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Conditional Law and Occupation Times of Two-Sided Sticky Brownian Motion

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Abstract: Sticky Brownian motion on the real line can be obtained as a weak solution of a system of stochastic differential equations. We find the conditional distribution of the process given the driving Brownian motion, both at an independent exponential time and at a fixed time $t > 0$. As a classical problem, we find the distribution of the occupation times of a half-line, and at 0, which is the sticky point for the process.

Keywords: sticky Brownian motion; resolvent density; Arcsine law; weak solution

1 Introduction

An important special case of diffusions with boundary conditions is sticky Brownian motion, defined by means of an infinitesimal generator on the half-line with a sticky boundary condition at 0 [5]. Ito and McKean [9] provided a construction of the process as a time-changed reflected Brownian motion. Ikeda and Watanabe [8] studied such diffusions with boundary conditions as solutions to stochastic differential equations. A sticky Brownian motion on the real line, where 0 is not at the boundary, is characterized as a solution of a system of stochastic differential equations. Recently, both versions have attracted further research interest motivated by applications [1, 2, 3, 4, 6, 11, 13]. In this paper, we derive the conditional law of the real-valued

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sticky Brownian motion at a snapshot in time, given the driving Brownian motion, as well as the occupation times at each half-line and 0 based on the results of [4, 13].

The one-sided, or reflecting, sticky Brownian motion is determined as a weak solution of the stochastic differential equation

$$dX_t = 1_{\{X_t>0\}}dB_t + \theta 1_{\{X_t=0\}}dt \tag{1}$$

where $\theta > 0$ is the stickiness parameter, with $X_0 \geq 0$. In [13], the conditional distribution of X_t given $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the driving Brownian motion B, has been found as

$$\mathbb{P}\{X_t < x|\mathcal{F}_t\} = e^{-2\theta(B_t+L_t-x)} \tag{2}$$

where $L_t = \sup_{s\leq t}\{-B_s \wedge 0\}$ and initial condition $(X_0, B_0) = (0, 0)$. This remarkable result is verified from the point of view of stochastic flows in [6]. On the other hand, the two-sided, or global, version of sticky Brownian motion is described by a system of stochastic differential equations given by

$$dX_t = 1_{\{X_t\neq 0\}}dB_t \tag{3}$$

$$1_{\{X_t=0\}}dt = \frac{1}{\theta}dl_t^0(X) \tag{4}$$

where $l_t^0(X)$ denotes the local time of X at 0. We will refer to the two-sided version as *the sticky Brownian motion* in this paper. It has been recently shown in [3] and [4] that there exists no strong solution to equations (3) and (4), but the system admits a weak solution uniquely determining the probability law of X. Moreover, if (X_t, B_t) solves (3)-(4), then $(|X_t|, \hat{B}_t)$ where $\hat{B}_t = \int_0^t \text{sgn}(X_s)dB_s$ is a one-sided sticky Brownian motion with the same parameter θ [4]. Based on this association, the positive and negative parts of X behave similar to the one-sided sticky Brownian and play a crucial role in our results.

Our first result is the characterization of the joint distribution of (X_t, B_t) through their resolvent density as given in Theorem 1. This is then used in Theorem 2 to derive the conditional law of X_T given B_T at an independent exponential time T with parameter λ as

$$\mathbb{P}\{X_T < x|B_T = b\} = \Big(\frac{\text{sgn}(b-x)}{4} + \text{sgn}(x)\frac{\gamma}{4\delta}\Big)e^{-\delta|b-x|+\gamma(|b|-sgn(x)x)} + \frac{i_x(b)}{2} \tag{5}$$

where $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$, and $i_x(b) = 1_{\{b<x\}}$. We invert the Laplace transforms involved in (5) to obtain $\mathbb{P}\{X_t < x|B_t = b\}$ in Corollary 1. Our approach is parallel to [13] by relying

on Knight's theorem. However, we condition on a single random variable B_t rather than the filtration $\mathcal{F}_t$.

As another result, we derive the distribution of the occupation times of the sticky Brownian motion in Theorem 3. Our exposure is along the lines of Brownian motion, for which the celebrated Arcsine law holds [12]. The counterparts for the one-sided case are derived based on the results of [13] for finding the distribution of the occupation time at 0. The occupation time at 0 during $[0,1]$, A_1^0, is nonzero as the process is sticky at 0, and satisfies

$$\mathbb{P}\{A_1^0 > t\} = \text{Erfc}(\frac{\theta t}{\sqrt{2(1-t)}})$$

for $0 < t < 1$. The distribution of the occupation time of the positive axis, A_1^+, is found as

$$\mathbb{P}\{A_1^+ > t\} = 1 - \mathbb{E}\left[\text{Erfc}\left(\frac{-Z^2 + Z\sqrt{Z^2 + \theta^2(1-t)}}{\theta\sqrt{2t}}\right)\right]$$

which approximates Arcsine law when $\theta \to \infty$ as expected.

The paper is organized as follows. Section 2 gives the resolvents involved and the conditional law for the sticky Brownian motion. The occupation distributions are found in Section 3.

2 Joint Resolvent and Conditional Distribution

In this section, we first obtain some fundamental identities using Skorohod representations, which will also be useful to derive the occupation time distributions in the next section. Then, we recover the resolvent of the sticky Brownian motion X found as a weak solution of (3)-(4), with $X_0 = 0$. As for the main results, we derive the joint resolvent of X and the driving Brownian motion B, and the conditional law of X_t given B_t for each $t > 0$.

The occupation times of X_t and their right continuous inverses are defined as follows

$$A_t^0 = \int_0^t 1_{\{X_s=0\}} ds \qquad \alpha_t^0 = \inf\{u : A_u^0 > t\} \tag{6}$$

$$A_t^+ = \int_0^t 1_{\{X_s>0\}} ds \qquad \alpha_t^+ = \inf\{u : A_u^+ > t\} \tag{7}$$

$$A_t^- = \int_0^t 1_{\{X_s<0\}} ds \qquad \alpha_t^- = \inf\{u : A_u^- > t\} \tag{8}$$

Note that A_t^0, A_t^- and A_t^+ are right continuous non-decreasing processes justifying the definition of the corresponding right inverses.

Let $\mathbb{P}^{(x,a)}$ and $\mathbb{E}^{(x,a)}$ denote the probability law and the expectation, respectively, given the initial condition $(X_0, B_0) = (x, a)$, $x, a \in \mathbb{R}$. We simply use $\mathbb{P}$ and $\mathbb{E}$ when $(X_0, B_0) = (0, 0)$. Let X^+ and X^- denote the positive and negative parts of the real-valued random variable X, respectively. The identities shown in the following lemma will be used in the sequel.

Lemma 1. *There exist independent Brownian motions W_t^+ and W_t^- such that*

$$X^+_{\alpha_t^+} = W_t^+ + \frac{\theta}{2} A^0_{\alpha_t^+} \quad , \quad X^-_{\alpha_t^-} = W_t^- + \frac{\theta}{2} A^0_{\alpha_t^-}$$

and

$$A^0_{\alpha_t^+} = \frac{2}{\theta} \sup_{s \le t}(-W_s^+) \quad , \quad A^0_{\alpha_t^-} = \frac{2}{\theta} \sup_{s \le t}(-W_s^-) \, .$$

Proof. Tanaka formulae [12, Thm.VI.1.2] for X satisfying equations (3)-(4) yields

$$\begin{aligned} X_t^+ &= \int_0^t 1_{\{X_s > 0\}} dB_s + \frac{\theta}{2} A_t^0 \\ X_t^- &= -\int_0^t 1_{\{X_s < 0\}} dB_s + \frac{\theta}{2} A_t^0 \, . \end{aligned} \tag{9}$$

As a result of a time change $t \to \alpha_t^+$, we get

$$X^+_{\alpha_t^+} = \int_0^{\alpha_t^+} 1_{\{X_s > 0\}} dB_s + \frac{\theta}{2} A^0_{\alpha_t^+}$$

Define

$$W_t^+ := \int_0^{\alpha_t^+} 1_{\{X_s > 0\}} dB_s \quad \text{and} \quad W_t^- := -\int_0^{\alpha_t^-} 1_{\{X_s < 0\}} dB_s$$

Then, $\langle W_t^+, W_t^+ \rangle = \langle W_t^-, W_t^- \rangle = t$ by definitions (7) and (8) implying that W^+ and W^- are Brownian motions by Levy's characterization theorem. Note that $\langle W^+_{A_t^+}, W^+_{A_t^+} \rangle = A_t^+$, $\langle W^-_{A_t^-}, W^+_{A_t^-} \rangle = A_t^-$ and $\langle W^-_{A_t^-}, W^+_{A_t^+} \rangle = 0$. Therefore, by Knight's theorem ([12, Thm.V.1.9]) W_t^+ and W_t^- are independent Brownian motions as $A_\infty^+ = A_\infty^- = \infty$. Furthermore, $\frac{\theta}{2} A_{\alpha_t^+}$ is a continuous and strictly increasing function of α_t^+ and constant on the set $\{t : X^+_{\alpha_t^+} > 0\}$. Thus, the result follows by Skorohod's Lemma [12, Lem.VI.2.1]. □

The following lemma gives the resolvent density of the sticky Brownian motion X starting from zero. Although it is available in [7, pg.23], where α^0 and analogous definition for the right continuous inverse of the time spent in $\mathbb{R} \setminus \{0\}$ are considered, we provide a proof referring to

the times spent by X in the three regions introduced above inspired by the proof of [13, Prop.13] for one-sided sticky. Our approach emphasizes the two-sided nature of X, also used in Section 4.

Lemma 2. *The resolvent density $p_\lambda(0, dy)$ of the sticky Brownian motion X starting from 0 is given as*

$$p_\lambda(0, dy) = \frac{\theta}{\theta\gamma + \lambda} e^{-\gamma|y|} dy + \frac{1}{\theta\gamma + \lambda} \delta_0(dy)$$

where $\gamma^2 = 2\lambda$.

Proof. Take exponentially distributed random times T_1, T_2 and T_3 with rate λ independently from (X, W), and define

$$T = \alpha^0_{T_1} \wedge \alpha^+_{T_2} \wedge \alpha^-_{T_3} \tag{10}$$

By Lemma 1, $(\theta/2)A^0_{\alpha^+_t}$ and $(\theta/2)A^0_{\alpha^-_t}$ are the running supremums of independent Brownian motions W^+ and W^-, respectively. Therefore, $A^0_{\alpha^+_{T_2}}$ and $A^0_{\alpha^-_{T_3}}$ are exponentially distributed with rate $\frac{\theta\gamma}{2}$, where $\gamma^2 = 2\lambda$. Note that T is also exponentially distributed with rate λ independent of X, similar to the one-sided case given in [13], essentially due to independence of T_i, $i = 1, \ldots, 3$. Now, we have $X_T = 0$ if and only if $T = \alpha^0_{T_1}$, which can be true only if $\alpha^0_{T_1} < \alpha^+_{T_2} \wedge \alpha^-_{T_3}$. Since A^0_t is a right continuous and nondecreasing process, if $\alpha^0_{T_1} < \alpha^+_{T_2} \wedge \alpha^-_{T_3}$ then $T_1 < A^0_{\alpha^+_{T_2}} \wedge A^0_{\alpha^-_{T_3}}$. Thus, we get

$$\begin{aligned}
\mathbb{P}\{X_T = 0\} &= \mathbb{P}\{T = \alpha^0_{T_1}\} \\
&= \mathbb{P}(T_1 < A^0_{\alpha^+_{T_2}} \wedge A^0_{\alpha^-_{T_3}}) \\
&= \frac{\lambda}{\lambda + \theta\gamma} .
\end{aligned}$$

For $y \neq 0$, we —consider only $y > 0$ since on negative axis the calculations are similar. If $X_T > 0$ then we should have $T = \alpha^+_{T_2}$ and this can be possible only if $A^0_{\alpha^+_{T_2}} < T_1 \wedge A^0_{\alpha^-_{T_3}}$. Then, we get

$$\begin{aligned}
P(X_T \in dy) &= P(X_{\alpha^+_{T_2}} \in dy) P(A^0_{\alpha^+_{T_2}} < T_1 \wedge A^0_{\alpha^-_{T_3}}) \\
&= P(X^+_{\alpha^+_{T_2}} \in dy) \frac{\frac{\theta\gamma}{2}}{\lambda + \theta\gamma} \\
&= \frac{\theta\lambda}{\lambda + \theta\gamma} e^{-\gamma|y|} dy
\end{aligned}$$

since $X^+_{\alpha^+_{T_2}}$ is a reflected Brownian motion independent from $A^0_{\alpha^+_{T_2}}$ and $A^0_{\alpha^-_{T_3}}$ by Lemma 1. For $y < 0$, we have the same expression due to symmetry. The result follows as $p_\lambda(0, dy) = (1/\lambda)P(X_T \in dy)$. □

We are ready to find the joint resolvent.

Theorem 1. *Let (X_t, B_t) be the solution to stochastic differential equations (3) - (4) with $X_0 = 0$. Then, there exists a Brownian motion W^0_t such that*

$$X_t + W^0_{A^0_t} = B_t \tag{11}$$

and the joint resolvent $\mathcal{V}_\lambda$ of the process (X_t, B_t) is given by

$$\begin{aligned}\mathcal{V}_\lambda f(x,a) &= \frac{1}{\theta}\int_{-\infty}^{\infty} \nu_\lambda(x,a,0,b) f(0,b)\, db + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nu_\lambda(x,a,y,b) f(y,b)\, db\, dy \\ &\quad + \int_{\{a+y=b+x\}} r^-_\lambda(x,y) f(y,b) dy\end{aligned}$$

where

$$\nu_\lambda(x,a,y,b) = \frac{\theta}{2\delta} e^{-\gamma(|y|+|x|)-\delta|b-y-a+x|}$$

with $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$, and

$$r^-_\lambda(x,y) = \gamma^{-1}(e^{-\gamma|y-x|} - e^{-\gamma|y+x|})$$

which is the resolvent density of Brownian motion killed at 0.

Proof. When $X_0 = 0$, (3) - (4) imply

$$X_t + \int_0^t 1_{\{X_s=0\}} dB_s = B_t$$

Define W^0_t as

$$W^0_t = \int_0^{\alpha^0_t} 1_{\{X_s=0\}} dB_s$$

Then, Levy's characterization and definition of α^0_t yields W^0_t is a Brownian motion and so decomposition (11) is shown. The resolvent of (X, B) can be derived by similar methods used for the resolvent density of X. Recall the definition of T in (10) as $T = \alpha^0_{T_1} \wedge \alpha^+_{T_2} \wedge \alpha^-_{T_3}$. Now, we have

$$A^0_T = T_1 \wedge A^0_{\alpha^+_{T_2}} \wedge A^0_{\alpha^-_{T_3}}$$

where $A^0_{\alpha^+_{T_2}}, A^0_{\alpha^-_{T_3}}$ and T_1 are independent exponential random variables with rates $\frac{\theta\gamma}{2}$, $\frac{\theta\gamma}{2}$ and λ, respectively. The distributions of $A^0_{\alpha^+_{T_2}}$ and $A^0_{\alpha^-_{T_3}}$, together with their independence follow from Lemma 1. Therefore, A^0_T is exponentially distributed with rate $\lambda + \theta\gamma$. Moreover, $A^0_{\alpha^+_{T_2}}$, $A^0_{\alpha^-_{T_3}}$, and W^0_t are all independent because W^+ and W^-, and W^0 are independent Brownian motions as their cross variations are 0. It follows that the W^0 is independent from A^0_T, and $W^0_{A^0_T}$ is double exponentially distributed with parameter $\delta := \sqrt{2(\lambda + \theta\gamma)}$. To find the joint resolvent, we first consider a starting point of the form $(X_0, W_0) = (0, a)$. Then, (11) becomes

$$X_t + a + W^0_{A^0_t} = B_t$$

Hence, we have

$$\begin{aligned}
\lambda \mathcal{V}_\lambda f(0,a) &= P^{(0,a)}\{X_T \in dy, B_T \in db\} \\
&= P^{(0,a)}\{X_T \in dy, W^0_{A^0_T} \in d(b-y-a)\} \\
&= \frac{\lambda}{2\delta}\big[\theta e^{-\gamma|y|}dy + \delta_0(dy)\big] e^{-\delta|b-y-a|}db
\end{aligned}$$

where we have used Lemma 2, and the independence of X_T and $W^0_{A^0_T}$. The latter independence is due to the following facts: i) $(X^+_{\alpha^+_t}, X^-_{\alpha^-_t}, W^0_t)$ is a 3-dimensional Brownian motion by Knight's theorem, ii) $X_{\alpha^+_t}$ ($X_{\alpha^-_t}$) is independent from $A^0_{\alpha^+_t}$ ($A^0_{\alpha^-_t}$) by Lemma 1; iii) depending on the value of T, X_T is equal to $X_{\alpha^0_{T_1}}$, $X_{\alpha^+_{T_2}}$, or $X_{\alpha^-_{T_3}}$. The details are similar to the proof of Lemma 2 where $y = 0$ and $y \neq 0$ are considered separately.

To find the resolvent of (X_t, B_t) starting from (x, a), let us define the time H_0 at which X_t hits zero for the first time, and use the strong Markov property of X_t at H_0. Since (X_t, B_t) moves lock step until H_0, the hitting time of X to 0, the law of the process is just that of a Brownian motion starting at x and killed at 0. If we define $\mathcal{R}^-_\lambda$ as the resolvent of Brownian motion killed at 0, with $\psi_\lambda(x) = \mathbb{E}^{(x,a)}[-\lambda H_0] = e^{-\gamma|x|}$, where $\delta = \sqrt{2\lambda}$, and $f_{x,a}(y) = f(y, a + y - x)$, we get

$$\mathcal{V}_\lambda f(x,a) = \mathcal{R}^-_\lambda f_{x,a}(x) + \psi_\lambda(x)\mathcal{V}_\lambda f(0, a-x)$$

which follows from the strong Markov property at the hitting time to 0; see e.g. [7, pg.24] for more details in the one-sided case. This yields the desired result. □

Next, we use the joint resolvent of the process (X_t, B_t) derived in Theorem 1 to obtain the conditional distribution at an independent exponential time.

Theorem 2. *Let (X_t, B_t) satisfy (3)-(4) with initial condition $(0,0)$ and let T be exponentially distributed with rate λ independent from (X,B). Then, we have*

$$\mathbb{P}\{X_T < x | B_T = b\} = \Big(\frac{sgn(b-x)}{4} + sgn(x)\frac{\gamma}{4\delta}\Big)e^{-\delta|b-x|+\gamma(|b|-sgn(x)x)} + \frac{i_x(b)}{2}$$

where $\gamma^2 = 2\lambda$, $\delta^2 = 2\lambda + 2\theta\gamma$ and $i_x(b) = 1_{\{b<x\}}$.

Proof. For a bounded Borel function g on $\mathbb{R}^2$, the resolvent $\mathcal{V}_\lambda$ of (X,B) satisfies $\mathcal{V}_\lambda g(0,0) = \mathbb{E}^{(0,0)}\, g(X_T, B_T)$. In particular, consider

$$g(y,b) = 1_{\{y\le x\}} f(b) =: i_x(y) f(b)$$

for a bounded Borel function f on $\mathbb{R}$. On the other hand, the projection property of conditional expectation implies

$$\mathbb{E}[g(X_T, B_T] = \mathbb{E}[1_{\{X_T\le x\}} f(B_T)] = \mathbb{E}[\mathbb{P}\{X_T \le x | B_T\} f(B_T)]\,.$$

Therefore, if we show that

$$\mathcal{V}_\lambda g(0,0) = [\mathcal{V}_\lambda(i_x f)](0,0) = [\mathcal{R}_\lambda(e_x f)](0)$$

for some $e_x : \mathbb{R} \to \mathbb{R}$, where R_λ is the resolvent of Brownian motion, we can conclude that $e_x(b) = \mathbb{P}\{X_T < x | B_T = b\}$ with $(X_0, B_0) = (0,0)$. Now, by Theorem 1, we get

$$\begin{aligned}
[\mathcal{V}_\lambda(i_x f)](0,0) &= \frac{1}{\theta}\int_{-\infty}^{\infty} \nu(0,0,0,b) i_x(0) f(b)\, db \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nu(0,0,y,b)\, i_x(y) f(b) dy\, db \\
&= i_x(0)\frac{1}{2\delta}\int_{-\infty}^{\infty} e^{-\delta|b|} f(b)\, db \\
&\quad + \frac{\theta}{2\delta}\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{-\gamma|y|-\delta|b-y|} f(b)\, dy\, db
\end{aligned}$$

Assume $x > 0$, then

$$\begin{aligned}
[\mathcal{V}_\lambda(i_x f)](0,0) &= \frac{1}{2\delta}\int_{-\infty}^{\infty} e^{-\delta|b|} f(b)db + \frac{\theta}{2\delta}\int_{-\infty}^{\infty} f(b)\int_{-\infty}^{x} e^{-\gamma|y|-\delta|b-y|}dydb \\
&= \frac{1}{2\delta}\int_{-\infty}^{\infty} e^{-\delta|b|} f(b)db + \frac{\theta}{2\delta}\int_{-\infty}^{\infty} f(b)\int_{-\infty}^{x} e^{-\gamma|y|-\delta|b-y|}dydb \\
&= \frac{1}{2\delta}\int_{-\infty}^{\infty} e^{-\delta|b|} f(b)db \\
&\quad + \frac{\theta}{2\delta}\int_{x}^{\infty} f(b)\Big(\int_{0}^{x} e^{-\gamma y-\delta(b-y)}dy + \int_{-\infty}^{0} e^{\gamma y-\delta(b-y)}dy\Big)db \\
&\quad + \frac{\theta}{2\delta}\int_{0}^{x} f(b)\Big(\int_{b}^{x} e^{-\gamma y-\delta(y-b)}dy + \int_{0}^{b} e^{-\gamma y-\delta(b-y)}dy \\
&\quad + \int_{-\infty}^{b} e^{\gamma y-\delta(b-y)}dy\Big)db \\
&\quad + \frac{\theta}{2\delta}\int_{-\infty}^{0} f(b)\Big(\int_{0}^{x} e^{-\gamma y-\delta(y-b)}dy + \int_{b}^{0} e^{\gamma y-\delta(y-b)}dy \\
&\quad + \int_{-\infty}^{b} e^{\gamma y-\delta(b-y)}dy\Big)db \\
&=: \int_{-\infty}^{\infty} \frac{1}{\gamma} e^{-\gamma|b|} e_x^{(1)}(b) f(b)db \;=\; [\mathcal{R}_\lambda(e_x^{(1)} f)](0)
\end{aligned}$$

where $\mathcal{R}_\lambda$ is the resolvent of Brownian motion and explicitly

$$e_x^{(1)}(b) = e^{-\delta|b-x|+\gamma(|b|-x)}\Big(\frac{\mathrm{sgn}(b-x)}{4} + \frac{\gamma}{4\delta}\Big) + \frac{i_x(b)}{2} \tag{12}$$

For the case $x \le 0$, we have

$$\begin{aligned}
[V_\lambda(i_x f)](0,0) &= \frac{\theta}{2\delta}\int_{-\infty}^{\infty} f(b)\int_{-\infty}^{x} e^{-\gamma|y|-\delta|b-y|}dy\,db \\
&= \frac{\theta}{2\delta}\int_{x}^{\infty} f(b)\int_{-\infty}^{x} e^{\gamma y-\delta(b-y)}dy\,db \\
&\quad + \frac{\theta}{2\delta}\int_{-\infty}^{x} f(b)\Big(\int_{b}^{x} e^{\gamma y-\delta(y-b)}dy + \int_{-\infty}^{b} e^{\gamma y-\delta(b-y)}dy\Big)db \\
&= \int_{-\infty}^{\infty} \frac{1}{\gamma} e^{-\gamma|b|} e_x^{(2)}(b) f(b)db \\
&= [\mathcal{R}_\lambda(e_x^{(2)} f)](0))
\end{aligned}$$

where

$$e_x^{(2)}(b) = e^{-\delta|b-x|+\gamma(|b|+x)}\Big(\frac{\mathrm{sgn}(b-x)}{4} - \frac{\gamma}{4\delta}\Big) + \frac{i_x(b)}{2}$$

Letting $e_x(b) = 1_{\{x>0\}} e_x^{(1)}(b) + 1_{\{x\leq 0\}} e_x^{(2)}(b)$, we obtain

$$\mathbb{E}[1_{\{X_T<x\}} f(B_T)] = \mathbb{E}[e_x(B_T) f(B_T)]$$

which completes the proof. □

Theorem 2 allows us to calculate the conditional distribution of X_t given B_t, for $t > 0$, next.

Corollary 1. *The conditional distribution of sticky Brownian motion X_t, given the driving Brownian motion B_t, both staring from zero, is given by*

$$\mathbb{P}\{X_t \leq x | B_t = b\} = \sqrt{2\pi t}\, e^{\frac{b^2}{2t}} \Big[\frac{sgn(b-x)}{4} f_1(x,b,t) * f_2(x,b,t) + \frac{sgn(x)}{4} f_3(x,b,t) * f_4(x,b,t) + \frac{i_x(b)}{2} f_2(x,b,t) \Big]$$

where $*$ is the convolution operator, and

$$\begin{aligned}
f_1(x,b,t) &= \frac{|b-x|}{\sqrt{2\pi t^3}} e^{-\frac{2|b-x|^2}{4t} - \theta|b-x|} \\
&\quad + \frac{\theta}{2\sqrt{2\pi t^3}} \int_{\sqrt{2}|b-x|}^{\infty} \xi\, e^{-\frac{\xi^2}{4t} - \frac{\theta\xi}{\sqrt{2}}} I_1\left(\theta\sqrt{\frac{\xi}{2} - |b-x|^2}\right) d\xi \\
f_2(x,b,t) &= \frac{1}{\sqrt{\pi t}} e^{-\frac{|x|^2}{2t}} \\
f_3(x,b,t) &= \frac{1}{2\sqrt{2\pi t^3}} \int_0^{\infty} \xi e^{-\frac{\xi^2}{4t} - \frac{\theta\xi}{\sqrt{2}}} d\xi \\
f_4(x,b,t) &= \frac{|x|}{\sqrt{2\pi t^3}}\, e^{-\frac{\sqrt{2}|x|^2}{4t}}
\end{aligned}$$

Proof. Under the initial condition $(X_0, B_0) = (0,0)$, we can write the joint distribution as

$$\begin{aligned}
&\mathbb{P}\{X_T \leq x, B_T \in db\} = \mathbb{P}\{X_T < x | B_T = b\} \mathbb{P}\{B_T \in db\} \\
&= \left[\frac{\gamma}{2} \Big(\frac{\mathrm{sgn}(b-x)}{4} + \mathrm{sgn}(x) \frac{\gamma}{4\delta} \Big) e^{-\delta|b-x| - \gamma|x|} + \frac{i_x(b)}{2} \frac{\gamma}{2} e^{-\gamma|b|} \right] db
\end{aligned} \tag{13}$$

where we have used the fact that B_T is double exponentially distributed with parameter $\gamma = \sqrt{2\lambda}$. By conditioning on T, we have

$$\begin{aligned}
\mathbb{P}\{X_T < x, B_T \in db\} &= \int_0^{\infty} \mathbb{P}\{X_T < x, B_T \in db | T = t\} \lambda e^{-\lambda t} dt \\
&= \int_0^{\infty} \mathbb{P}\{X_t < x, B_t \in db\} \lambda e^{-\lambda t} dt
\end{aligned}$$

Thus, we can obtain $\mathbb{P}\{X_t < x, B_t \in db\}$ by Laplace inversion of (13). Denoting the Laplace transform operator by $\mathcal{L}_\lambda$, we get

$$\begin{aligned}
\mathbb{P}\{X_t < x, B_t \in db\} &= \mathcal{L}_\lambda^{-1}\Big(\frac{\operatorname{sgn}(b-x)}{4\gamma}\exp\{-\delta|b-x|-\gamma|x|\}\Big)\,db \\
&\quad + \mathcal{L}_\lambda^{-1}\Big(\frac{1}{4\delta}\operatorname{sgn}(x)\exp\{-\delta|b-x|-\gamma|x|\}\Big)\,db \\
&\quad + \mathcal{L}_\lambda^{-1}\Big(\frac{i_x(b)}{2\gamma}e^{-\gamma(|b|)}\Big)\,db \\
&= \frac{\operatorname{sgn}(b-x)}{4}\mathcal{L}_\lambda^{-1}\Big(\exp\{-\delta|b-x|\}\frac{1}{\gamma}\exp\{-\gamma|x|\}\Big)\,db \\
&\quad + \frac{\operatorname{sgn}(x)}{4}\mathcal{L}_\lambda^{-1}\Big(\frac{1}{\delta}\exp\{-\delta|b-x|\}\exp\{-\gamma|x|\}\Big)\,db \\
&\quad + \frac{i_x(b)}{2}\mathcal{L}_\lambda^{-1}\Big(\frac{1}{\gamma}e^{-\gamma(|b|)}\Big)\,db
\end{aligned}$$

Let the inverse Laplace transforms $f_i : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ be assigned as

$$f_1(x,b,t) = \mathcal{L}_\lambda^{-1}\Big(\exp\{-\delta|b-x|\}\Big)[t]\,, \quad f_2(x,b,t) = \mathcal{L}_\lambda^{-1}\Big(\frac{1}{\gamma}\exp\{-\gamma|x|\}\Big)[t]$$
$$f_3(x,b,t) = \mathcal{L}_\lambda^{-1}\Big(\frac{1}{\delta}\exp\{-\delta|b-x|\}\Big)[t]\,, \quad f_4(x,b,t) = \mathcal{L}_\lambda^{-1}\Big(\exp\{-\gamma|x|\}\Big)[t]$$

Then, the inverse Laplace formulae in [10, pp. 978-979] yield the result. □

3 Distribution of Occupation Times

In this section, we find the distributions of occupation times of positive and negative axes by the two-sided sticky Brownian motion. We also find the distribution of the time spent at 0, which we show to be equal to that of the one-sided case. The desired distribution will be calculated via the relation between the two-sided and the one-sided sticky Brownian motions.

We first recover the distribution of the time spent at 0 by the one-sided sticky Brownian motion, which satisfies Equation (1). This result is cited in [13], but not given explicitly. For the sake of completeness, we derive the distribution of both occupation times in the following lemma.

Lemma 3. *Let $\bar{X}_t$ denote the one-sided sticky Brownian motion satisfying* (1)*, and let*

$$\bar{A}_t^0 = \int_0^t 1_{\{\bar{X}_s=0\}}ds \quad \textit{and} \quad \bar{A}_t^+ = \int_0^t 1_{\{\bar{X}_s>0\}}ds\,.$$

Then, we have

$$\begin{aligned} i)\ \mathbb{P}\{\bar{A}^0_s < t\} &= 1 - Erfc(\frac{\theta t}{\sqrt{2(s-t)}}) \\ ii)\ \mathbb{P}\{\bar{A}^+_s < t\} &= Erfc(\frac{\theta(s-t)}{\sqrt{2t}}) \end{aligned}$$

for $t \le s$, *where* $Erfc(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\xi^2} d\xi$.

Proof. Let $\bar{\alpha}^+_t = \inf\{s : \bar{A}^+_s \ge t\}$. It has been shown in [13] that $\bar{A}^0_{\bar{\alpha}^+_T}$ is exponentially distributed with rate $\theta\gamma = \theta\sqrt{2\lambda}$. Formally, we can write

$$\begin{aligned} \theta\sqrt{2\lambda}\, e^{-\theta\sqrt{2\lambda}\, x}\, dx = \mathbb{P}\{\bar{A}^0_{\alpha^+_T} \in dx\} &= \int_0^\infty \mathbb{P}\{\bar{A}^0_{\alpha^+_t} \in dx | T = t\}\mathbb{P}\{T \in dt\} \\ &= \int_0^\infty \mathbb{P}\{\bar{A}^0_{\alpha^+_t} \in dx\}\lambda e^{-\lambda t} dt. \end{aligned}$$

That is, $(1/\lambda)\mathbb{P}\{\bar{A}^0_{\alpha^+_T} \in dx\} = \mathcal{L}_\lambda(f(\cdot, x))$ where $f(t,x)\, dx = \mathbb{P}\{\bar{A}^0_{\alpha^+_t} \in dx\}$. The formula [10, (10) p.964] yields

$$\mathbb{P}\{\bar{A}^0_{\bar{\alpha}^+_t} \in dx\} = \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 x^2}{2t}} dx \tag{14}$$

for $x \ge 0$, which is half normal distribution with parameter $\sigma = \sqrt{t}/\theta$. Recall that $t = \bar{A}^0_t + \bar{A}^+_t$ for each $t > 0$. So, the distribution of $\bar{\alpha}^+_t$ can be calculated as

$$\begin{aligned} \mathbb{P}\{\bar{\alpha}^+_t < x\} &= \mathbb{P}\{t + \bar{A}^0_{\bar{\alpha}^+_t} < x\} = \mathbb{P}\{\bar{A}^0_{\bar{\alpha}^+_t} < x - t\} \\ &= \int_0^{x-t} \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 y^2}{2t}} dy = \int_t^x \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 (y-t)^2}{2t}} dy \end{aligned}$$

in view of (14). Then, the density is given by

$$\mathbb{P}\{\bar{\alpha}^+_t \in dy\} = \frac{\theta\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\theta^2 (y-t)^2}{2t}} dy \qquad \text{for} \quad y \ge t$$

which is also half Normal, but shifted to t. The result i) follows from the definition of $\bar{\alpha}^+_t$ since

$$\mathbb{P}\{\bar{A}^+_s < t\} = \mathbb{P}\{s < \bar{\alpha}^+_t\} \tag{15}$$

Then, ii) follows from the fact that $\mathbb{P}\{A^0_s < t\} = \mathbb{P}\{s - A^+_s < t\} = 1 - \mathbb{P}\{A^+_s \le (s-t)\}$. □

Recall that the occupation times of the sticky Brownian motion have been defined in equations (6) to (8). Clearly, A^+ and A^- have the same distribution, which is given in the following main result of this section. The distribution deviates from the Arcsine law corresponding to Brownian motion [12, Thm.VI.2.7]. Although our method of proof is borrowed from that for Brownian motion, we take into account that A_t^0 is positive with positive probability in the sticky case.

Theorem 3. *Let X be the sticky Brownian motion satisfying* (3)-(4) *with $X_0 = 0$. For $0 < t < 1$, the occupation times of X satisfy*

$$\begin{aligned} i)\ \mathbb{P}\{A_1^0 > t\} &= Erfc(\frac{\theta t}{\sqrt{2(1-t)}}) \\ ii)\ \mathbb{P}\{A_1^+ > t\} &= 1 - \mathbb{E}\left[Erfc\left(\frac{-Z^2 + Z\sqrt{Z^2 + \theta^2(1-t)}}{\theta\sqrt{2t}}\right)\right] \end{aligned}$$

for $0 < t < 1$, where Z is a standard Gaussian random variable.

Proof. i) The relation between the two-sided and one-sided sticky Brownian motion has been given in [4]. Through the specific construction of (X_t, B_t) given in [4, Sec.2] satisfying (3)-(4), the pair $(|X_s|, \hat{B}_t)$ satisfies Equation (1) for the one-sided sticky Brownian motion, where $\hat{B}_t = \int_0^t \operatorname{sgn}(X_s)\, dB_s$. Clearly, $l_t^0(X) = \frac{1}{2} l_t^0(|X|)$. As a result, we obtain

$$A_t^0 = \frac{1}{\theta} l_t^0(X) = \frac{1}{2\theta} l_t^0(|X|) = \bar{A}_t^0$$

as in [4, Eq.3.8]. This implies that A_t^0 and $\bar{A}_t^0$ have the same distribution that is given in Lemma 3.

ii) Note that $t = A_t^0 + A_t^- + A_t^+$, which implies

$$\alpha_t^+ = t + A^0_{\alpha_t^+} + A^-_{\alpha_t^+}\ . \tag{16}$$

Since $\mathbb{P}\{A_1^+ > t\} = \mathbb{P}\{1 > \alpha_t^+\}$, we will study α_t^+ to get the distribution of A_1^+. Recall from Lemma 1 that $\frac{\theta}{2} A^0_{\alpha_t^+}$ is the running supremum of the Brownian motion $-W^+$. Hence, we have

$$A^0_{\alpha_t^+} = \frac{2}{\theta} \sup_{s \le t}(-W_s^+) =: \frac{2}{\theta}\, S_t^+ \tag{17}$$

at time t, where S_t^+ has the law of a reflected Brownian motion. Now, for studying the term $A^-_{\alpha_t^+}$ in (16), define the stopping times

$$\tau_t = \inf\{s > 0 : l_s^0(X) > t\} \quad \text{and} \quad T^{W^-}(a) = \inf\{t : W_t^- > a\}$$

for $a > 0$. Note that (9) implies

$$0 = X^-_{\tau_t} = -W^-_{A^-_{\tau_t}} + \frac{\theta}{2} A^0_{\tau_t} = -W^-_{A^-_{\tau_t}} + \frac{l^0_{\tau_t}(X)}{2} = -W^-_{A^-_{\tau_t}} + \frac{t}{2}$$

in view of (4). This yields $W^-_{A^-_{\tau_t}} = \frac{t}{2}$. Then, we have

$$A^-_{\tau_t} = T^{W^-}(t/2)$$

for each $t \geq 0$, which follows along the same lines of proof for Brownian motion in [12, Thm.VI.2.7]. Replacing t with $l^0_{\alpha^+_t}$ above yields

$$A^-_{\alpha^+_t} = T^{W^-}(l^0_{\alpha^+_t}/2) = T^{W^-}\left(\frac{\theta}{2} A^0_{\alpha^+_t}\right) = T^{W^-}(S^+_t) \tag{18}$$

where the last equality follows from (17).

We are now ready to compute $\mathbb{P}\{A^+_1 > t\}$. Using (16), (17) and (18), we get

$$\begin{aligned}
\mathbb{P}\{A^+_1 > t\} &= \mathbb{P}\{1 > \alpha^+_t\} = \mathbb{P}\{0 > t - 1 + A^0_{\alpha^+_t} + A^-_{\alpha^+_t}\} \\
&= \mathbb{P}\{0 > t - 1 + \frac{2}{\theta} S^+_t + T^{W^-}(S^+_t)\} \\
&= \mathbb{E}[\mathbb{P}\{0 > t - 1 + \frac{2}{\theta} S^+_t + T^{W^-}(S^+_t) \,|\, S^+_t\}] \\
&= \mathbb{E}[\mathbb{P}\{0 > t - 1 + \frac{2}{\theta} S^+_t + \frac{(S^+_t)^2}{(W^-_1)^2} \,|\, S^+_t\}] \qquad (19) \\
&= \mathbb{P}\left\{0 > t - 1 + \frac{2}{\theta} S^+_t + \frac{(S^+_t)^2}{(W^-_1)^2}\right\}
\end{aligned}$$

where (19) follows by the independence of S^+ and W^- from [12, III.Prop.3.10], which states that

$$T^{W^-}(a) \stackrel{(d)}{=} \frac{a^2}{(W^-_1)^2} \qquad a > 0\,.$$

As S^+ and W^- are independent, we will only need the roots of the polynomial $P(x) = t - 1 + \frac{2}{\theta} x + \frac{x^2}{w^2}$. The positive root is given by $(w^2/\theta)(-1 + \sqrt{1 + \theta^2(1-t)/w^2})$. As a result, we get

$$\begin{aligned}
\mathbb{P}\{A^+_1 > t\} &= \int_{-\infty}^{\infty} \left(\int_0^{(x^2/\theta)(-1+\sqrt{1+\theta^2(1-t)/x^2})} \frac{2\, e^{-y^2/(2t)}}{\sqrt{2\pi t}}\, dy \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_0^{\frac{x^2}{\theta\sqrt{t}}(-1+\sqrt{1+\theta^2(1-t)/x^2})} \frac{2\, e^{-y^2/2}}{\sqrt{2\pi}}\, dy \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx
\end{aligned}$$

since S^+ has the law of a reflected Brownian motion. □

As a final remark, we observe from (19) that the distribution of A_1^+ converges to the Arcsine law as $\theta \to \infty$, by similar computations that would follow as in the proof of [12, Thm.VI.2.7] for Brownian motion. This is consistent with the fact that the two-sided sticky Brownian motion approximates a standard Brownian motion in this case.

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