hypotheses of Theorem 5.3.10(iii) of loc. cit. hold thanks to (A1)-(A3) (as we have also demonstrated above). □

Theorem 2.20 will be used for applications towards the main conjectures of Iwasawa theory for totally real fields. We remark once again that Theorem 2.20 does not rely on any conjecture; however, to link the statement of Theorem 2.20 with the relevant L-values, we will need to construct a Λ-adic Kolyvagin system from the (conjectural) Rubin-Stark elements. Furthermore, we will need Leopoldt's conjecture in order to prove that the Λ-adic Kolyvagin system constructed this way is non-trivial.

3. Λ-adic Kolyvagin systems of Stark units

In this section, we review Rubin's integral refinement of Stark's conjectures and construct Kolyvagin systems for the modified Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ on $T\otimes\Lambda$, coming from the Stark elements of Rubin. We note that the existence of Kolyvagin systems for $\mathcal{F}_{\mathbb{L}_\infty}$ on $T\otimes\Lambda$ was proved *unconditionally* in the previous section, building on the main result of [Büy07b].

For the rest of this paper, we assume that the Rubin-Stark conjecture [Rub96, Conjecture B′] holds.

Before we give an outline of Rubin's conjectures, we set some notation. Let $k, k_\infty, \chi, f_\chi$ and L be as above. For a cycle τ of the number field k, let $k(\tau)$ be the maximal p-extension inside the ray class field of k modulo τ. Let k_n be the unique sub-extension of k_∞/k which has degree p^n over k. For any other number field F, we define $F(\tau)$ as the composite of $k(\tau)$ and F; and F_n as the composite of k_n and F. Let

$$\mathcal{K} = \{L_n(\tau) : \tau \text{ is a (finite) cycle of } k \text{ prime to } f_\chi p \text{ and } n\in\mathbb{Z}_{\geqslant 0}\}$$

and

$$\mathcal{K}_0 = \{k_n(\tau) : \tau \text{ is a (finite) cycle of } k \text{ prime to } f_\chi p \text{ and } n\in\mathbb{Z}_{\geqslant 0}\}$$

be two collections of abelian extensions of k.

3.1 Stark elements and Euler systems (of rank r) for $\mathbb{Z}_p(1)$

Fix a (finite) set S of places of k that does *not* contain any prime above p, but contains the set of infinite places S_∞ and all primes λ that divide the conductor f_χ of χ. Assume that $|S|\geqslant r+1$. For each $K\in\mathcal{K}$, let

$$S_K = S\cup\{\text{places of } k \text{ at which } K \text{ is ramified}\}$$

be another set of places of k. Let $\mathcal{O}^\times_{K,S_K}$ denote the S_K units of K (i.e., the elements of $K^\times$ that are units away from the primes of K above S_K), and Δ_K (resp., δ_K) denote $\mathrm{Gal}(K/k)$ (resp., $|\mathrm{Gal}(K/k)|$). Conjecture B′ of [Rub96] predicts the existence of certain elements[4]

$$\tilde{\varepsilon}_{K,S_K}\in\Lambda_{K,S_K}\subset\frac{1}{\delta_K}\bigwedge^r\mathcal{O}^\times_{K,S_K}$$

where the module Λ_{K,S_K} is defined in [Rub96, §2.1], and has the property that for any homomorphism

$$\tilde{\psi}\in\mathrm{Hom}_{\mathbb{Q}_p[\Delta_K]}(\bigwedge^r\mathcal{O}^{\times,\wedge}_{K,S_K}\otimes\mathbb{Q}_p,\mathcal{O}^{\times,\wedge}_{K,S_K}\otimes\mathbb{Q}_p)$$

which is induced from a homomorphism

$$\psi\in\mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(\bigwedge^r\mathcal{O}^{\times,\wedge}_{K,S_K},\mathcal{O}^{\times,\wedge}_{K,S_K}),$$

one has $\tilde{\psi}(\Lambda_{K,S_K})\subset\mathcal{O}^{\times,\wedge}_{K,S_K}$. We remark that the r-th exterior power $\bigwedge^r\mathcal{O}^{\times,\wedge}_{K,S_K}$ (and all other exterior powers which appear below) is taken in the category of $\mathbb{Z}_p[\Delta_K]$-modules.

Remark 3.1. Rubin's conjecture predicts that the elements $\tilde{\varepsilon}_{K,S_K}$ should in fact lie inside the module $\frac{1}{\delta_K}\bigwedge^r\mathcal{O}^\times_{K,S_K,\mathcal{T}}$, where $\mathcal{T}$ is a finite set of primes disjoint from S_K, chosen in a way that the group $\mathcal{O}^\times_{K,S_K,\mathcal{T}}$ of S_K-units which are congruent to 1 modulo all the primes in $\mathcal{T}$ is torsion-free. However, in our case, any set $\mathcal{T}$ which contains a prime other than 2 will suffice (since all the fields

[4]Note that what we call here $\tilde{\varepsilon}_{K,S_K}$ is denoted by ε_{K,S_K} in [Rub96, Büy07a]. For an explanation for the change of notation, see Remark 3.3 and Remark 4.1 below.

which appear in our paper are totally real). We may therefore fix $\mathcal{T} = \{\mathfrak{q}\}$, a set consisting of a single prime $\mathfrak{q}$ which is prime to 2.

Further, $\mathcal{T} = \{\mathfrak{q}\}$ may be chosen in a way that the extra factors that appear in the definition of the $(S_K, \mathcal{T})$-modified *zeta function* for K (c.f. [Rub96, §1] for details about these zeta functions) will be prime to p, when they are evaluated at 0. (This could be accomplished, for example, by choosing the prime $\mathfrak{q}$ so that $\mathbf{N}\mathfrak{q} - 1$ is prime to p.) We note that for such $\mathcal{T}$, we have $\mathcal{O}^{\times,\wedge}_{K,S_K,\mathcal{T}} = \mathcal{O}^{\times,\wedge}_{K,S_K}$, for example, by the exact sequence (1) in [Rub96]. Since in our paper we will only work with the p-adic completion of the group of units, we will safely exclude $\mathcal{T}$ from our notation.

One minor point arises because of the appearance of the set $\mathcal{T} = \{\mathfrak{q}\}$: One should remove all the fields $k_n(\tau)$ (and $L_n(\tau)$) from the collections $\mathcal{K}_0$ (and $\mathcal{K}$, respectively) for which $\mathfrak{q}|\tau$. This is not a problem for our purposes either.

Definition 3.2. For $F = k$ or $F = L$, we set

$$\varepsilon_{F_n(\tau),S_{F_n(\tau)}} = \mathbf{N}^r_{F_{n+1}(\tau)/F_n(\tau)} \left(\tilde{\varepsilon}_{F_{n+1}(\tau),S_{F_{n+1}(\tau)}} \right)$$

where $\mathbf{N}^r_{F_{n+1}(\tau)/F_n(\tau)}$ denotes the norm map induced on the r-th exterior power. It follows from [Rub96, Proposition 6.1] that

$$\varepsilon_{F_n(\tau),S_{F_n(\tau)}} = \tilde{\varepsilon}_{F_n(\tau),S_{F_n(\tau)}}, \text{ if } n \geqslant 1.$$

Note that Proposition 6.1 of loc.cit. says something also when $n = 0$. We will return to this point in Remark 3.3 below.

The collection $\{\varepsilon_{K,S_K}\}_{K\in\mathcal{K}}$ (which we call the collection of Rubin-Stark elements) satisfies, thanks to [Rub96, Proposition 6.1], the distribution relation which is ought to be satisfied by an *Euler system of rank r* (in the sense of [PR98]). Since S is fixed (therefore S_K, too), we will often drop S or S_K from notation and denote ε_{K,S_K} by ε_K; or sometimes use S instead of S_K and denote $\mathcal{O}_{K,S_K}$ by $\mathcal{O}_{K,S}$.

For any number field K, Kummer theory gives a canonical isomorphism

$$H^1(K, \mathbb{Z}_p(1)) \cong K^{\times,\wedge} := \varprojlim_n K^\times/(K^\times)^{p^n}.$$

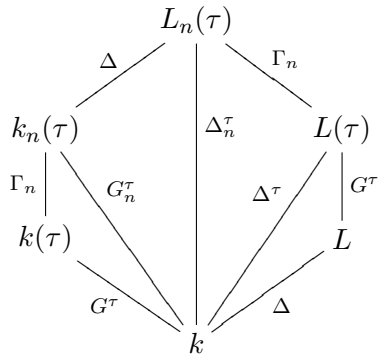
Under this identification, we view each ε_{K,S_K} as an element of $\frac{1}{\delta_K}\bigwedge^r H^1(K, \mathbb{Z}_p(1))$. The distribution relation satisfied by the Stark elements ([Rub96, Proposition 6.1]) shows that the collection $\{\varepsilon_{K,S_K}\}_{K\in\mathcal{K}}$ is an Euler system of rank r in the sense of [PR98] (except for the *denominators* $1/\delta_K$; we ignore this subtlety for the moment, as these denominators disappear once we 'pass' to an Euler system of rank one by the defining property of these elements; see Proposition 3.14 below).

3.2 Twisting by the character χ

Following the formalism of [Rub00, §II.4], we may *twist* the Euler system $\{\varepsilon_{K,S_K}\}_{K\in\mathcal{K}}$ for the representation $\mathbb{Z}_p(1)$, in order to obtain an Euler system for the representation $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$.

Define the Galois groups $\Gamma_n := \mathrm{Gal}(k_n/k)$, $G^\tau := \mathrm{Gal}(k(\tau)/k)$, $\Delta^\tau := \mathrm{Gal}(L(\tau)/k) = G^\tau \times \Delta$, and finally $G^\tau_n := \mathrm{Gal}(k_n(\tau)/k) = G^\tau \times \Gamma_n$, which is the p-part of $\Delta^\tau_n := \mathrm{Gal}(L_n(\tau)/k) \cong G^\tau_n \times \Delta = G^\tau \times \Gamma_n \times \Delta$. (These canonical factorizations of the Galois groups follow easily from the fact that $|\Delta|$ is prime to p and from ramification considerations.) The array of fields and Galois groups below

summarizes this paragraph:



Let χ be as above, and let ϵ_χ denote the idempotent $\frac{1}{|\Delta|}\sum_{\sigma\in\Delta}\chi(\sigma)\sigma^{-1}$. We may (and we will) regard this element as an element of the groups ring $\mathbb{Z}_p[\Delta_n^\tau]$ via the factorization above. For notational simplicity, we set $\delta = \delta_{k_n(\tau)}$ (note that δ also equals $\delta_{L_n(\tau)}$ up to multiplication by a p-adic unit). δ will appear as a denominator below; although it does depend on n and τ, we will allow ourselves to be sloppy with the notation we use for these denominators, as they will not be present when the Stark elements are utilized for our main purposes (i.e., when they are used to construct a Λ-adic Kolyvagin system for the Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$).

For any cycle τ which is prime to pf_χ, we define

$$\varepsilon^\chi_{L_n(\tau)} := \epsilon_\chi\varepsilon_{L_n(\tau),S} \in \frac{1}{\delta}\epsilon_\chi\bigwedge^r H^1(L_n(\tau),\mathbb{Z}_p(1)) \tag{4}$$

$$= \frac{1}{\delta}\bigwedge^r \epsilon_\chi H^1(L_n(\tau),\mathbb{Z}_p(1)) \tag{5}$$

$$= \frac{1}{\delta}\bigwedge^r H^1(L_n(\tau),\mathbb{Z}_p(1))^\chi. \tag{6}$$

We note that the equality between the lines (4) and (5) above

$$\left(\bigwedge^r H^1(L_n(\tau),\mathbb{Z}_p(1))\right)^\chi = \epsilon_\chi\bigwedge^r H^1(L_n(\tau),\mathbb{Z}_p(1))$$

$$= \bigwedge^r \epsilon_\chi H^1(L_n(\tau),\mathbb{Z}_p(1))$$

$$= \bigwedge^r H^1(L_n(\tau),\mathbb{Z}_p(1))^\chi$$

simply holds because $\epsilon_\chi^r = \epsilon_\chi$.

The Hochschild-Serre spectral sequence gives rise to an exact sequence

$$H^1(\Delta,T) \longrightarrow H^1(k_n(\tau),T) \longrightarrow H^1(L_n(\tau),T)^\Delta \longrightarrow H^2(\Delta,T) \tag{7}$$

where $H^1(L_n(\tau),T)^\Delta$ stands for the largest submodule of $H^1(L_n(\tau),T)$ on which Δ acts trivially. On the other hand, since $|\Delta|$ is prime to p, it follows that the very first and the very last terms in (7) vanish. We therefore have an isomorphism

$$H^1(k_n(\tau),\mathbb{Z}_p(1)\otimes\chi^{-1}) \longrightarrow H^1(L_n(\tau),\mathbb{Z}_p(1)\otimes\chi^{-1})^\Delta.$$

On the other hand, since $G_{L_n(\tau)}$ is in the kernel of χ,

$$H^1(L_n(\tau),\mathbb{Z}_p\otimes\chi^{-1}) \cong H^1(L_n(\tau),\mathbb{Z}_p(1))\otimes\chi^{-1},$$

hence

$$H^1(k_n(\tau),T) \xrightarrow{\sim} H^1(L_n(\tau),T)^\Delta \cong H^1(L_n(\tau),\mathbb{Z}_p(1))^\chi. \tag{8}$$

This induces an isomorphism

$$\bigwedge^r H^1(k_n(\tau), T) \xrightarrow{\sim} \bigwedge^r H^1(L_n(\tau), \mathbb{Z}_p(1))^\chi. \tag{9}$$

The inverse image of the element $\varepsilon^\chi_{L_n(\tau)}$ (which was defined in (4)) under the isomorphism induced from (9) above will be denoted by $\varepsilon^\chi_{k_n(\tau)}$. The collection $\{\varepsilon^\chi_{k_n(\tau)}\}_{n,\tau} = \{\varepsilon^\chi_K\}_{K\in\mathcal{K}_0}$ we be called the *Stark element Euler system of rank r*.

Remark 3.3. Suppose (A3) holds. By our definition of $\varepsilon_{L(\tau)}$ and [Rub96, Proposition 6.1], it follows that

$$\varepsilon_{L(\tau)} = \prod_{\wp|p}(1 - \mathrm{Frob}_\wp^{-1})\tilde{\varepsilon}_{L(\tau)},$$

where the product is over the primes $\wp$ of k which lie above p, and $\mathrm{Frob}_\wp$ is the Frobenius element at $\wp$ inside G_k. Hence, after restricting to the χ-parts, we see that

$$\varepsilon^\chi_k = \prod_{\wp|p}(1 - \chi^{-1}(\mathrm{Frob}_\wp))\tilde{\varepsilon}^\chi_k.$$

(We warn the reader that we still use the additive notation). It follows from our assumption (A3) that the element ε^χ_k used in this paper differs from the element $\tilde{\varepsilon}^\chi_k$ (which is identical to the element ε^χ_k utilized in [Büy07a]) only by a unit $u \in \mathbb{Z}_p^\times$:

$$\varepsilon^\chi_k = u \cdot \tilde{\varepsilon}^\chi_k.$$

See also Remark 4.1 below, where we use this fact to compare the Kolyvagin systems derived from these two families of Rubin-Stark elements.

Next, we construct an *Euler system of rank one* (i.e., an Euler system in the sense of [Rub00]) using ideas from [Rub96, §6] and [PR98, §1.2.3]. The main point is that, if one applied the arguments of [Rub96, PR98] directly, all one would get (after applying Kolyvagin's descent) would be a Λ-adic Kolyvagin system for the coarser Selmer structure $\mathcal{F}_\Lambda$ on $T \otimes \Lambda$. In Section §3.4, we overcome this difficulty and obtain a Λ-adic Kolyvagin system for the finer Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ on $T \otimes \Lambda$.

3.3 Choosing homomorphisms

For any field $K \in \mathcal{K}_0$, recall that $\Delta_K := \mathrm{Gal}(K/k)$ and write $\delta = |\Delta_K|$. Using the elements of

$$\varprojlim_{K\in\mathcal{K}_0} \bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K,T), \mathbb{Z}_p[\Delta_K]) \tag{10}$$

(or the images of the elements of

$$\varprojlim_{K\in\mathcal{K}_0} \bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K_p,T), \mathbb{Z}_p[\Delta_K])$$

under the canonical map

$$\varprojlim_{K\in\mathcal{K}_0} \textstyle\bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K_p,T), \mathbb{Z}_p[\Delta_K]) \longrightarrow \varprojlim_{K\in\mathcal{K}_0} \textstyle\bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K,T), \mathbb{Z}_p[\Delta_K])$$

which is induced from the localization at p) and the Rubin-Stark elements above, Rubin [Rub96, §6] (see also [PR98, §1.2.3]) shows how to obtain an Euler system (in the sense of [Rub00]) for the G_k-representation T. In this section, we show how to choose these homomorphisms (which belong to the inverse limit (10)) carefully, so that the resulting Euler system gives rise to a Kolyvagin system for the $\mathbb{L}_\infty$-modified Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$ on $T \otimes \Lambda$. We remark also that for $\Psi = \{\psi_K\}$ which

belongs to the module (10),

$$\psi_K(\varepsilon_K^\chi) \in H^1(K,T)$$

by the defining (integrality) property of the elements

$$\varepsilon_K^\chi \in \frac{1}{\delta}\bigwedge^r H^1(K,T),$$

i.e., the denominators δ will disappear once we apply the homomorphisms from (10) on the Rubin-Stark elements.

One may again identify $H^1(k_n(\tau),T)$ by $(L_n(\tau)^\times)^\chi$, using the isomorphism (8) and Kummer theory. Similarly, one may identify the semi-local cohomology group $H^1(k_n(\tau)_p,T)$ by $(L_n(\tau)_p^\times)^\chi$, where $L_n(\tau)_p := L_n(\tau)\otimes\mathbb{Q}_p$. Let $V_{L_n(\tau)}$ denote the p-adic completion of the local units of $L_n(\tau)_p$.

The proof of [Rub00, Proposition III.2.6(ii)] shows:

Lemma 3.4. *Assume that* (A3) *holds. Then for every* $k_n(\tau)\in\mathcal{K}_0$,

$$H^1(k_n(\tau)_p,T) \cong (L_n(\tau)_p^\times)^\chi \cong V_{L_n(\tau)}^\chi$$

for all $k_n(\tau)\in\mathcal{K}_0$.

Recall that $r=[k:\mathbb{Q}]$ and note that all the fields that appear in this paper (namely, the elements of the collections $\mathcal{K}$ and $\mathcal{K}_0$) are all totally real. Further, assuming (A1) (i.e., p is prime to f_χ), it follows that $L(\tau)/k$ is unramified at all primes above p. Therefore, Krasner's Lemma [Kra39] on the structure of 1-units implies:

Lemma 3.5.(Krasner) *If* (A1) *holds, then* $V_{L(\tau)}$ *is a free* $\mathbb{Z}_p[\Delta^\tau]$*-module of rank* r.

Corollary 3.6. *Assume that* (A1) *and* (A3) *hold. Then for every* $k(\tau)\in\mathcal{K}_0$, *the* $\mathbb{Z}_p[G^\tau]$*-module* $H^1(k(\tau)_p,T)=V_{L(\tau)}^\chi$ *is free of rank* r.

Until the end of this paper, we assume that (A1)-(A3) are true.

Proposition 3.7. *For every* $k(\tau)\in\mathcal{K}_0$,

(i) *the* Λ*-module* $H^1(k(\tau)_p,T\otimes\Lambda)$ *is free of rank* $[k(\tau):\mathbb{Q}]=r\cdot|G^\tau|$,

(ii) *the canonical map* $H^1(k(\tau)_p,T\otimes\Lambda)\to H^1(k(\tau)_p,T)$ *is surjective.*

Proof. Let $\mathcal{Q}$ be any prime of $k(\tau)$ lying above p, and let $k(\tau)_{\mathcal{Q},\infty}$ be the cyclotomic $\mathbb{Z}_p$-extension of $k(\tau)_\mathcal{Q}$. We set

$$H_{k(\tau)_\mathcal{Q}} = \mathrm{Gal}(\overline{k(\tau)}_\mathcal{Q}/k(\tau)_{\mathcal{Q},\infty}),$$

following the notation in the Appendix. Thanks to the assumption (A2), the extension $k(\tau)$ is unramified at p, and we may therefore identify Γ by $\mathrm{Gal}(k(\tau)_{\mathcal{Q},\infty}/k(\tau)_\mathcal{Q})$.

It is easy to see that $H^0(k(\tau)_\mathcal{Q},T)=0$. Therefore, Theorem A.8 applied with the field $k(\tau)$ proves (i). To prove (ii), we need to check that

$$0=\mathrm{coker}\left(H^1(k(\tau)_p,T\otimes\Lambda)\longrightarrow H^1(k(\tau)_p,T)\right)=H^2(k(\tau)_p,T\otimes\Lambda)[\gamma-1].$$

As in the proof of Proposition 2.9(ii), this follows (by local duality) from the vanishing

$$H^0(k(\tau)_p,T^*)=H^0(k_p,T^*)=0,$$

where the second equality holds thanks to (A3), and the first equality follows from the proof of [Rub00, Lemma IV.2.5(i)] and the remark following [Rub00, Conjecture VIII.2.6]. □

Remark 3.8. The statement of Proposition 3.7(ii) is equivalent to the statement that the map

$$H^1(k(\tau)_p, T \otimes \Lambda)/(\gamma - 1) \longrightarrow H^1(k(\tau)_p, T)$$

is an isomorphism.

Let $\Lambda_\tau := \mathbb{Z}_p[[G^\tau \times \Lambda]] = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G^\tau]$ and let $\mathbb{M} = \bigcup_{F \in \mathcal{K}_0} F$.

Proposition 3.9. *The Λ_τ-module $H^1(k(\tau)_p, T \otimes \Lambda)$ is free of rank r.*

Proof. Let $e = \{e_1, \ldots, e_r\}$ be a $\mathbb{Z}_p[G^\tau]$-basis for $H^1(k(\tau)_p, T)$. By Proposition 3.7(ii) and Nakayama's Lemma, there exists a set of generators $\mathbb{E} = \{\mathbb{E}_1, \ldots, \mathbb{E}_r\}$ of the Λ_τ-module $H^1(k(\tau)_p, T \otimes \Lambda)$, which lifts the basis e (with respect to the surjection of Proposition 3.7(ii)).

We claim that $\mathbb{E}$ is a basis for the Λ_τ-module $H^1(k(\tau)_p, T \otimes \Lambda)$. Assume the contrary, and suppose that there is a non-trivial relation

$$\sum_{i=1}^{r} a_i \mathbb{E}_i = 0 \tag{11}$$

with $a_i \in \Lambda_\tau$. Since $H^1(k(\tau)_p, T \otimes \Lambda)$ is Λ-torsion free (by Proposition 3.7(i)), we may assume without loss of generality that $a_1 \notin (\gamma - 1)$. This implies that the relation (11), reduced modulo $(\gamma - 1)$, gives rise to a non-trivial relation among $\{e_1, \ldots, e_r\}$ over $\mathbb{Z}_p[G^\tau]$. This is a contradiction, hence the set of generators $\mathbb{E}$ is a Λ_τ-basis for $H^1(k(\tau)_p, T \otimes \Lambda)$. □

Corollary 3.10. *The $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]$-module $\mathbb{V} := \varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) = \varprojlim_{\tau} H^1(k(\tau)_p, T \otimes \Lambda)$ is free of rank r.*

Proof. Immediate after Proposition 3.9. □

Corollary 3.11. *The $\mathbb{Z}_p[G_n^\tau]$-module $H^1(k_n(\tau)_p, T)$ is free of rank r.*

Proof. This follows from Proposition 3.9 and from the fact that the map

$$H^1(k(\tau)_p, T \otimes \Lambda) \longrightarrow H^1(k_n(\tau)_p, T)$$

is surjective (as the relevant H^2 vanishes). □

Choose *any* $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]$-line $\mathbb{L}$ inside $\mathbb{V}$ such that the quotient $\mathbb{V}/\mathbb{L}$ is also free (of rank $r-1$) as a $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]$-module.

Remark 3.12. Although the main results of this article do not "see" the choice of the line $\mathbb{L} \subset \mathbb{V}$, our methods require, in an essential way, to make this somewhat unnatural choice.

Choose any decomposition

$$\mathbb{V} = \bigoplus_{i=1}^{r} \mathbb{L}_i \tag{12}$$

of $\mathbb{V}$ into rank-one $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]$-modules. This in return fixes a decomposition

$$H^1(k_p, T) = \bigoplus_{i=1}^{r} \mathcal{L}_i \tag{13}$$

into rank-one $\mathbb{Z}_p$-modules $\mathcal{L}_i$; with $\mathbb{L}_i \mapsto \mathcal{L}_i$ under the obvious map $\mathbb{V} \to H^1(k_p, T)$. Consider

$$\sum_{i=1}^{r} \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}) \subset \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P}). \tag{14}$$

It is not hard to see (c.f., [Büy08, Remark 1.27]) that the sum in (14) is in fact a direct sum. In view of the fact that the main applications of the Kolyvagin systems for the modified Selmer structure $\mathcal{F}_{\mathcal{L}_i}$ are independent of the choice of $\mathcal{L}_i$, it seems natural to raise the following:
Question. Does the rank-r submodule

$$\bigoplus_{i=1}^{r} \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}) \subset \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

depend on the decomposition (12)?

Unfortunately, we are not able to answer this question at this point. Note that, one would have a natural rank-r submodule of $\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$, if the answer to this question was affirmative. See [Büy08, Theorem 1.28] for further analysis of this matter.

Definition 3.13. For all $k_n(\tau) = K \in \mathcal{K}_0$, let $\mathcal{L}_K$ be the image of $\mathbb{L}$ under the (surjective) projection map $\mathbb{V} \to H^1(K_p, T)$.

Note that $\mathcal{L}_K$ is a free $\mathbb{Z}_p[\Delta_K]$-module of rank one, for all $K \in \mathcal{K}_0$, and that

$$(\mathcal{L}_{K'})^{\mathrm{Gal}(K'/K)} = \mathcal{L}_K$$

for all $K \subset K'$. We will simply write $\mathcal{L}$ for $\mathcal{L}_k$; and $\mathcal{L}_n^\tau$ for $\mathcal{L}_{k_n(\tau)}$. We denote the image of $\mathbb{L}$ under the projection $\mathbb{V} \twoheadrightarrow H^1(k(\tau)_p, T \otimes \Lambda)$ by $\mathbb{L}_\infty^\tau$. When $\tau = 1$, we simply write $\mathbb{L}_\infty$ for $\mathbb{L}_\infty^\tau$, and $\mathcal{L}_n$ for $\mathcal{L}_n^\tau$.

We define

$$\bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]) := \varprojlim_{K \in \mathcal{K}_0} \bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K_p, T), \mathbb{Z}_p[\Delta_K]),$$

where the inverse limit is taken with respect to the natural maps induced from the restriction map

$$H^1(K_p, T) \longrightarrow H^1(K'_p, T)^{\mathrm{Gal}(K'/K)},$$

(which may easily be verified to be injective, e.g., by using the identifications of the semi-local cohomology groups in question by semi-local units) and the isomorphism

$$\begin{array}{ccc} \mathbb{Z}_p[\Delta_{K'}]^{\mathrm{Gal}(K'/K)} & \xrightarrow{\sim} & \mathbb{Z}_p[\Delta_K] \\ \mathbf{N}_K^{K'} & \longmapsto & 1 \end{array}$$

for $K \subset K'$.

The localization map at p gives rise to a map $H^1(K, T) \xrightarrow{\mathrm{loc}_p} H^1(K_p, T)$, which induces a canonical map

$$\bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]) \longrightarrow \varprojlim_{K \in \mathcal{K}_0} \bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K, T), \mathbb{Z}_p[\Delta_K]). \tag{15}$$

We will still write Φ for the image of $\Phi \in \bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$ under this map. Choose an arbitrary $\{\phi_K\}_{K \in \mathcal{K}_0} = \Phi \in \bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$, and define[5]

$$H^1(K, T) \ni \varepsilon_{K,\Phi}^\chi := \phi_K(\varepsilon_K^\chi),$$

where $\{\varepsilon_K^\chi\}_{K \in \mathcal{K}_0}$ is the Stark element Euler system of rank r, which was defined in §3.2.

[5]To make sense of our notation, one first needs to use the map (15) in order to view ϕ_K as an element of $\wedge^{r-1}\mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K,T), \mathbb{Z}_p[\Delta_K])$; and then use [Rub96, Equation (4)], to view it as an element of $\mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(\wedge^r H^1(K,T), H^1(K,T))$.

Proposition 3.14. *The collection $\{\varepsilon^{\chi}_{K,\Phi}\}_{K\in\mathcal{K}_0}$ (which we henceforth call the Euler system of Φ-Stark elements) is an Euler system for the G_k-representation $T=\mathbb{Z}_p(1)\otimes\chi^{-1}$, in the sense of [Rub00, Definition II.1.1].*

We will sometimes write $\{\varepsilon^{\chi}_{k_n(\tau),\Phi}\}_{n,\tau}$ for the Euler system $\{\varepsilon^{\chi}_{K,\Phi}\}_{K\in\mathcal{K}_0}$.

Proof. This is [Rub96, Proposition 6.6]. See also [PR98, 1.2.3] for a more general application of this idea. □

Proposition 3.15. *For any $K\in\mathcal{K}_0$ the projection map*

$$\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])\longrightarrow\bigwedge^{r-1}\mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K_p,T),\mathbb{Z}_p[\Delta_K])$$

is surjective.

Proof. Obvious after Corollary 3.11. □

If the *Euler system to Kolyvagin system map* of Mazur and Rubin (c.f., [MR04, Theorem 5.3.3]) is applied on the Euler system of Φ-Stark elements, then all one gets is a Λ-adic Kolyvagin system for the coarser Selmer structure $\mathcal{F}_\Lambda$ on $T\otimes\Lambda$. Below, we will choose a particular element $\Phi_0^{(\infty)}\in\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$, so that the Kolyvagin system (coming from the $\Phi_0^{(\infty)}$-Stark elements) will be a Kolyvagin system for the finer Selmer structure $\mathcal{F}_{\mathbb{L}_\infty}$.

Definition 3.16. We say that an element

$$\{\phi^{\tau}_n\}_{n,\tau}=\Phi\in\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$$

satisfies $\mathrm{H}_{\mathbb{L}}$, if for any $K=k_n(\tau)\in\mathcal{K}_0$, one has $\phi^{\tau}_n(\bigwedge^r H^1(K_p,T))\subset\mathcal{L}^{\tau}_n$.

We will next construct an element

$$\Phi_0^{(\infty)}\in\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$$

which satisfies $\mathrm{H}_{\mathbb{L}}$, and which lifts the element Φ_0 of [Büy07a, §2.3], with respect to the (surjective) map

$$\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])\longrightarrow\varprojlim_{\tau}\bigwedge^{r-1}\mathrm{Hom}_{\mathbb{Z}_p[G^\tau]}(H^1(k(\tau)_p,T),\mathbb{Z}_p[G^\tau])\ni\Phi_0.$$

The element Φ_0 is used in [Büy07a] to construct a *primitive* Kolyvagin system for the Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T.

Fix a basis $\left\{\Psi^{(i)}_{\mathbb{L}}\right\}_{i=1}^{r-1}$ of the free $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]$-module

$$\mathrm{Hom}_{\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]}\left(\mathbb{V}/\mathbb{L},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]\right)$$

of rank $r-1$. This in return fixes a basis $\left\{\psi^{(i)}_{\mathcal{L}^\tau_n}\right\}_{i=1}^{r-1}$ for the free $\mathbb{Z}_p[G^\tau_n]$-module

$$\mathrm{Hom}_{\mathbb{Z}_p[G^\tau_n]}\left(H^1(k_n(\tau)_p,T)/\mathcal{L}^\tau_n,\,\mathbb{Z}_p[G^\tau_n]\right)$$

for all $k_n(\tau)\in\mathcal{K}_0$; such that the homomorphisms $\left\{\psi^{(i)}_{\mathcal{L}^\tau_n}\right\}_{n,\tau}$ are compatible with respect to the surjections

$$\mathrm{Hom}_{\mathbb{Z}_p[\Delta_{K'}]}(H^1(K'_p,T)/\mathcal{L}_{K'},\mathbb{Z}_p[\Delta_{K'}])\twoheadrightarrow\mathrm{Hom}_{\mathbb{Z}_p[\Delta_K]}(H^1(K_p,T)/\mathcal{L}_K,\mathbb{Z}_p[\Delta_K])$$

for $k_n(\tau) = K \subset K' = k_{n'}(\tau')$. Furthermore, the homomorphism

$$\bigoplus_{i=1}^{r-1} \psi_{\mathcal{L}_n^\tau}^{(i)} : H^1(k_n(\tau)_p, T)/\mathcal{L}_n^\tau \longrightarrow \mathbb{Z}_p[G_n^\tau]^{r-1}$$

is an isomorphism of $\mathbb{Z}_p[G_n^\tau]$-modules, for every n and τ.

Let $\psi_{n,\tau}^{(i)}$ denote the image of $\psi_{\mathcal{L}_n^\tau}^{(i)}$ under the canonical injection

$$\mathrm{Hom}_{\mathbb{Z}_p[G_n^\tau]}(H^1(k_n(\tau)_p, T)/\mathcal{L}_n^\tau, \mathbb{Z}_p[G_n^\tau]) \lhook\joinrel\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G^\tau]}(H^1(k_n(\tau)_p, T), \mathbb{Z}_p[G_n^\tau]).$$

Note then that the map

$$\Psi_n^\tau := \bigoplus_{i=1}^{r-1} \psi_{n,\tau}^{(i)} : H^1(k_n(\tau)_p, T) \longrightarrow \mathbb{Z}_p[G_n^\tau]^{r-1}$$

is surjective and $\ker(\Psi_n^\tau) = \mathcal{L}_n^\tau$.

Define

$$\phi_n^\tau := \psi_{n,\tau}^{(1)} \wedge \psi_{n,\tau}^{(2)} \wedge \cdots \wedge \psi_{n,\tau}^{(r-1)} \in \bigwedge^{r-1} \mathrm{Hom}(H^1(k_n(\tau)_p, \mathbb{Z}_p[G_n^\tau]).$$

(When $\tau = 1$, we drop τ from the notation and simply write ϕ_n for ϕ_n^τ, etc.) Note once again that, for $\tau|\tau'$ and $n \leqslant n'$, the element $\phi_{n'}^{\tau'}$ maps to the element ϕ_n^τ under the surjective (thanks to Corollary 3.11) homomorphism

$$\textstyle\bigwedge^{r-1} \mathrm{Hom}(H^1(k_{n'}(\tau')_p, T), \mathbb{Z}_p[G_{n'}^{\tau'}]) \twoheadrightarrow \bigwedge^{r-1} \mathrm{Hom}(H^1(k_n(\tau)_p, T), \mathbb{Z}_p[G_n^\tau]).$$

We may therefore regard $\Phi_0^{(\infty)} := \{\phi_n^\tau\}_{n,\tau}$ as an element of the module

$$\varprojlim_{k_n(\tau)\in\mathcal{K}_0} \bigwedge^{r-1} \mathrm{Hom}(H^1(k_n(\tau)_p, T), \mathbb{Z}_p[G_n^\tau]) = \bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]]).$$

Proposition 3.17. *Let $\{\phi_n^\tau\}_{n,\tau} = \Phi_0^{(\infty)}$ be as above. Then for every n and τ, ϕ_n^τ induces an isomorphism*

$$\phi_n^\tau : \bigwedge^r H^1(k_n(\tau)_p, T) \xrightarrow{\sim} \ker(\Psi_n^\tau) = \mathcal{L}_n^\tau.$$

In particular, $\Phi_0^{(\infty)}$ satisfies $\mathrm{H}_{\mathbb{L}}$.

Proof. The proof is identical to the proof of (the easy half of) [Büy07b, Lemma 3.1], which also follows the proof of [MR04, Lemma B.1] almost line by line. □

Remark 3.18. It is easy to see that the element $\Phi_0^{(\infty)}$ lifts, by construction, the element Φ_0 of [Büy07b, §2.3], in the sense we explained above.

Remark 3.19. Since the map

$$H^1(k_p, T \otimes \Lambda) = \varprojlim_n H^1((k_n)_p, T) \longrightarrow H^1((k_n)_p, T)$$

is surjective for every n, and the Λ-module $H^1(k_p, T \otimes \Lambda)$ (resp., the $\mathbb{Z}_p[\Gamma_n]$-module $H^1((k_n)_p, T)$) is free of rank r, it follows that the natural map

$$\textstyle\bigwedge^r H^1(k_p, T \otimes \Lambda) = \bigwedge^r \varprojlim_n H^1((k_n)_p, T) \longrightarrow \varprojlim_n \bigwedge^r H^1((k_n)_p, T)$$

is an isomorphism. This, combined with Proposition 3.17, shows that the map $\phi_\infty := \{\phi_n\}_n$ induces an isomorphism

$$\bigwedge^r H^1(k_p, T\otimes\Lambda) \cong \varprojlim_n \bigwedge^r H^1((k_n)_p, T) \xrightarrow{\phi_\infty} \varprojlim_n \mathcal{L}_n = \mathbb{L}_\infty.$$

3.4 Kolyvagin systems for the $\mathbb{L}_\infty$-modified Selmer triple - II

For notational simplicity, write $\mathbb{T} := T\otimes\Lambda$, and for a fixed topological generator γ of $\Gamma = \mathrm{Gal}(k_\infty/k)$, set $\gamma_n = \gamma^{p^n}$. Let $\mathcal{M}$ denote the maximal ideal of Λ. Throughout this section, we assume (A1)-(A3), as well as the Rubin-Stark conjecture [Rub96, Conjecture B′].

In §2.5, we proved under the hypotheses (A1)-(A3) that:

(i) the Λ-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$ is free of rank one,
(ii) the $\mathbb{Z}_p$-module $\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ is free of rank one,
(iii) the natural map $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}) \to \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ is surjective.

In [Büy07a], a particular generator $\kappa^{\Phi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ is constructed using the Rubin-Stark elements. The main goal of this section is to "lift" κ^{Φ_0} to a Λ-adic Kolyvagin system $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$, so that $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ maps to κ^{Φ_0} under the surjection of (iii) above.

Definition 3.20. For $\mathbb{F} = \mathcal{F}_\Lambda$ or $\mathcal{F}_{\mathbb{L}_\infty}$, we set

$$\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P}) := \varprojlim_{m,n} \varinjlim_j \mathbf{KS}(\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}, \mathbb{F}, \mathcal{P}_j),$$

where $\mathbf{KS}(\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}, \mathbb{F}, \mathcal{P}_j)$ is the $\Lambda/(p^m, \gamma_n - 1)$-module of Kolyvagin systems (in the sense of [MR04, Definition 3.1.3]) for the propagated Selmer structure $\mathbb{F}$ on the quotient $\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}$.

Remark 3.21. We introduced the module $\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P})$ above because, after applying Kolyvagin's descent procedure [Rub00, §IV] (modified appropriately in [MR04, Appendix A]), one obtains elements of $\overline{\mathbf{KS}}'(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P})$. On the other hand, it is not hard to see that the module $\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P})$ defined above is naturally isomorphic to the module $\overline{\mathbf{KS}}(\mathbb{T}, \mathbb{F}, \mathcal{P})$ of Definition 2.18(ii), using the fact that each of the collections $\{p^m, \gamma_n - 1\}_{m,n}$ and $\{p^m, \mathbf{X}^n\}_{m,n}$ forms a base of neighborhoods at *zero*. Furthermore, using the fact that the collection $\{\mathcal{M}^\alpha\}_{\alpha\in\mathbb{Z}^+}$ also forms a base of neighborhoods at *zero*, one may identify these two modules Kolyvagin systems by the generalized module of Kolyvagin systems defined in [MR04, Definition 3.1.6]. By slight abuse, we will write $\overline{\mathbf{KS}}(\mathbb{T}, \mathbb{F}, \mathcal{P})$ for any of the three modules of Kolyvagin systems given by three different definitions (i.e., by Definition 2.18 and 3.20 here; and [MR04, Definition 3.1.6]). For our purposes in this section, we will use Definition 3.20 to define this module.

Let $\mathbf{ES}(T)$ denote the collection of Euler systems for T, in the sense of [Rub00]. The *Euler system to Kolyvagin system map* of [MR04, Theorem 5.3.3] gives a map

$$\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P}).$$

Fix an arbitrary $\{\phi_n^\tau\}_{n,\tau} = \Phi \in \bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$ which satisfies $\mathrm{H}_\mathbb{L}$, and let $\boldsymbol{\kappa}^\Phi = \left\{\left\{\kappa_\tau^\Phi(m,n)\right\}_{\tau\in\mathcal{N}}\right\}_{m,n}$ be the image of the Euler system $\left\{\varepsilon^\chi_{k_n(\tau),\Phi}\right\}_{n,\tau}$ of Proposition 3.14. Thanks to [MR04, Theorem 5.3.3], we know that

$$\begin{array}{ccc} \kappa_1^\Phi & \overset{\text{def}}{=\!=} & \varprojlim_{m,n} \kappa_1^\Phi(m,n) \in \varprojlim_{m,n} H^1(k, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) = H^1(k, \mathbb{T}) \\ \| & & \\ \{\varepsilon^\chi_{k_n,\Phi}\}_n & \overset{\text{def}}{=\!=} & \left\{\phi_n(\varepsilon^\chi_{k_n})\right\}_n \in \varprojlim_n H^1(k_n, T) = H^1(k, \mathbb{T}). \end{array}$$

Remark 3.22. For every (rational) prime ℓ, Shapiro's Lemma shows that

$$H^1(k(\tau),\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})\cong H^1(k_n(\tau),T/p^mT) \text{ and,} \tag{16}$$

$$H^1(k(\tau)_\ell,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})\cong H^1(k_n(\tau)_\ell,T/p^mT). \tag{17}$$

See [MR04, Lemma 5.3.1] for the first isomorphism and [Rub00, Appendix B.5] for its semi-local version. This way, we may talk about the propagation of a local condition $H^1_{\mathcal{F}}(k_\ell,\mathbb{T})\subset H^1(k_\ell,\mathbb{T})$ at ℓ to a local condition

$$H^1_{\mathcal{F}}((k_n)_\ell,T/p^mT)\subset H^1((k_n)_\ell,T/p^mT)\cong H^1(k_\ell,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T}),$$

i.e., we define $H^1_{\mathcal{F}}((k_n)_\ell,T/p^mT)$ as the isomorphic copy of $H^1_{\mathcal{F}}(k_\ell,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})$ under the isomorphism (17) of Shapiro's Lemma.

Theorem 3.23. *For any* $\Phi\in\bigwedge^{r-1}\mathbf{Hom}(\mathbb{V},\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$ *which satisfies* $\mathrm{H}_{\mathbb{L}}$*,*

$$\boldsymbol{\kappa}^\Phi:=\Big\{\big\{\kappa^\Phi_\tau(m,n)\big\}_{\tau\in\mathcal{N}}\Big\}_{m,n}\in\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}_\infty},\mathcal{P}).$$

For the rest of this section, the integers m and n will be fixed, and we will simply denote the element $\kappa^\Phi_\tau(m,n)\in H^1(k,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})$ by κ^Φ_τ. Theorem 3.23 claims that, for each $\tau\in\mathcal{N}_{m+n}$, $\kappa^\Phi_\tau\in H^1_{\mathcal{F}_{\mathbb{L}_\infty}(\tau)}(k,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})$. Here, $\mathcal{F}_{\mathbb{L}_\infty}(\tau)$ is the modified Selmer structure defined in [MR04, Example 2.1.8]. Here we only remark that the Selmer structures $\mathcal{F}_{\mathbb{L}_\infty}$ and $\mathcal{F}_{\mathbb{L}_\infty}(\tau)$ determine the same local conditions away from τ, in particular, they agree at p. On the other hand, [MR04, Theorem 5.3.3] already shows that $\kappa^\Phi_\tau\in H^1_{\mathcal{F}_\Lambda(\tau)}(k,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})$. Since $\mathcal{F}_{\mathbb{L}_\infty}$ and $\mathcal{F}_\Lambda$ determine the same local conditions away from p, Theorem 3.23 follows from the following:

Proposition 3.24. *Let*

$$\begin{array}{rccc}\mathrm{loc}_p: & H^1(k,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T}) & \longrightarrow & H^1(k_p,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})\\ & \downarrow\cong & & \downarrow\cong\\ & H^1(k_n,T/p^mT) & \longrightarrow & H^1((k_n)_p,T/p^mT)\end{array}$$

be the localization map into the semi-local cohomology at p (where the vertical isomorphisms follow from Remark 3.22). Then,

$$\mathrm{loc}_p(\kappa^\Phi_\tau)\in\mathcal{L}_n/p^m\mathcal{L}_n\subset H^1((k_n)_p,T/p^mT).$$

Proposition 3.24 will be proved below. We first remark that

$$\mathcal{L}_n/p^m\mathcal{L}_n=H^1_{\mathcal{F}_{\mathbb{L}_\infty}}((k_n)_p,T/p^mT),$$

i.e., $\mathcal{L}_n/p^m\mathcal{L}_n$ is the propagation of the local condition

$$H^1_{\mathcal{F}_{\mathbb{L}_\infty}}((k_n)_p,T)=H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k_p,\mathbb{T}/(\gamma_n-1)\mathbb{T})=\mathcal{L}_n \tag{18}$$

at p. We note that the first equality in (18) is explained in Remark 3.22.

Let

$$\big\{\tilde\kappa^\Phi_\tau(m,n)\in H^1((k_n)_p,T/p^mT)\big\}_{\tau\in\mathcal{N}_{m+n}}$$

be the collection that Rubin [Rub00, Definition IV.4.10] associates to the Euler system $\Big\{\varepsilon^\chi_{k_n(\tau),\Phi}\Big\}_{n,\tau}$. Here, we write $\tilde\kappa^\Phi_\tau(m,n)$ for the class which was denoted by $\kappa_{[k_n,\tau,m]}$ in [Rub00]. Since we have already fixed m and n, we will safely drop m and n from the notation and denote $\tilde\kappa^\Phi_\tau(m,n)$ by $\tilde\kappa^\Phi_\tau$, when

there is no danger of confusion. Note that Equation (33) in [MR04, Appendix A] relates the class $\tilde{\kappa}_\tau^\Phi$ to the class κ_τ^Φ.

Lemma 3.25. *If* $\mathrm{loc}_p(\tilde{\kappa}_\tau^\Phi) \in \mathcal{L}_n/p^m\mathcal{L}_n$, *then* $\mathrm{loc}_p(\kappa_\tau^\Phi) \in \mathcal{L}_n/p^m\mathcal{L}_n$ *as well.*

Proof. Obvious using Equation (33) in [MR04, Appendix A]. □

Let D_τ denote the derivative operator, defined as in [Rub00, Definition IV.4.1]. Rubin defines [Rub00, Definition IV.4.10 and Remark IV.4.3] the class $\tilde{\kappa}_\tau^\Phi$ as the inverse image of $D_\tau\varepsilon^\chi_{k_n(\tau),\Phi} \pmod{p^m}$ under the restriction map[6]

$$H^1(k_n, T/p^mT) \longrightarrow H^1(k_n(\tau), T/p^mT)^{G^\tau}.$$

Therefore, $\mathrm{loc}_p(\tilde{\kappa}_\tau^\Phi)$ maps to $\mathrm{loc}_p(D_\tau\varepsilon^\chi_{k_n(\tau),\Phi}) \pmod{p^m}$ under the following map (which is also an isomorphism thanks to Remark 4.4.3, Proposition B.5.1 and Proposition B.4.2 of [Rub00]):

$$H^1((k_n)_p, T/p^mT) \longrightarrow H^1(k_n(\tau)_p, T/p^mT)^{G^\tau}.$$

Under this isomorphism, $\mathcal{L}_n/p^m\mathcal{L}_n$ is mapped isomorphically onto the rank one $\mathbb{Z}/p^m\mathbb{Z}\,[\Gamma_n]$-module $(\mathcal{L}_n^\tau/p^m\mathcal{L}_n^\tau)^{G^\tau}$, by the definition of $\mathcal{L}_n^\tau$ and by the fact that $\mathcal{L}_n^\tau$ is a free $\mathbb{Z}_p[G_n^\tau]$-module. The diagram below summarizes the discussion in this paragraph:

$$\begin{array}{ccc} H^1((k_n)_p, T/p^mT) & \xrightarrow{\sim} & H^1(k_n(\tau)_p, T/p^mT)^{G^\tau} \\ \uparrow & & \uparrow \\ \mathcal{L}_n/p^m\mathcal{L}_n & \xrightarrow{\sim} & (\mathcal{L}_n^\tau/p^m\mathcal{L}_n^\tau)^{G^\tau} \end{array}$$

Proposition 3.26. *If* Φ *satisfies* $\mathrm{H}_\mathbb{L}$, *then* $\mathrm{loc}_p(\tilde{\kappa}_\tau^\Phi) \in \mathcal{L}_n/p^m\mathcal{L}_n$.

Proof. Since loc_p is Galois equivariant, $\mathrm{loc}_p(D_\tau\varepsilon^\chi_{k_n(\tau),\Phi}) = D_\tau\mathrm{loc}_p(\varepsilon^\chi_{k_n(\tau),\Phi})$. Furthermore,

$$\mathrm{loc}_p(\varepsilon^\chi_{k_n(\tau),\Phi}) \in \mathcal{L}_n^\tau,$$

since Φ satisfies $\mathrm{H}_\mathbb{L}$. On the other hand, by [Rub00, Lemma 4.4.2], $D_\tau\varepsilon_{k_n(\tau),\Phi} \pmod{p^m}$ is fixed by G^τ, which in return implies that

$$\mathrm{loc}_p(\varepsilon^\chi_{k_n(\tau),\Phi}) \,(\mathrm{mod}\, p^m) \in (\mathcal{L}_n^\tau/p^m\mathcal{L}_n^\tau)^{G^\tau}.$$

This proves that $\mathrm{loc}_p(\tilde{\kappa}_\tau^\Phi)$ belongs to $\mathcal{L}_n/p^m\mathcal{L}_n$ by the discussion above. □

Proof of Proposition 3.24. Immediate from Lemma 3.25 and Proposition 3.26. □

By the discussion following the statement of Theorem 3.23, this also completes the proof of Theorem 3.23.

[6]Note that $(\mu_{p^\infty} \otimes \chi^{-1})^{G_{k_n(\tau)}}$ is trivial, since, for example, the complex conjugation cannot act by χ on μ_{p^∞} as χ is an even character. This proves that the restriction map in question is an isomorphism, by [Rub00, Remark 4.4.3].

4. Applications to the main conjectures

Assume throughout this section that the finite set of places S of k does not contain any non-archimedean prime which splits completely in L/k. Suppose that the hypotheses (A1)-(A3) hold, as well as the Rubin-Stark conjecture. In this section, we assume that Leopoldt's conjecture holds true, see Remark 4.1 below for a discussion regarding this assumption. Let

$$\{\phi_n^\tau\}_{n,\tau} = \Phi_0^{(\infty)} \in \bigwedge^{r-1} \mathbf{Hom}(\mathbb{V}, \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{M}/k)]])$$

be as in §3.3 (recall that we write ϕ_n for ϕ_n^τ when $\tau = 1$, and we set $\phi_\infty = \{\phi_n\}$), and let

$$\boldsymbol{\kappa}^{\Phi_0^{(\infty)}} = \left\{ \kappa_\tau^{\Phi_0^{(\infty)}} \right\}_\tau \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P})$$

be the Λ-adic Kolyvagin system of $\Phi_0^{(\infty)}$-Stark elements (which we introduced in §3.4).

In [Büy07a], the author determines an explicit generator, which descends from the Euler systems of Stark elements, for the cyclic $\mathbb{Z}_p$-module of Kolyvagin systems $\overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P})$. Following [Büy07a], we denote this generator by $\kappa^{\Phi_0} = \{\kappa_\tau^{\Phi_0}\}$, where $\Phi_0 \in \varprojlim_\tau \bigwedge^{r-1} \mathrm{Hom}_{\mathbb{Z}_p[G^\tau]}(H^1(k(\tau)_p, T), \mathbb{Z}_p[G^\tau])$ is a certain element which is used in [Büy07a], and which we also recalled in §3.3 above.

Remark 4.1.

1. As in [Büy07a, §3], we use Leopoldt's conjecture to ensure that the Kolyvagin system $\kappa^{\Phi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P})$ is *primitive* (in particular, non-trivial). Aside from Proposition 2.12 and Corollary 2.13, this is our only reason to assume Leopoldt's conjecture.
2. By Remark 3.3 and Theorem 2.19, what we call κ^{Φ_0} here differs from what [Büy07a] calls κ^{Φ_0} by a unit $u \in \mathbb{Z}_p^\times$, where u is as in Remark 3.3. We remind the reader that this difference is due to the fact that we use Rubin-Stark elements $\{\varepsilon_K^\chi\}_{K\in\mathcal{K}}$ in this paper to construct Euler systems, whereas [Büy07a] uses the Rubin-Stark elements $\{\tilde{\varepsilon}_K^\chi\}_{K\in\mathcal{K}}$ for the same purpose. Since κ^{Φ_0} of [Büy07a] is a primitive Kolyvagin system, it follows that κ^{Φ_0} that appears in this paper is a primitive Kolyvagin system as well.

It is clear from our construction that $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ maps to the element κ^{Φ_0} under the surjective map

$$\overline{\mathbf{KS}}(\mathbb{T}) := \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}_\infty}, \mathcal{P}) \twoheadrightarrow \overline{\mathbf{KS}}(T, \mathcal{F}_\mathcal{L}, \mathcal{P}) =: \overline{\mathbf{KS}}(T).$$

Proposition 4.2. *The Λ-adic Kolyvagin system of $\Phi_0^{(\infty)}$-Stark elements $\kappa^{\Phi_0^{(\infty)}}$ is Λ-primitive, in the sense of [MR04, Definition 5.3.9].*

Proof. Let $\overline{T}$ be the residual representation $\mathbb{T}/\mathcal{M}\mathbb{T} = T/pT$. For $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T})$ (resp., $\kappa \in \overline{\mathbf{KS}}(T)$), let $\overline{\boldsymbol{\kappa}}$ (resp., $\overline{\kappa}$) denote the image of $\boldsymbol{\kappa}$ (resp., κ) under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \mathbf{KS}(\overline{T})$ (resp., under the map $\overline{\mathbf{KS}}(T) \to \mathbf{KS}(\overline{T})$). Since $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ maps to the element κ^{Φ_0} under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \overline{\mathbf{KS}}(T)$, it is clear that $\overline{\boldsymbol{\kappa}}^{\Phi_0^{(\infty)}} = \overline{\kappa}^{\Phi_0}$, and we henceforth write $\overline{\kappa}$ for both. Since κ^{Φ_0} is primitive[7], it follows from [MR04, Definition 4.5.5 and Theorem 5.2.10(ii)] that $\overline{\kappa} \neq 0$. This proves that the image of $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \overline{\mathbf{KS}}(\mathbb{T}/\mathfrak{p}\mathbb{T})$ is non-zero for any height-one prime $\mathfrak{p} \subset \Lambda$; since we

[7]We remark that our definition of a primitive Kolyvagin system is *a priori* different from the definition [MR04, Definition 4.5.5] of Mazur and Rubin. However, [MR04, Theorem 5.2.10(ii)] shows that these two definitions are equivalent.

have a commutative diagram

$$\begin{array}{ccc} \boldsymbol{\kappa}^{\Phi_0^{(\infty)}} & \in & \overline{\mathbf{KS}}(\mathbb{T}) \\ \downarrow & & \downarrow \\ \overline{\kappa} & \in & \mathbf{KS}(\overline{T}) \end{array} \quad \begin{array}{c} \searrow \\ \\ \swarrow \end{array} \quad \overline{\mathbf{KS}}(\mathbb{T}/\mathfrak{p}\mathbb{T})$$

and $\overline{\kappa} \neq 0$.

□

Let $\mathrm{char}(\mathbb{A})$ denote the characteristic ideal of a torsion Λ-module $\mathbb{A}$ and reecall that $A^\vee$ denotes the Pontryagin dual of an abelian group A. The main application of the (Λ-primitive) Kolyvagin system $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ is the following:

Theorem 4.3. *We have*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}_\infty}}(k,\mathbb{T}^*)^\vee\right) = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}_\infty}}(k,\mathbb{T})/\Lambda\cdot\kappa_1^{\Phi_0^{(\infty)}}\right)$$

Proof. This follows from Theorem 2.20 and Proposition 4.2. □

Corollary 2.13 and Theorem 4.3 (applied with $c = \kappa_1^{\Phi_0^{(\infty)}}$) imply that:

Corollary 4.4. *We have*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{str}}}(k,\mathbb{T}^*)^\vee\right) = \mathrm{char}\left(\mathbb{L}_\infty/\Lambda\cdot\kappa_1^{\Phi_0^{(\infty)}}\right).$$

Remark 4.5. We know a priori only that

$$\varepsilon^\chi_{k_n(\tau)} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r H^1(k_n(\tau), T).$$

Let $\mathrm{loc}_p^{(r)}$ denote the map

$$\bigwedge^r H^1(k_n(\tau),T) \longrightarrow \bigwedge^r H^1((k_n(\tau))_p, T) \cong \bigwedge^r V^\chi_{L_n(\tau)},$$

so that

$$\mathrm{loc}_p^{(r)}\left(\varepsilon^\chi_{k_n(\tau)}\right) \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r V^\chi_{L_n(\tau)}.$$

It follows from the defining (integrality) property of the Rubin-Stark elements that:

CLAIM. *For any*

$$\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_r = \psi \in \bigwedge^r \mathrm{Hom}\left(V^\chi_{L_n(\tau)}, \mathbb{Z}_p[G_n^\tau]\right),$$

we have

$$\psi\left(\mathrm{loc}_p^{(r)}\left(\varepsilon^\chi_{k_n(\tau)}\right)\right) \in \mathbb{Z}_p[G_n^\tau],$$

where we we still write ψ for the map on $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r V^\chi_{L_n(\tau)}$ induced from ψ.

On the other hand, we know (by Corollary 3.11) that the $\mathbb{Z}_p[G_n^\tau]^\chi$-module $V^\chi_{L_n(\tau)}$ is free of rank r, hence the Claim above implies that $\mathrm{loc}_p^{(r)}\left(\varepsilon^\chi_{k_n(\tau)}\right) \in \bigwedge^r V^\chi_{L_n(\tau)}$. (This has already been pointed out in [Rub96] Example 1 following Proposition 1.2.)

Let $\mathbf{c}^{\mathrm{stark}}_{k_\infty} \in \bigwedge^r H^1(k_p,\mathbb{T})$ be the image of the element

$$\{\varepsilon^{\chi}_{k_n}\}_n \in \varprojlim_n \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\bigwedge^r H^1(k_n,T) \xrightarrow{\mathrm{loc}_p^{(r)}} \varprojlim_n \bigwedge^r H^1((k_n)_p,T) \overset{(3.19)}{\cong} \bigwedge^r H^1(k_p,\mathbb{T})$$

under the localization-at-p map $\mathrm{loc}_p^{(r)}$ composed with the isomorphism from of Remark 3.19. This is the element which showed up in the statement of Theorem A in §1.2.

The homomorphism $\phi_\infty=\{\phi_n\}$ maps $\bigwedge^r H^1(k_p,\mathbb{T})$ isomorphically onto $\mathbb{L}_\infty$ by Remark 3.19; and the element $\mathbf{c}^{\mathrm{stark}}_{k_\infty} \in \bigwedge^r H^1(k_p,\mathbb{T})$ to the class $\kappa_1^{\Phi_0^{(\infty)}} \in \mathbb{L}_\infty$, by construction. We therefore conclude that

$$\bigwedge^r H^1(k_p,\mathbb{T})/\Lambda\cdot\mathbf{c}^{\mathrm{stark}}_{k_\infty} \cong \mathbb{L}_\infty/\Lambda\cdot\kappa_1^{\Phi_0^{(\infty)}},$$

which, combined with Corollary 4.4, implies the following:

Corollary 4.6. *We have*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{str}}}(k,\mathbb{T}^*)^\vee\right) = \mathrm{char}\left(\bigwedge^r H^1(k_p,\mathbb{T})/\Lambda\cdot\mathbf{c}^{\mathrm{stark}}_{k_\infty}\right).$$

Using the explicit description of the Galois cohomology groups in question (c.f., [Rub00, §I.6.3]), one may identify $H^1_{\mathcal{F}^*_{\mathrm{str}}}(k,\mathbb{T}^*)^\vee$ by $\mathrm{Gal}(M_\infty/L_\infty)^\chi$, where M_∞ is the maximal abelian p-extension of L_∞ which is unramified outside the primes above p. This is the Iwasawa module which is involved in the formulation of the "main conjectures" in this setting. Let $\mathcal{L}^\chi_k$ denote (an appropriate normalization of) the Deligne-Ribet [DR80] p-adic L-function attached to the character χ. As a consequence of the work of Wiles [Wil90], we therefore deduce:

Theorem 4.7. *The* Λ*-module* $\mathrm{char}\left(\bigwedge^r H^1(k_p,\mathbb{T})/\Lambda\cdot\mathbf{c}^{\mathrm{stark}}_{k_\infty}\right)$ *is generated by* $\mathcal{L}^\chi_k$.

It would be very desirable to prove Theorem 4.7 without appealing to [Wil90], and therefore obtain a proof of the "main" conjecture. It seems this is also plausible: The statement of Theorem 4.7 is much in the spirit of [PR94a, PR94b, PR95, Col98b]; and when $k=\mathbb{Q}$ (i.e., when $r=1$) Theorem 4.7 is a classical result of Iwasawa [Iwa64]. Note that the Stark elements are obtained from the cyclotomic units when $k=\mathbb{Q}$. One key observation we remark here is that the cyclotomic units demonstrate the complex Stark conjecture and the p-adic Stark conjecture (c.f., [Sol02, Sol04] for Solomon's version of the p-adic Stark conjecture) simultaneously. However, it would not be reasonable to expect to prove Theorem 4.7 using *only* the properties of the Rubin-Stark elements (which are solutions for the Rubin-Stark conjecture for the complex L-functions), since Rubin's conjecture only predicts the values of a certain *complex*-valued regulator evaluated at these elements. In fact, it is plausible that one would also need to utilize the solutions to an appropriate p-adic Stark conjecture.

In a sequel to this paper, we hope to discuss the relation between the solutions to the complex and p-adic Stark conjectures via the rigidity offered by Theorem 2.19, and prove Theorem 4.7 without assuming Wiles' work, thus deduce the main conjectures. One big obstacle with which the author is faced is the lack of an *integral* p-adic Stark conjecture (either at $s=1$ or $s=0$) along the cyclotomic $\mathbb{Z}_p$-tower.

Nevertheless, Theorem 4.7 is true, and this fact already hints at a relation between the solutions of the complex and p-adic Stark conjectures. This relation should be understood as an analogy to the fact that the cyclotomic units give solutions to both the complex Stark conjecture and the p-adic Stark conjecture (when $k=\mathbb{Q}$).

Appendix A. Local Conditions at p over an Iwasawa algebra via the theory of (φ,Γ)-modules

In this Appendix, we give an overview of certain results due to Benois, Colmez, Fontaine, Herr, and Perrin-Riou. We use these results to determine the structure of the semi-local cohomology groups $H^1(k_n(\tau)_p, T)$ in §3.3. This probably could have been achieved without appealing to the theory of (φ,Γ)-modules, however, this very general approach may help one to generalize the methods of this paper to apply in many other settings.

Throughout this Appendix, let K denote a finite extension of $\mathbb{Q}_p$ and set $\tilde{K}_n := K(\mu_{p^n})$, $\tilde{K}_\infty := \bigcup_n \tilde{K}_n$. Define the Galois groups $\tilde{H}_K := \mathrm{Gal}(\overline{K}/\tilde{K}_\infty)$ and $\tilde{\Gamma}_K := G_K/\tilde{H}_K = \mathrm{Gal}(\tilde{K}_\infty/K)$. Let $\tilde{\gamma}$ be a topological generator of the pro-cyclic group $\tilde{\Gamma}_K$, and let $\tilde{\Lambda}_K := \mathbb{Z}_p[[\tilde{\Gamma}_K]]$. Let $\tilde{\gamma}_n$ be a fixed topological generator of $\mathrm{Gal}(\tilde{K}_\infty/\tilde{K}_n) := \tilde{\Gamma}^{(n)}$ for $n \in \mathbb{Z}^+$, which is chosen in a way that $\tilde{\gamma}_n = \tilde{\gamma}_1^{p^{\alpha_n}}$, where $\alpha_n \in \mathbb{Z}^+$ is such that $[\tilde{K}_n : \tilde{K}] = p^{\alpha_n}$.

Let K_n be the maximal p-extension of K inside $\tilde{K}_n$, and let $\bigcup_n K_n =: K_\infty \subset \tilde{K}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of K. We set $\Gamma_K := \mathrm{Gal}(K_\infty/K)$ and $\Lambda_K := \mathbb{Z}_p[[\Gamma_K]]$. Note then that

$$\tilde{\Gamma}_K = W \times \Gamma_K, \text{ and that } \tilde{\Lambda}_K = \mathbb{Z}_p[W] \otimes_{\mathbb{Z}_p} \Lambda_K,$$

where W is a finite group whose order is prime to p. (In fact W can be identified by $\mathrm{Gal}(K(\mu_p)/K)$.) Let γ denote the restriction of $\tilde{\gamma}$ to K_∞; so that the element γ is a topological generator of Γ_K. Let γ_n denote the image of $\tilde{\gamma}_n$ under the natural isomorphism

$$\mathrm{Gal}(\tilde{K}_\infty/\tilde{K}_n) \cong \mathrm{Gal}(K_\infty/K_n),$$

and set $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ (so that $H_K/\tilde{H}_K \cong W$).

In [Fon90], Fontaine[8] introduces the notion of a (φ,Γ)-module over a certain period ring which he denotes by $\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}}$ (and which is the ring of integers of the field $\widehat{\varepsilon^{\mathrm{nr}}}$). We set $\mathcal{O}_{\varepsilon(K)} := (\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}})^{H_K}$. We will not include a detailed discussion of these objects here, we refer the reader to [Fon90, A.3.1-3.2] for the definitions and the basic properties of these rings. Briefly, a (φ,Γ)-module over $\mathcal{O}_{\varepsilon(K)}$ is a finitely generated $\mathcal{O}_{\varepsilon(K)}$-module with semi-linear continuous and commuting actions of φ and $\Gamma := \tilde{\Gamma}_K$. A (φ,Γ)-module D over $\mathcal{O}_{\varepsilon(K)}$ is called *étale* if $\varphi(D)$ generates D as an $\mathcal{O}_{\varepsilon(K)}$-module.

Using his theory, Fontaine established an equivalence between the category of $\mathbb{Z}_p$-representations of the absolute Galois group G_K of K, and the category of étale (φ,Γ)-modules over $\mathcal{O}_{\varepsilon(K)}$. This equivalence is given by

$$\begin{array}{rcl} T & \longmapsto & D(T) := (\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}} \otimes_{\mathbb{Z}_p} T)^{H_K} \\ (\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}} \otimes_{\mathcal{O}_{\varepsilon(K)}} D)^{\varphi=1} =: T(D) & \longleftarrow & D \end{array}$$

See [Fon90, A.1.2.4-1.2.6] for details.

Suppose T is any $\mathbb{Z}_p[[G_K]]$-module which is free of finite rank over $\mathbb{Z}_p$. In [Her98], Herr makes use of the theory of (φ,Γ)-modules to compute the Galois cohomology groups $H^*(K,T)$. One of the benefits of his approach is that the complex he constructs with cohomology $H^*(K,T)$ is quite explicit. This allows one to compute certain local Galois cohomology groups of p-adic fields. In [Her01], Herr gives a proof of the local Tate duality, where the local the pairing (c.f., [Her01, §5]) is explicitly defined in terms of the residues of the differential forms on $\mathcal{O}_{\varepsilon(K)}$. The rest of this Appendix is a survey of Herr's results and their applications [Ben00, CC99] to Iwasawa theory.

[8]However, one should be cautious as Fontaine uses K_n for our $\tilde{K}_n$, etc. For example, his Γ_K is our $\tilde{\Gamma}_K$.

In Fontaine's theory of (φ,Γ)-modules, there is an important operator[9]

$$\psi : \mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}} \longrightarrow \mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}},$$

$$\psi(x) := \frac{1}{p}\varphi^{-1}(\mathrm{Tr}_{\widehat{\varepsilon^{\mathrm{nr}}}/\varphi(\widehat{\varepsilon^{\mathrm{nr}}})}(x)),$$

which is crucial for what follows. The map ψ is a left inverse of φ, and its action on $\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}}$ commutes with the action of G_K. It induces an operator (which we still denote by ψ)

$$\psi : D(T) \longrightarrow D(T)$$

for any G_K-representation T.

Let $C_{\psi,\tilde{\gamma}}$ be the complex

$$C_{\psi,\tilde{\gamma}} : 0 \longrightarrow D(T) \xrightarrow{(\psi-1,\tilde{\gamma}-1)} D(T)\oplus D(T) \xrightarrow{(\tilde{\gamma}-1)\ominus(\psi-1)} D(T) \longrightarrow 0$$

The main result of [Her98] is the following:

Theorem A.1. *The complex $C_{\psi,\tilde{\gamma}}$ computes the G_K-cohomology of T:*

(i) $H^0(K,T) \cong D(T)^{\psi=1,\tilde{\gamma}=1}$,

(ii) $H^2(K,T) \cong D(T)/(\psi-1,\tilde{\gamma}-1)$,

(iii) *There is an exact sequence*

$$0 \longrightarrow \frac{D(T)^{\psi=1}}{\tilde{\gamma}-1} \longrightarrow H^1(K,T) \longrightarrow \left(\frac{D(T)}{\psi-1}\right)^{\tilde{\gamma}=1} \longrightarrow 0.$$

All the isomorphisms and maps that appear above are functorial in T and K.

Definition A.2. Let

$$H^1_{\tilde{\mathrm{Iw}}}(K,T) := \varprojlim_n H^1(\tilde{K}_n,T) \text{ and,}$$

$$H^1_{\mathrm{Iw}}(K,T) := \varprojlim_n H^1(K_n,T),$$

where the inverse limits are taken with respect to the corestriction maps.

Remark A.3. Since the order of W is prime to p, it follows that

$$H^1_{\mathrm{Iw}}(K,T) \xrightarrow{\sim} H^1_{\tilde{\mathrm{Iw}}}(K,T)^W$$

by the Hochschild-Serre spectral sequence.

We now determine the structure of $H^1_{\mathrm{Iw}}(K,T)$ using Theorem A.1.

Proposition A.4. *Define $\tau_n := 1+\tilde{\gamma}_{n-1}+\cdots+\tilde{\gamma}_{n-1}^{p-1} \in \mathbb{Z}_p[[\tilde{\Gamma}_K]]$. There is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
C_{\psi,\tilde{\gamma}_n}(\tilde{K}_n,T): & 0 & \longrightarrow & D(T) & \longrightarrow & D(T)\oplus D(T) & \longrightarrow & D(T) & \longrightarrow 0 \\
& \vdots\,\tau_n^* & & \downarrow \tau_n & & \downarrow \tau_n\oplus\mathrm{id} & & \downarrow \mathrm{id} & \\
C_{\psi,\tilde{\gamma}_{n-1}}(\tilde{K}_{n-1},T): & 0 & \longrightarrow & D(T) & \longrightarrow & D(T)\oplus D(T) & \longrightarrow & D(T) & \longrightarrow 0
\end{array}$$

[9]This definition makes sense because:

- $\mathrm{Tr}_{\widehat{\varepsilon^{\mathrm{nr}}}/\varphi(\widehat{\varepsilon^{\mathrm{nr}}})}(\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}}) \subset p\mathcal{O}_{\widehat{\varepsilon^{\mathrm{nr}}}}$,
- φ is injective.

Furthermore, the map induced from the morphism τ_n^* on the cohomology of $C_{\psi,\tilde{\gamma}_n}(\tilde{K}_n, T)$ coincides with the corestriction map under Herr's identification $H^*(C_{\psi,\tilde{\gamma}_n}(\tilde{K}_n, T)) \cong H^*(\tilde{K}_n, T)$ of Theorem A.1.

Proof. This follows from the fact that τ_n^* is a cohomological functor and induces $\mathrm{Tr}_{\tilde{K}_n/\tilde{K}_{n-1}}$ on H^0, hence it induces corestrictions on H^i. □

Using Proposition A.4, one may compute $H^*_{\tilde{\mathrm{Iw}}}(K, T)$:

Theorem A.5.

(i) $H^i_{\tilde{\mathrm{Iw}}}(K, T) = 0$ *if* $i \neq 1, 2$,

(ii) $H^1_{\tilde{\mathrm{Iw}}}(K, T) \xrightarrow{\sim} D(T)^{\psi=1}$,

(iii) $H^2_{\tilde{\mathrm{Iw}}}(K, T) \xrightarrow{\sim} D(T)/(\psi - 1)$.

See [CC99, §II.3] for a proof of this theorem.

Remark A.6. The isomorphism

$$\exp^* : H^1_{\tilde{\mathrm{Iw}}}(K, T) \xrightarrow{\sim} D(T)^{\psi=1}$$

of Theorem A.5(ii) can be considered as a vast generalization of Coleman's map [Col79]. The isomorphism $\exp^*$ conjecturally gives rise to the (conjectural) p-adic L-function attached to T. This viewpoint we gain is one of the important benefits of using the theory of (φ, Γ)-modules to compute Galois cohomology.

Let $\mathcal{C}(T) := (\varphi - 1)D(T)^{\psi=1}$. Since ψ is a left inverse of φ, it follows that

$$\ker\{D(T)^{\psi=1} \xrightarrow{\varphi-1} \mathcal{C}(T)\} = D(T)^{\varphi=1},$$

hence we have an exact sequence:

$$0 \longrightarrow D(T)^{\varphi=1} \longrightarrow D(T)^{\psi=1} \xrightarrow{\varphi-1} \mathcal{C}(T) \longrightarrow 0. \tag{19}$$

Using techniques from the theory of (φ, Γ)-modules, one can determine the structure of $\mathcal{C}(T)$:

Proposition A.7. *The $\tilde{\Lambda}_K$-module $\mathcal{C}(T)$ is free of rank $[K : \mathbb{Q}_p] \cdot \mathrm{rank}_{\mathbb{Z}_p} T$.*

One may also check that $D(T)^{\varphi=1} \cong T^{\tilde{H}_K}$. In particular, $D(T)^{\varphi=1}$ is finitely generated over $\mathbb{Z}_p$, hence is a torsion $\mathbb{Z}_p[[\tilde{\Gamma}_K]]$-module. Thus, it follows from Proposition A.7 and Theorem A.5 that $D(T)^{\varphi=1} = H^1_{\tilde{\mathrm{Iw}}}(K, T)_{\mathrm{tors}}$, the torsion submodule of $H^1_{\tilde{\mathrm{Iw}}}(K, T)$.

If we now take the W-invariance of the exact sequence (19) (and use the fact that taking W-invariance is an exact functor) and apply Remark A.3 along with Theorem A.5 and Proposition A.7, we see that:

Theorem A.8.(Cherbonnier and Colmez [CC99])

(i) *For the Λ_K-torsion submodule $H^1_{\mathrm{Iw}}(K, T)_{\mathrm{tors}}$ of $H^1_{\mathrm{Iw}}(K, T)$, we have $H^1_{\mathrm{Iw}}(K, T)_{\mathrm{tors}} \cong T^{H_K}$.*

(ii) *The Λ_K-module $H^1_{\mathrm{Iw}}(K, T)/H^1_{\mathrm{Iw}}(K, T)_{\mathrm{tors}}$ is free of rank $[K : \mathbb{Q}_p] \cdot \mathrm{rank}_{\mathbb{Z}_p} T$.*

References

Ben00 Denis Benois. On Iwasawa theory of crystalline representations. *Duke Math. J.*, 104(2):211–267, 2000.

Büy07a Kâzım Büyükboduk. Kolyvagin systems of Stark units. *J. Reine Angew. Math. (Crelle's), to appear*, 2007. Available at http://arxiv.org/abs/math.NT/0703426.

Büy07b Kâzım Büyükboduk. Λ-adic Kolyvagin systems, 2007. 56pp., preprint. http://arxiv.org/abs/0706.0377v1.

Büy08 Kâzım Büyükboduk. Stickelberger elements and Kolyvagin systems, 2008. 36pp., *available upon request.*

CC99 Frédéric Cherbonnier and Pierre Colmez. Théorie d'Iwasawa des représentations p-adiques d'un corps local. *J. Amer. Math. Soc.*, 12(1):241–268, 1999.

Col79 Robert F. Coleman. Division values in local fields. *Invent. Math.*, 53(2):91–116, 1979.

Col98a Pierre Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

Col98b Pierre Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

DR80 Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

dS87 Ehud de Shalit. *Iwasawa theory of elliptic curves with complex multiplication*, volume 3 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1987. p-adic L functions.

Fon90 Jean-Marc Fontaine. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

Gre94 Ralph Greenberg. Trivial zeros of p-adic L-functions. In *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 149–174. Amer. Math. Soc., Providence, RI, 1994.

Her98 Laurent Herr. Sur la cohomologie galoisienne des corps p-adiques. *Bull. Soc. Math. France*, 126(4):563–600, 1998.

Her01 Laurent Herr. Une approche nouvelle de la dualité locale de Tate. *Math. Ann.*, 320(2):307–337, 2001.

Iwa64 Kenkichi Iwasawa. On some modules in the theory of cyclotomic fields. *J. Math. Soc. Japan*, 16:42–82, 1964.

Kra39 Marc Krasner. Sur la représentation exponentielle dans les corps relativement galoisiens de nombers $\mathfrak{p}$-adiques. *Acta Arith.*, 3:133–173, 1939.

Mil86 J. S. Milne. *Arithmetic duality theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1986.

MR04 Barry Mazur and Karl Rubin. Kolyvagin systems. *Mem. Amer. Math. Soc.*, 168(799):viii+96, 2004.

MTT86 B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986.

PR94a Bernadette Perrin-Riou. La fonction L p-adique de Kubota-Leopoldt. In *Arithmetic geometry (Tempe, AZ, 1993)*, volume 174 of *Contemp. Math.*, pages 65–93. Amer. Math. Soc., Providence, RI, 1994.

PR94b Bernadette Perrin-Riou. Théorie d'Iwasawa des représentations p-adiques sur un corps local. *Invent. Math.*, 115(1):81–161, 1994. With an appendix by Jean-Marc Fontaine.

PR95 Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198, 1995.

PR98 Bernadette Perrin-Riou. Systèmes d'Euler p-adiques et théorie d'Iwasawa. *Ann. Inst. Fourier (Grenoble)*, 48(5):1231–1307, 1998.

Rub92 Karl Rubin. Stark units and Kolyvagin's "Euler systems". *J. Reine Angew. Math.*, 425:141–154, 1992.

Rub96 Karl Rubin. A Stark conjecture "over $\mathbf{Z}$" for abelian L-functions with multiple zeros. *Ann. Inst. Fourier (Grenoble)*, 46(1):33–62, 1996.

Rub00 Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

Sol02 David Solomon. On p-adic abelian Stark conjectures at $s = 1$. *Ann. Inst. Fourier (Grenoble)*, 52(2):379–417, 2002.

Sol04 D. Solomon. Abelian conjectures of Stark type in $\mathbb{Z}_p$-extensions of totally real fields. In *Stark's conjectures: recent work and new directions*, volume 358 of *Contemp. Math.*, pages 143–178. Amer. Math. Soc., Providence, RI, 2004.

Wil90 Andrew Wiles. The Iwasawa conjecture for totally real fields. *Ann. of Math. (2)*, 131(3):493–540, 1990.

Kâzım Büyükboduk `kazim@math.stanford.edu`
Web page: `http://math.stanford.edu/∼kazim`
Kazim Buyukboduk
Department of Mathematics
450 Serra Mall, Bldg 380
Stanford, CA, 94305
USA
Current address:
IHÉS, Le Bois-Marie, 35,
Route de Chartres
Bures-sur-Yvette, F-91440
France