Asymptotically Optimal Energy Consumption and Inventory Control in a Make-to-Stock Manufacturing System

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We study energy consumption and inventory control in a make-to-stock manufacturing system. We consider a Markovian system with a single server. The server consumes energy, and the associated energy cost depends on the server state: a busy server consumes more power than an idle server does, and an idle server consumes more power than a turned-off server does. However, when a turned-off server is turned on, it must complete a set-up process which is costly and lasts for a while. The objective of the system controller is to minimize the long-run average inventory holding, backorder, and energy consumption costs by deciding when to produce, when to idle or turn-off the server, and when to turn on a turned-off server. We consider the asymptotic regime in which the server is in the conventional heavy-traffic regime. We formulate a Brownian control problem (BCP) which is a mixed impulse-singular control problem. Depending on the system parameters, the optimal BCP solution is of either control-band or barrier type. From the optimal BCP solution, we propose a simple heuristic control policy and prove its asymptotic optimality. Finally, we demonstrate the close-to-optimal performance of the heuristic control policy by numerical experiments.

Key words: Make-to-stock manufacturing, energy consumption, inventory control, asymptotic optimality

1. Introduction

The manufacturing industry was the highest energy consuming sector in the U.S. in 2021 by accounting for 32.7% of all energy consumption in the U.S (see EIA (2022)). As the energy costs increase, efficient energy consumption becomes more and more important. The traditional manufacturing objectives such as maximizing the revenue and maximizing the throughput ignore the energy consumption costs. However, energy costs can be significant and there is a room for improvement in the efficiency of energy consumption. For example, Gutowski et al. (2005) observed that in machining operations in a Toyota factory, 85.2% of the total energy consumption was done when the machines were "idle". Mouzon et al. (2007) observed that a machine stayed idle on average 16%

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of the time in a manufacturing facility and if the machines were turned off during the breaks and idle times, at least 13% energy savings could be achieved. Ford reduced energy consumption by 14% in an assembly plant in one year by shutting down equipment during non-production periods (see Galitsky and Worrell (2008)). Therefore, more and more machines are equipped with different energy-saving modes to reduce energy consumption (see Eberspächer and Verl (2013)).

We study energy consumption and inventory control in a make-to-stock manufacturing system. We consider a Markovian manufacturing system with a single server. The demand arrives in the system with respect to a Poisson process and the manufacturing times are exponentially distributed. Upon a demand arrival, if there is on-hand inventory, the demand is fulfilled immediately from the inventory. Otherwise, the demand is backordered. The system incurs inventory holding cost per finished product per unit time and backorder cost per demand per unit time. Furthermore, the server consumes energy and the associated energy cost depends on the server state. Specifically, a busy server consumes more power than an idle server does and an idle server consumes more power than a turned-off server does. However, when a turned-off server is turned on, it must complete a set-up process before it becomes available for manufacturing. The set-up process lasts for an exponential amount of time and the server consumes more power in the set-up state than it does in the busy state. On the one hand, turning-off the server reduces the energy consumption cost per unit time (that is, the power consumption cost) more than idling the server does, which makes turning-off the server appealing. On the other hand, the set-up process is costly and lasts for a while, which makes turning-off the server unappealing. The objective of the system controller is to minimize the long-run average inventory holding, backorder, and energy consumption costs by deciding when to produce, when to idle or turn-off the server, and when to turn on a turned-off server.

The exact analytical characterization of the optimal control policy is challenging and not available for the system we consider in this study. Therefore, we use asymptotic analyses. Specifically, we consider the conventional heavy-traffic regime for the server, in which the demand arrival rate

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approaches to the manufacturing rate from below such that the server is fully utilized in the limit. The heavy-traffic regime results in a tractable *Brownian control problem* (BCP). On the one hand, because the limiting idle-time process can increase at uncountably many time points, it makes the BCP a *singular (or instantaneous) control problem* (see Harrison and Taksar (1983)). On the other hand, because the diffusion limit of the cumulative amount of time the server is turned-off (which we call as the turned-off-time process) is a pure-jump process, it makes the BCP an *impulse control problem* (see Harrison et al. (1983)). Consequently, unlike the BCPs that we see in the literature on the control of make-to-stock manufacturing systems (which are generally singular control problems, see for example Wein (1992) and Ata and Barjesteh (2022)), our BCP is jointly an impulse control problem and a singular control problem; or with respect to the terminology of Dai and Yao (2013a), the BCP is a *mixed impulse-singular control problem*. Therefore, depending on the system parameters, the optimal BCP solution is either a *control-band policy* or a *barrier policy*. Under a control-band policy, whenever the inventory process reaches an upper threshold level, it makes a downward jump to a lower threshold level (see section 7.2 of Harrison (2013)). The control-band policy uses the turned-off-time process to control the inventory level. Under a barrier policy, whenever the inventory process reaches an upper barrier level, it is reflected downwards and thus the inventory process is never allowed to exceed the barrier level (see section 7.7 of Harrison (2013)). The barrier policy uses the idle time process to control the inventory level.

Next, we show that the optimal objective function value of the BCP is an asymptotic lower bound on the long-run average cost under any admissible sequence of control policies under diffusion scaling. Furthermore, based on the optimal solution to the BCP, we formulate a heuristic control policy for the original (non-asymptotic) system, to which we refer to as the "pre-limit" system.

- If the optimal BCP solution is a control-band policy, our heuristic control policy is a simple *turning-off policy* in which the server can be in busy, turned-off, or set-up states. Specifically, the busy server is turned off when the inventory level reaches a specific upper threshold level and the turned-off server is turned on when the inventory level decreases to a specific lower threshold level.

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After the set-up process, the server manufactures non-stop until the inventory level reaches the upper threshold level.

- If the optimal BCP solution is a barrier policy, our heuristic control policy is a simple *idling policy* in which the server can be in busy or idle states. Specifically, the busy server is idled when the inventory level reaches a specific upper threshold level. While the server is idle, after the inventory level decreases to a specific lower threshold level, the server manufactures non-stop until the inventory level reaches the upper threshold level.

We show that the diffusion-scaled long-run average cost under our proposed heuristic control policy achieves the asymptotic lower bound in the limit, implying that it is asymptotically optimal. Finally, we show that the heuristic control policy performs very well in numerical experiments.

We use the classical lower-bound approach to solve the BCP (Harrison and Taksar (1983), Harrison et al. (1983), Dai and Yao (2013a)). First, we present the so-called verification theorem (see Lemma 2) which presents a lower bound on the optimal BCP solution. Then, we show that depending on the system parameters, either a control-band policy or a barrier policy achieves that lower bound (see Theorem 1). The main technical challenge is to prove the asymptotic optimality of our heuristic control policy. The reason is that even though the turned-off-time process is both continuous and piecewise linear in the pre-limit, it converges to a pure-jump process, which is discontinuous. To overcome this problem, we prove that in the pre-limit, the set of admissible policies (that is, deterministic Markov policies with a stationary distribution) is performance-wise dominated by three types of control policies, namely turning-off, idling, and hybrid type of policies (see Lemma 1). Because the three types of policies have a relatively simple regenerative structure, we use the renewal theory (especially the renewal-reward theorem) to prove the asymptotic lower-bound result (see Theorem 2) and the asymptotic optimality of our heuristic control policy (see Theorem 3). Our main contributions are as follows.

1. We present an asymptotic optimality result in diffusion scale for energy consumption and inventory control in a make-to-stock manufacturing system. To the best of our knowledge, this is the first optimality result proven for the aforementioned system.

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2. To the best of our knowledge, in the literature on the Brownian inventory models, our study is the first one proving an asymptotic optimality result when the underlying BCP is a mixed impulse-singular control problem.
3. We derive a simple decision rule stating whether our heuristic control policy is of turning-off or idling type. Specifically, we compare the area under a function with the limiting cost of a set-up process. That function is related to the derivative of the relative value function and the area can be numerically computed easily. If that area is greater than the limiting set-up cost, then a specific control-band policy is an optimal BCP solution and thus a specific sequence of turning-off policies is asymptotically optimal. Otherwise, a specific barrier policy is an optimal BCP solution and thus a specific sequence of idling policies is asymptotically optimal.

   The intuitive explanation of the decision rule is as follows. First, the relative value function is decreasing in the inventory level. The reason is that if the inventory level is high, then the system controller can decrease the power consumption cost by turning off or idling the server and the inventory level will eventually decrease in the meantime due to the demand. Therefore, if the server is turned off, energy saving will be done but inventory related costs will increase due to the decrease in the inventory level. The aforementioned area represents the total saving in the energy cost done when the server is turned off minus the increase in the inventory related costs. If that net saving is less than the cost of a set-up process, then turning off the server is not beneficial.
4. In the asymptotic optimality proofs, we jointly use the theory of stochastic integration and the renewal theory. Such a simple but effective solution approach may also be used for stochastic control problems in different contexts.

The rest of the paper is organized as follows. We present a literature review in Section 2 and the model in Section 3. We introduce the asymptotic analyses in Section 4. Then, we present and solve the BCP in Section 5. We present the asymptotic optimality result in Section 6. Next, we present numerical experiments in Section 7. Finally, we present some future research directions in Section 8. All the proofs are presented in the online appendix (OA).

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2. Literature Review

The literature on the energy consumption control in queueing or manufacturing systems has significantly grown in the recent years. Down (2022) presents some challenges in the control of energy-aware queueing systems and presents some open problems. Badian-Pessot et al. (2021) studies the energy consumption, inventory, and service-rate control in an $M/M/1$ queue. By formulating an MDP model, they prove that there exists an optimal control policy which is among two types of policies. Under the first policy type, the server is never turned off and operates in a work-conserving fashion. Under the second policy type, the server is turned off when the system becomes empty and the server is turned on when the queue length reaches a certain threshold. They also prove that the optimal service rate is increasing in the queue length. uit het Broek et al. (2020) study energy consumption and queue-length control in an $M/G/1$ queue with state-dependent set-up times and costs. Their motivation comes from a circuit board manufacturer which needs to maintain a certain temperature for production. The system controller can turn off the heater to decrease energy consumption. There is a set up process associated with turning on the heater and its length and cost depend on the temperature of the heater when it is turned on. They derive the structure of the optimal control policy in a deterministic fluid model. Then, they study performance evaluation of certain class of control policies in the stochastic setting. They observe that turning-off policies perform very well in numerical experiments. Frigerio and Matta (2016) numerically demonstrate the efficiency of turning-off type policies in an $M/M/1/K$ queue. A closely related study to ours is Tan et al. (2022). They study a make-to-stock manufacturing system which is more general than ours because i) they use Markov arrival processes to formulate the stochastic primitives and ii) they consider both a model with lost sales and a model with backorders. However, they derive the optimal policies numerically based on the solution of an MDP model rather than proving structural results. In the numerical experiments of Tan et al. (2022), at the instances where the stochastic primitives are exponentially distributed, the optimal control policy is either turning-off or idling type. Similarly, our asymptotically optimal proposed heuristic control policy is of either turning-off or idling type. For a recent literature review on energy consumption control in manufacturing

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systems, see Tan et al. (2022). The main difference of our study from the aforementioned studies is that we not only prove that there exist an asymptotically optimal control policy which is of either turning-off or idling type but also derive a simple rule which determines the type and the parameters of that control policy.

Our paper is closely related to the literature on the control of make-to-stock manufacturing systems. The seminal paper Wein (1992) studies scheduling control of a multi-class make-to-stock system with the objective of minimizing the long-run average holding and backorder costs. By assuming that the system operates in the heavy-traffic regime, Wein (1992) formulates and solves a BCP, where the optimal BCP solution is a barrier policy. Later, Ata and Barjesteh (2022) extend the model of Wein (1992) by including dynamic pricing and outsourcing to the set of controls. Similar to Wein (1992), Ata and Barjesteh (2022) formulate and solve a BCP. The optimal BCP solution is a barrier policy with a two-sided barrier. When the inventory level reaches an upper barrier level, the server idles and keeps the inventory level below the barrier level. When the inventory level reaches a lower barrier level, the system controller keeps the inventory level above the lower barrier level by outsourcing the demand. The system controller uses dynamic pricing continuously over time to maximize the long-run average profit. Both Wein (1992) and Ata and Barjesteh (2022) derive heuristic control policies from their optimal BCP solutions and demonstrate the close-to-optimal performance of those policies by numerical experiments. However, none of them prove the asymptotic optimality of the heuristic control policies. Gao and Huang (2022) study the same problem as Ata and Barjesteh (2022) do and they rigorously prove the asymptotic optimality of their proposed heuristic control policy. For a comprehensive literature review on the control of make-to-stock manufacturing systems, see Ata and Barjesteh (2022).

There are three major differences between our study and Wein (1992), Ata and Barjesteh (2022), and Gao and Huang (2022). First, those three studies consider a multi-class manufacturing system whereas we consider a single-class one. Because the BCP is multi-dimensional in a multi-class setting, each of those three studies derives a single-dimensional equivalent workload formulation

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(EWF) of the BCP and solves the EWF. The derivation of the EWF is relatively straightforward and uses the fact that the workload allocation in which the inventory holding, backordering, and outsourcing costs are incurred only due to the cheapest product types is optimal. However, in our setting, when the server is turned-off, inventory level of each product type decreases and thus inventory holding cost or backorder cost is incurred due to all product types (see OA A.6 for details). Consequently, in our setting, if there are multiple product types, it is not possible to obtain a single-dimensional EWF by the conventional techniques. Second, the BCPs in those three studies are all singular control problems and thus barrier type policies are optimal solutions to the BCPs. However, our BCP is a mixed impulse-singular control problem and thus either a control-band policy or a barrier policy is an optimal BCP solution depending on the system parameters. Third, similar to Gao and Huang (2022), we prove the asymptotic optimality of our heuristic control policy. Because Gao and Huang (2022) study a multi-class setting, the main technical challenge that they overcome is about the steady-state analysis of the underlying multi-dimensional Markov chains. They use the Foster-Lyapunov criteria to prove the existence of stationary distributions and then establish necessary tightness results by deriving tail bounds for the stationary distributions. On the one hand, because we consider a single-class setting, the resulting Markov chains are two-dimensional, which facilitates the analyses. On the other hand, because our BCP is a mixed impulse-singular control problem, we need to prove the convergence of a continuous and piecewise linear control process to a discontinuous one in the limit, which complicates the analyses.

There is a rich literature on the Brownian inventory models. One of the earliest examples is the seminal paper of Bather (1966). Since then, Brownian inventory models are used to study stochastic storage models (Harrison and Taksar (1983), Harrison et al. (1983)), make-to-stock manufacturing systems (Wein (1992), Ata and Barjesteh (2022), Gao and Huang (2022)), make-to-order manufacturing systems (Rubino and Ata (2009)), repair facility control problems (Özkan and van Houtum (2021)) etc. If a control process can increase uncountably many times, the BCP becomes a singular (or instantaneous) control problem (Harrison and Taksar (1983), Wein (1992),

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Rubino and Ata (2009), Özkan and van Houtum (2021), Ata and Barjesteh (2022), Gao and Huang (2022)). If a control process is allowed to change finitely many times on finite time intervals with upward or downward jumps, the BCP becomes an impulse control problem (Bather (1966), Harrison et al. (1983), Örmeci et al. (2008)). Associated with each jump, there are both a fixed cost and a variable cost which is dependent on the size of the jump. If the singular and impulse controls exist jointly, then the BCP becomes a mixed impulse-singular control problem (Dai and Yao (2013a,b)) and our BCP belongs to this class. To the best of our knowledge, in the literature on the Brownian inventory models, our study is the first one proving an asymptotic optimality result when the underlying BCP is a mixed impulse-singular control problem. Furthermore, in the literature, the downward inventory adjustments in the impulse control problems are generally explained as curtailed shipments (Örmeci et al. (2008)) or salvage sales or returns to the supplier (Federgruen et al. (2020)). However, in our context, those adjustments appear in the diffusion limit due to the server shut-downs and have a slightly different structure: The fixed cost appears due to the set-up process and the variable component is a revenue rather than a cost due to the energy saving done when the server is turned off. A comprehensive literature review on Brownian inventory models can be found in Dai and Yao (2013a).

Our study is also related to the papers in the queueing literature in which the diffusion limit of the state process is a discontinuous process (Krylov and Liptser (2002), Atar and Lev-Ari (2018)). Recently, Atar et al. (2022a,b) prove the first asymptotic optimality result in a queueing control problem where the limiting workload process is a diffusion process with discontinuous drift and diffusion coefficients. Kushner and Ramachandran (1989) study the control of a tandem queueing system. Because their controls are associated with fixed costs, the resulting BCP is an impulse control problem. The system controller can cut off the external arrivals and the arrivals to the second server coming from the first server. Furthermore, there is a fixed cost associated with stopping the production in a server. Because the resulting BCP is multi-dimensional, only numerical solutions are available. Kushner and Ramachandran (1989) prove that if there exists an ϵ-optimal BCP solution with a certain structure, there exists a sequence of control policies in the pre-limit whose discounted cost converges to the objective function value of the ϵ-optimal BCP solution.

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3. Model

We let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$ denote the set of integers, non-negative integers, and strictly positive integers, respectively. We let $\mathbb{R}$ denote the real line, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$. We consider a sequence of make-to-stock manufacturing systems indexed by $n \in \mathbb{N}_+$. Each make-to-stock manufacturing system is formulated as a single-server queue. In the nth system, demand arrives with respect to a Poisson process with rate λ^n and we let A^n denote the associated Poisson process, that is, $A^n(t)$ is the cumulative number of demands arrived up to time t for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. Manufacturing time for the products are independent and identically distributed (i.i.d.), independent of n, and exponentially distributed with mean $1/\mu$. We let S denote the Poisson process associated with the manufacturing process, that is, S has rate μ and $S(t)$ is the cumulative number of products manufactured if the server produces nonstop on the interval $[0, t]$ for all $t \in \mathbb{R}_+$.

The server can be in four different states, namely, busy, idle, turned-off, and set-up states. On the one hand, the power consumption in the idle and turned-off states is lower than the one under the busy state. On the other hand, the power consumption in the set-up state is higher than the one under the busy state. All the state transitions of the server happen instantaneously. When a turned-off server is turned on, the server enters the set-up state where the set-up times are i.i.d. and exponentially distributed with mean $1/\tau$. We let κ_i denote the ith set-up time for all $i \in \mathbb{N}_+$ and thus $\kappa_i \sim \exp(\tau)$.

We assume that $\{A^n, n \in \mathbb{N}_+\}$, S, and $\{\kappa_i, i \in \mathbb{N}_+\}$ are mutually independent. We have the following notation defined for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$.

- $Y^n(t) \in \mathbb{Y} := \{\text{busy, idle, turned-off, set-up}\}$ is the state of the server at time t.
- $Q^n(t)$ is the finished-goods inventory level at time t. $Q^n(t) \in \mathbb{Z}$ and $Q^n(t) < 0$ implies that there are backordered demands. We let $Q^n(0)$ denote the initial inventory level and $\{Q^n(0), n \in \mathbb{N}_+\}$ is a sequence of random variables independent of all other stochastic primitives.
- $T^n(t)$ is the cumulative amount of time the server manufactures products up to time t.
- $I^n(t)$ is the cumulative amount of time the server is idle up to time t.

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- $F^n(t)$ is the cumulative amount of time the server is turned-off up to time t.
- $U^n(t)$ is the cumulative amount of time the server spends for the set-up process up to time t.

We have the following constraints for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$.

$$Q^n(t) = Q^n(0) + S(T^n(t)) - A^n(t). \tag{1a}$$

$$T^n(t) + I^n(t) + F^n(t) + U^n(t) = t. \tag{1b}$$

$$T^n,\ I^n,\ F^n, \text{ and } U^n \text{ are continuous and nondecreasing.} \tag{1c}$$

$$Y^n(t) = \begin{cases} \text{busy}, & \text{if } \mathrm{d}T^n(t) > 0, \\ \text{idle}, & \text{if } \mathrm{d}I^n(t) > 0, \\ \text{turned-off}, & \text{if } \mathrm{d}F^n(t) > 0, \\ \text{set-up}, & \text{if } \mathrm{d}U^n(t) > 0. \end{cases} \tag{1d}$$

$$\text{If } Y^n(t) \in \{\text{busy, idle}\}, \text{ then } Y^n(t-) \neq \text{turned-off}. \tag{1e}$$

$$\text{If } Y^n(t) = \text{turned-off, then } Y^n(t-) \neq \text{set-up}. \tag{1f}$$

$$\text{If } Y^n(t) = \text{set-up, then } Y^n(t-) \notin \{\text{busy, idle}\}. \tag{1g}$$

$$\text{For all } x \in \{T^n, I^n, F^n, U^n\}, \text{ if } \mathrm{d}x(t) > 0 \text{ for some } t \in \mathbb{R}_+, \text{ then } \mathrm{d}x(t) = \mathrm{d}t. \tag{1h}$$

$$\text{The length of the } i\text{th interval on which } \mathrm{d}U^n(t) > 0 \text{ is equal to } \kappa_i \text{ for all } i \in \mathbb{N}_+. \tag{1i}$$

Constraints (1d) - (1i) restrict the transitions of the server state. For example, (1e) and (1i) imply that a server cannot be turned on instantaneously. Constraints (1b), (1c), and (1h) imply that the server cannot be in multiple different states at the same time.

We let $\mathfrak{X}^n(t) := (Q^n(t), Y^n(t)) \in \mathbb{Z} \times \mathbb{Y}$ denote the state of the system at time $t \in \mathbb{R}_+$. We let π^n denote a control policy in the nth system for all $n \in \mathbb{N}_+$. For notational convenience, we denote the process X in the nth system under the control policy π^n by $X^{\pi,n}$.

Definition 1. (Admissible policy) For a fixed $n \in \mathbb{N}_+$, a control policy $\pi^n : \mathbb{Z} \times \mathbb{Y} \to \mathbb{Y}$ is admissible if the process $(\mathfrak{X}^{\pi,n}, T^{\pi,n}, I^{\pi,n}, F^{\pi,n}, U^{\pi,n})$ satisfies (1) and the state process $\mathfrak{X}^{\pi,n}$ is a Markov process with a stationary distribution.

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In Definition 1, we focus on deterministic Markov control policies. Under an admissible control policy, the state process becomes a continuous-time Markov chain (CTMC) which has a stationary distribution. We let Π^n denote the set of admissible policies in the nth system for all $n \in \mathbb{N}_+$.

The objective is to minimize the long-run average relative cost. Let $h \in \mathbb{R}_{++}$ denote the holding cost per product per unit time and $b \in \mathbb{R}_{++}$ denote the backorder cost per demand per unit time. Next, we will introduce the costs related to energy consumption. Those costs are associated with the server states idle, turned-off, and set-up such that they represent the relative costs with respect to the server in the busy state. Specifically,

- p^n denotes the difference between the power consumption cost of being in the busy state and in the idle state. We assume that p^n is nonnegative and thus it represents the energy cost saving per unit time done by idling the server.
- r^n denotes the difference between the power consumption cost of being in the busy state and in the turned-off state. We assume that r^n is strictly positive and thus it represents the energy cost saving per unit time done by turning-off the server. We also assume that $r^n > p^n$ for all $n \in \mathbb{N}_+$, that is, the energy cost saving under the turned-off server state per unit time is strictly greater than the one under the idle server state.
- c^n denotes the difference between the power consumption cost in a set-up process and the power consumption cost of being in the busy state. We assume that c^n is strictly positive, that is, we assume that the server consumes more power in the set-up state than it does in the busy state (see figure 7 of Frigerio and Matta (2015) and figure 2 of Weinert et al. (2011) to see power consumption levels of a CNC machining center and a laser welding machine, respectively, in different states). If the set-up process must be handled by a set-up specialist, the associated labor cost per unit time is embedded in c^n without loss of generality. Observe that c^n/τ is the expected total relative cost of a set-up process with respect to the server being busy for the same duration.

For all $x \in \mathbb{R}$, let $(x)^+ := \max\{0, x\}$ and $(x)^- := \max\{0, -x\}$. For all $t \in \mathbb{R}_+$, $n \in \mathbb{N}_+$, and the control policy $\pi^n \in \Pi^n$, we let $J^{\pi,n}(t)$ denote the average cost up to time t such that

$$J^{\pi,n}(t) := \frac{1}{t}\mathbf{E}\left[c^n U^{\pi,n}(t) - r^n F^{\pi,n}(t) - p^n I^{\pi,n}(t) + \int_0^t \left(h\left(Q^{\pi,n}(s)\right)^+ + b\left(Q^{\pi,n}(s)\right)^-\right) \mathrm{d}s\right].$$

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For a fixed n, the objective is to find a control policy in the set

$$\arg\min_{\pi^n \in \Pi^n} \limsup_{t\to\infty} J^{\pi,n}(t). \tag{2}$$

4. Asymptotic Analyses

Because solving the optimization problem (2) is quite challenging, we will simplify it by first considering an asymptotic regime and then reducing the set of admissible policies without sacrificing the optimal performance. We assume that the system is in the conventional heavy-traffic regime.

ASSUMPTION 1. *(Asymptotic regime) There exists a constant $\theta \in \mathbb{R}_{++}$ such that $\lambda^n = \mu - \theta/\sqrt{n}$ for all $n \in \mathbb{N}_+$. Therefore, $\sqrt{n}\,(\mu - \lambda^n) = \theta$ for all $n \in \mathbb{N}_+$.*

Let $\rho^n := \lambda^n/\mu$ denote the load on the server in the nth system. Assumption 1 implies that $\rho^n \to 1$ as $n \to \infty$, that is, the server is fully utilized in the limit.

Next, we introduce three classes of admissible control policies.

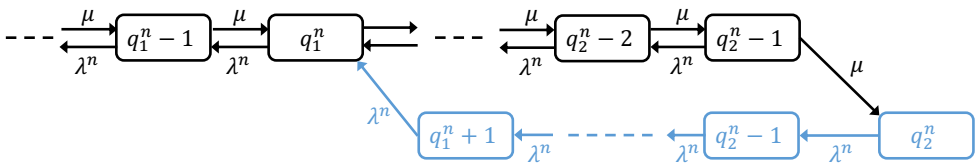
DEFINITION 2. Fix an arbitrary $n \in \mathbb{N}_+$ and let $(q_0^n, q_1^n, q_2^n) \in \mathbb{Z}^3$ be such that $q_0^n < q_1^n < q_2^n$.

1. *Idling policy* with the parameters (q_1^n, q_2^n): Whenever the inventory level reaches q_2^n, the server is idled until the inventory level decreases to q_1^n. At that time, the server starts manufacturing until the inventory level reaches q_2^n again.
2. *Turning-off policy* with the parameters (q_1^n, q_2^n): Whenever the inventory level reaches q_2^n, the server is turned off until the inventory level decreases to q_1^n. At that time, the server is turned on and enters the set-up stage. As soon as the set-up is over, the server starts manufacturing until the inventory level reaches q_2^n again.
3. *Hybrid policy* with the parameters (q_0^n, q_1^n, q_2^n): Whenever the inventory level reaches q_2^n, the server is turned off until the inventory level decreases to q_1^n. At that time, the server is turned on and enters the set-up stage.
   (a) When the set-up is over, if the inventory level is in the set $\{q_0^n + 1, q_0^n + 2, \ldots, q_1^n\}$, the server is idled until the inventory level decreases to q_0^n. At that time, the server starts manufacturing until the inventory level reaches q_2^n again.

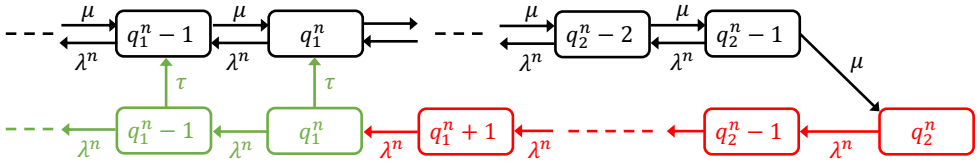
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(b) When the set-up is over, if the inventory level is less than or equal to q_0^n, the server starts manufacturing until the inventory level reaches q_2^n again.

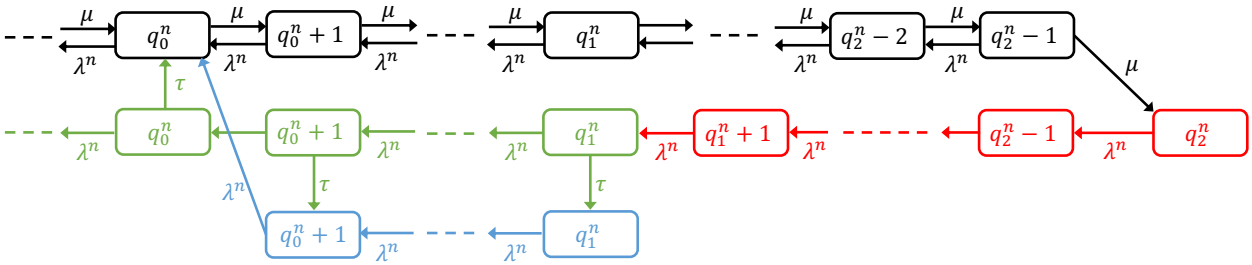
Figure 1 presents the CTMC transition diagrams of the idling, turning-off, and hybrid policies. The following lemma states that we can focus on the three classes of policies in Definition 2.



(a) The idling policy with the parameters (q_1^n, q_2^n).



(b) The turning-off policy with the parameters (q_1^n, q_2^n).



(c) The hybrid policy with the parameters (q_0^n, q_1^n, q_2^n).

Figure 1 (Color online) CTMC transition diagrams of the idling, turning-off, and hybrid policies. The black, blue, red, and green colors denote the busy, idle, turned-off, and set-up states for the server, respectively.

LEMMA 1. *Fix an arbitrary $n \in \mathbb{N}_+$.*

1. *All idling, turning-off, and hybrid policies induce an irreducible and positive recurrent CTMC, and thus they are admissible.*
2. *For any given admissible control policy π^n, there exists a control policy which is of idling, turning-off, or hybrid type with less than or equal long-run average cost.*

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The proof of Lemma 1 is presented in the OA A.1. The first part of Lemma 1 implies that under each idling, turning-off, or hybrid policy, the state process $\{\mathfrak{X}^n(t), t \in \mathbb{R}_+\}$ has a limiting distribution, which is the unique stationary distribution.

We make the following assumption for the initial system state.

ASSUMPTION 2. *(Initial system state) Fix an arbitrary $n \in \mathbb{N}_+$. Under all the idling, turning-off, and hybrid policies, the system starts with the induced limiting distribution.*

An immediate consequence of Assumption 2 is that the system state is stationary at all times. For example $\mathfrak{X}^n(t)$ has the same distribution with $\mathfrak{X}^n(0)$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$.

Next, we introduce the fluid and diffusion scaled processes. Let $\mathbb{D}$ denote the set of functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are right continuous with left limits. For all $Z^n \in \mathbb{D}$, the fluid scaled version of it is denoted by $\bar{Z}^n(t) := Z^n(nt)/n$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. We define the diffusion scaled processes such that

$$\hat{S}^n(t) := \frac{S(nt) - \mu nt}{\sqrt{n}}, \qquad \hat{A}^n(t) := \frac{A^n(nt) - \lambda^n nt}{\sqrt{n}},$$

and for all other $Z^n \in \mathbb{D}$, the diffusion scaled version of it is denoted by $\hat{Z}^n(t) := Z^n(nt)/\sqrt{n}$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Let

$$\hat{X}^n(t) := \hat{S}^n \circ \bar{T}^n(t) - \hat{A}^n(t), \qquad \forall t \in \mathbb{R}_+, n \in \mathbb{N}_+, \tag{3}$$

where '$\circ$' denotes the composition map. By Assumption 1, we have

$$\hat{Q}^n(t) = \hat{Q}^n(0) + \hat{X}^n(t) + \theta t - \mu \left(\hat{I}^n(t) + \hat{F}^n(t) + \hat{U}^n(t) \right), \quad \forall t \in \mathbb{R}_+, n \in \mathbb{N}_+. \tag{4}$$

We consider a specific scaling of the cost parameters.

ASSUMPTION 3. *(Cost scaling) There exists $c, r \in \mathbb{R}_{++}$ and $p \in \mathbb{R}_+$ such that for all $n \in \mathbb{N}_+$,*

$$c^n = n^{3/2}\tau c, \qquad p^n = np, \qquad r^n = nr, \qquad r > p \geq 0.$$

Assumption 3 implies that the energy cost saving per unit time in the idle and turned-off server states are $O(n)$, where $O(\cdot)$ is the big-O notation. Furthermore, the relative cost of a set-up process

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is $O(n^{3/2})$. Under the cost scalings in Assumption 3, the cost savings under the idle and turned-off states and the cost of a set-up process appear in the resulting BCP. By Assumption 3, the diffusion-scaled average cost up to time $t \in \mathbb{R}_+$ is as follows.

$$\hat{J}^n(t) = \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - p\hat{I}^n(t) + \int_0^t \left(h\left(\hat{Q}^n(s)\right)^+ + b\left(\hat{Q}^n(s)\right)^-\right) \mathrm{d}s\right]. \tag{5}$$

5. The Brownian Control Problem

By considering the limits of the diffusion scaled processes, we obtain a BCP. Let X be a driftless Brownian motion with variance $\Sigma \in \mathbb{R}_{++}$ and $X(0) = 0$. Let us consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ satisfying the usual conditions (see appendix A.1 of Harrison (2013) for details) such that $\mathbb{F}$ is the filtration generated by X. For all $x \in \mathbb{D}$ and $t \in \mathbb{R}_{++}$, let $\Delta x(t) := x(t) - x(t-)$. The associated BCP is as follows.

$$\min \limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\left(\sum_{0<s\le t:\, \Delta F(s)>0} (c - r\Delta F(s))\right) - pI(t) + \int_0^t \left(h\left(Q(s)\right)^+ + b\left(Q(s)\right)^-\right) \mathrm{d}s\right] \tag{6a}$$

$$\text{s.t. } Q(t) = Q(0) + X(t) + \theta t - \mu I(t) - \mu F(t), \qquad \forall t \in \mathbb{R}_+, \tag{6b}$$

$$I(0) = F(0) = 0 \text{ and both } I \text{ and } F \text{ are nondecreasing, } \mathbb{F}\text{-adapted, and in } \mathbb{D}, \tag{6c}$$

$$F \text{ is piecewise constant and } \Delta I(t)\Delta F(t) = 0 \text{ for all } t \in \mathbb{R}_+, \tag{6d}$$

$$\mathbf{E}\left[\left(Q(0)\right)^2\right] < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\left(Q(t)\right)^2\right] = 0, \tag{6e}$$

where the decision variable is the process (Q, I, F) and the sum in the objective function (6a) is over the jump epochs of the process F. Constraint (6d) states that the server cannot be in both idle and turned-off states simultaneously. Constraint (6e) helps us to avoid uninteresting technical complexities by eliminating some ineffective controls.

Recall that Σ is variance parameter of the Brownian motion X. Hence, Σ is the limiting variance of the process $\hat{X}^n$ defined in (3) and thus $\Sigma = \mu\left(c_a^2 + c_s^2\right)$ by Assumption 1, where c_a and c_s are the coefficient of variations of the inter-arrival and manufacturing times, respectively. Therefore, in the Markovian system described in Section 3, we have $\Sigma = 2\mu$. Because we will derive a heuristic

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control policy from the optimal BCP (6) solution which is also implementable in non-Markovian systems, we use the general notation Σ rather than 2μ in the BCP (6).

The BCP (6) has a simpler structure than the diffusion-scaled optimization problem (4) - (5). First, because the cumulative amount of time spent for set-up (that is, the process U^n) does not appear in (6b), the queue length constraint (6b) is simpler than the diffusion-scaled queue length constraint (4). Therefore, in the BCP (6), the effect of a set-up process appears only in the objective function (6a) as a cost. An intuitive explanation for the aforementioned situation is as follows. Recall that we scale the time with n and the space with $1/\sqrt{n}$ under diffusion scaling, e.g., $\hat{Q}^n(t) = Q^n(nt)/\sqrt{n}$. The average length of a set-up process is $1/\tau = O(1)$. If the server starts a set-up process at time nt, the set-up will end at time $nt + O(1)$, which implies that from the perspective of a diffusion-scaled process, the ending time of the set-up process is $t + O(1)/n \approx t$ when n is large. Therefore, it is as if the set-up process happens instantaneously over the diffusion-scaled time.

Second, the cumulative amount of time the server is turned-off, that is, the process F, is a pure jump process in the BCP (6) unlike the situation in the pre-limit (recall that the process F^n is continuous and piecewise linear). Therefore, in the BCP (6), turning off the server causes a downward jump in the inventory process and the goal is to control the timing and the magnitude of those jumps. By (6a), the cost incurred due to a jump of the control process F with a magnitude $x > 0$ is equal to $c - rx$, where $c > 0$ is the fixed cost due to set-up and $r > 0$ is the energy saving per unit jump (recall Assumption 3). An intuitive explanation for the aforementioned situation is as follows. Under the conventional heavy-traffic regime (recall Assumption 1), the inventory level is expected to be $O(\sqrt{n})$. Therefore, when the server is turned-off, the decrease in the inventory level should be $O(\sqrt{n})$, which implies that the duration of the time interval on which the server is turned off should be $O(\sqrt{n})$. Hence, if the server is turned off at time nt, it will be turned on at time $nt + O(\sqrt{n})$, implying that from the perspective of a diffusion-scaled process, the turning-on time is $t + O(\sqrt{n})/n \approx t$ when n is large. Therefore, it is as if the server is turned off instantaneously over the diffusion-scaled time. Furthermore, because the inventory level is $O(\sqrt{n})$ and decreases by $O(\sqrt{n})$ when the server is turned-off, the inventory process makes a downward jump.

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On the one hand, because the control process F is a pure jump process, the BCP (6) becomes an impulse control problem (see section 7.1 of Harrison (2013)). On the other hand, because the control process I is allowed to increase at uncountably many points, the BCP (6) becomes a singular control problem (see section 7.7 of Harrison (2013)). Consequently, the BCP (6) is a mixed impulse-singular control problem (see Dai and Yao (2013a)).

Next, we will use the lower-bound approach (see Dai and Yao (2013a)) to solve the BCP (6). First, we will derive a lower bound on the optimal BCP (6) objective function value in Section 5.1. Then, we will prove that depending on the system parameters, either a control-band policy or a barrier policy achieves that lower bound in Section 5.2.

5.1. The Lower-Bound Approach

The following lemma is known as the "verification theorem" in the literature (see Dai and Yao (2013a)) and presents a lower bound on the optimal BCP (6) objective function value. Let $\mathbb{C}_1$ denote the set of functions $f:\mathbb{R}_+\to\mathbb{R}$ which have a continuous derivative and that derivative is not differentiable only at finitely many points. Let f' denote the derivative of f and f'' denote the second derivative such that f'' is defined appropriately on the points where f' is not differentiable.

LEMMA 2. *Let $\gamma\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be such that $f\in\mathbb{C}_1$, f'' is piecewise continuous, and*

$$\frac{1}{2}\Sigma f''(x)+\theta f'(x)+h(x)^+ +b(x)^- \geq \gamma, \qquad \forall x\in\mathbb{R}, \tag{7a}$$

$$f'(x)\leq -\frac{p}{\mu}, \qquad \forall x\in\mathbb{R}, \tag{7b}$$

$$f(x)-f(y)\geq \frac{r}{\mu}(y-x)-c, \qquad \forall x\leq y, \tag{7c}$$

$$|f(x)|\leq C_1+C_2x^2, \qquad \forall x\in\mathbb{R}, \tag{7d}$$

where C_1 and C_2 are strictly positive constants. Then, γ is a lower bound on the optimal objective function value of the BCP (6).

The proof of Lemma 2 is presented in the OA A.3.1. Next, we will find a pair (f,γ) which satisfies the condition (7). For a given $\gamma\in\mathbb{R}$, consider the parametric ordinary differential equation (ODE)

$$\frac{1}{2}\Sigma g_\gamma'(x)+\theta g_\gamma(x)+h(x)^+ +b(x)^- =\gamma, \qquad \forall x\in\mathbb{R}. \tag{8}$$

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Observe that

$$g_\gamma(x) := \begin{cases} \frac{\gamma}{\theta} - \frac{b\Sigma}{2\theta^2} + \frac{b}{\theta}x, & \text{if } x \le 0, \\ \frac{\gamma}{\theta} + \frac{h\Sigma}{2\theta^2} - \frac{h}{\theta}x - \frac{(h+b)\Sigma}{2\theta^2}\mathrm{e}^{-\frac{2\theta}{\Sigma}x}, & \text{if } x > 0, \end{cases} \tag{9}$$

is a solution to the ODE (8) and

$$g'_\gamma(x) = \begin{cases} \frac{b}{\theta}, & \text{if } x \le 0, \\ -\frac{h}{\theta} + \frac{h+b}{\theta}\mathrm{e}^{-\frac{2\theta}{\Sigma}x}, & \text{if } x > 0. \end{cases}$$

Let

$$f_\gamma(x) := \int_0^x g_\gamma(u)\mathrm{d}u, \qquad \forall (\gamma, x) \in \mathbb{R}^2. \tag{10}$$

Then, $f'_\gamma(x) = g_\gamma(x)$ for all $x \in \mathbb{R}$ and

$$f_\gamma(x) = \begin{cases} \left(\frac{\gamma}{\theta} - \frac{b\Sigma}{2\theta^2}\right)x + \frac{b}{2\theta}x^2, & \text{if } x \le 0, \\ \left(\frac{\gamma}{\theta} + \frac{h\Sigma}{2\theta^2}\right)x - \frac{h}{2\theta}x^2 - \frac{(h+b)\Sigma^2}{4\theta^3}\left(1 - \mathrm{e}^{-\frac{2\theta}{\Sigma}x}\right), & \text{if } x > 0. \end{cases} \tag{11}$$

Let

$$q^* := \frac{\Sigma}{2\theta}\ln\left(1 + \frac{b}{h}\right).$$

Observe that if $\gamma \le hq^*$, then

$$\lim_{x\to-\infty} f_\gamma(x) = +\infty, \qquad \lim_{x\to+\infty} f_\gamma(x) = -\infty,$$

$$\lim_{x\to-\infty} g_\gamma(x) = -\infty, \qquad \lim_{x\to+\infty} g_\gamma(x) = -\infty,$$

$$g_\gamma \text{ is concave and has a unique maximum, and } g_\gamma(x) \le 0 \text{ for all } x \in \mathbb{R}, \tag{12a}$$

$$\arg\max_{x\in\mathbb{R}} g_\gamma(x) = q^*, \qquad g'_\gamma(q^*) = 0, \tag{12b}$$

$$g'_\gamma(x) < (>)\, 0, \qquad \forall x > (<)\, q^*, \tag{12c}$$

$$\max_{x\in\mathbb{R}} g_\gamma(x) = g_\gamma(q^*) = (\gamma - hq^*)/\theta. \tag{12d}$$

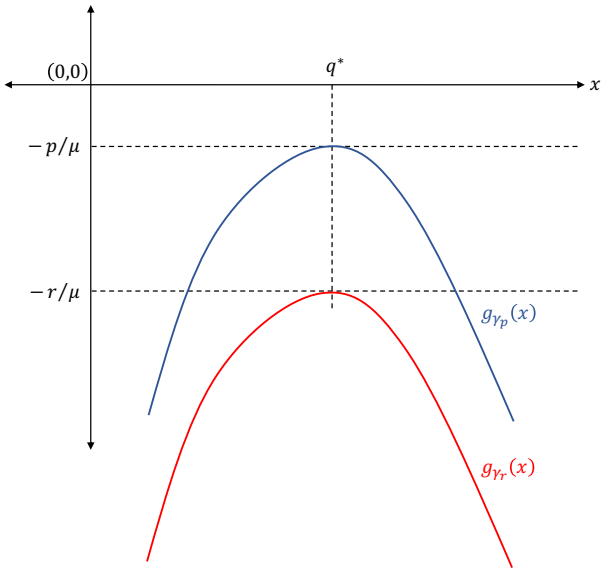
Observe that as γ increases, the function $g_\gamma(\cdot)$ shifts up but its shape does not change. Recall that we have $r > p$ by Assumption 3. Let

$$\gamma_r := hq^* - r\theta/\mu, \tag{13a}$$

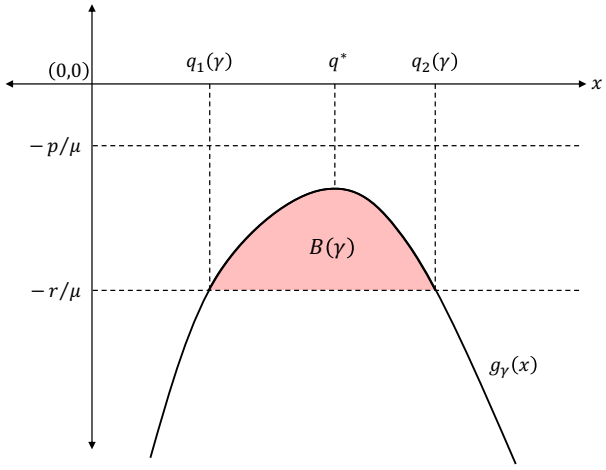
$$\gamma_p := hq^* - p\theta/\mu. \tag{13b}$$

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For all $\gamma \in [\gamma_r, \gamma_p]$, we have $\max_{x\in\mathbb{R}} g_\gamma(x) \in [-r/\mu, -p/\mu]$ by (12d). See Figure 2a for an illustration.



(a) Illustration of the functions g_{γ_r} and g_{γ_p}.



(b) Illustration of $B(\gamma)$ for some $\gamma \in (\gamma_r, \gamma_p)$.

Figure 2 (Color online) Illustration of the functions g_{γ_r}, g_{γ_p}, and $B(\gamma)$ for some $\gamma \in (\gamma_r, \gamma_p)$.

Observe that for all $\gamma \in (\gamma_r, \gamma_p]$, there are exactly two points at which $g_\gamma(x) = -r/\mu$. Let

$$q_1(\gamma) := \inf\{x \in \mathbb{R} : g_\gamma(x) = -r/\mu\}, \qquad \forall \gamma \in (\gamma_r, \gamma_p], \tag{14a}$$

$$q_2(\gamma) := \sup\{x \in \mathbb{R} : g_\gamma(x) = -r/\mu\}, \qquad \forall \gamma \in (\gamma_r, \gamma_p]. \tag{14b}$$

As depicted in Figure 2b, for all $\gamma \in (\gamma_r, \gamma_p]$,

$$q_1(\gamma) < q^* < q_2(\gamma). \tag{15}$$

Next, let

$$B(\gamma) := \frac{r}{\mu}\left(q_2(\gamma) - q_1(\gamma)\right) + \int_{q_1(\gamma)}^{q_2(\gamma)} g_\gamma(x)\mathrm{d}x, \qquad \forall \gamma \in (\gamma_r, \gamma_p]. \tag{16}$$

Observe that $B(\gamma)$ is the area depicted in Figure 2b. Furthermore,

$$B(\gamma) \text{ is strictly increasing and continuous in } \gamma \text{ on the interval } (\gamma_r, \gamma_p], \tag{17a}$$

$$\inf_{\gamma\in(\gamma_r,\gamma_p]} B(\gamma) = 0, \qquad \max_{\gamma\in(\gamma_r,\gamma_p]} B(\gamma) = B(\gamma_p). \tag{17b}$$

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Let

$$\gamma^* := \begin{cases} \gamma \in (\gamma_r, \gamma_p) \text{ such that } B(\gamma) = c, & \text{if } B(\gamma_p) > c, \\ \gamma_p, & \text{if } B(\gamma_p) \leq c. \end{cases} \tag{18}$$

By (17), γ^* exists and is well defined.

We will prove that γ^* is the optimal objective function value of the BCP (6). But first, we will construct a function f which together with γ^* satisfy the condition (7). By recalling (11), (14b), and (18), let

$$f(x) := \begin{cases} f_{\gamma^*}(x), & \text{if } x < q_2(\gamma^*), \\ f_{\gamma^*}(q_2(\gamma^*)) - \frac{r}{\mu}(x - q_2(\gamma^*)), & \text{if } x \geq q_2(\gamma^*). \end{cases} \tag{19}$$

LEMMA 3. *The pair* (γ^*, f) *defined in* (18) *and* (19) *satisfies the condition* (7). *Furthermore,*

$$\frac{1}{2}\Sigma f''(x) + \theta f'(x) + h(x)^+ + b(x)^- = \gamma^*, \qquad \forall x \leq q_2(\gamma^*). \tag{20a}$$

$$\text{If } \gamma^* \in (\gamma_r, \gamma_p), \text{ then } f(q_1(\gamma^*)) - f(q_2(\gamma^*)) = \frac{r}{\mu}\left(q_2(\gamma^*) - q_1(\gamma^*)\right) - c. \tag{20b}$$

The proof of Lemma 3 is presented in the OA A.3.2.

Next, we will prove that there exists a feasible solution to the BCP (6) which achieves γ^* as the objective function value, implying that γ^* is the optimal objective function value of the BCP (6). Furthermore, f is the relative cost function for an optimal BCP (6) solution.

5.2. An optimal solution to the BCP (6)

Depending on the parameters, the optimal solution of the BCP (6) is either a control-band or a barrier policy. A control-band policy is specified by the parameter pair $(q_1, q_2) \in \mathbb{R}^2$ such that $q_1 < q_2$. Whenever the inventory process, Q, reaches the inventory level q_2, the control process F makes a jump with the magnitude $(q_2 - q_1)/\mu$ such that the inventory process makes a downward jump to the inventory level q_1. Under the control-band policy, $I(t) = 0$ for all $t \in \mathbb{R}_+$, that is, the server is never idled.

A barrier policy is specified by a single parameter $q \in \mathbb{R}$ such that the inventory process is never allowed to exceed the barrier level q. The control process I keeps the inventory process below the

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barrier level with the minimum amount of push. Formally, the control process I can be expressed with the regulator component of the conventional one-sided regulator mapping (see chapter 2.2 of Harrison (2013)). Specifically, under the barrier policy, given that $Q(0) \leq q$,

$$\mu I(t) = \sup_{0 \leq s \leq t} \left(Q(0) + X(s) + \theta s - q\right)^+, \qquad \forall t \in \mathbb{R}_+, \tag{21}$$

and $F(t) = 0$ for all $t \in \mathbb{R}_+$, that is, the server is never turned off.

The following theorem presents an optimal solution to the BCP (6).

THEOREM 1. *The optimal objective function value of the BCP (6) is equal to γ^*.*

1. *Suppose that $B(\gamma_p) > c$ and thus $\gamma^* \in (\gamma_r, \gamma_p)$ by (18). Then, the control-band policy with the parameters $(q_1(\gamma^*), q_2(\gamma^*))$ and the initial inventory level $Q(0) = q_1(\gamma^*)$ is an optimal solution to the BCP (6).*
2. *Suppose that $B(\gamma_p) \leq c$ and thus $\gamma^* = \gamma_p$ by (18). Then, the barrier policy with the parameter q^* and the initial inventory level $Q(0) = q^*$ is an optimal solution to the BCP (6).*

The proof of Theorem 1 is presented in the OA A.3.4. The comparison $B(\gamma_p)$ vs. c determines whether the optimal solution is of control-band or barrier type. Recall that γ_p is defined in (13b) and $B(\gamma_p)$ is the area depicted in Figure 2b when $\gamma = \gamma_p$. By (16),

$$B(\gamma_p) = \frac{r}{\mu}\left(q_2(\gamma_p) - q_1(\gamma_p)\right) - \int_{q_1(\gamma_p)}^{q_2(\gamma_p)} \left(-g_{\gamma_p}(x)\right) \mathrm{d}x.$$

Because $g_\gamma(\cdot)$ is a non-positive function for all $\gamma \in (\gamma_r, \gamma_p]$ (recall (12a)), the relative value function is non-increasing in the inventory level. The first term in the right-hand side (RHS) of the equality above is the total saving in the energy cost done by turning-off the server. The second term is the increase in the relative value function due to the decrease in the inventory level. Therefore, intuitively, $B(\gamma_p)$ represents the net saving when the server is turned-off. If that net saving is less than the limiting cost of a set-up process, then turning-off the server is not beneficial.

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6. Asymptotic Optimality of the Turning-off and Idling Policies

First, we prove an asymptotic lower bound on the long-run average cost under any sequence of admissible policies. Then, we prove that depending on the comparison $B(\gamma_p)$ vs. c, either a sequence of turning-off policies or a sequence of idling policies achieves the aforementioned asymptotic lower bound, implying the asymptotic optimality of that sequence.

The following theorem states that the optimal BCP (6) objective function value is an asymptotic lower bound on the long-run average cost under any sequence of admissible policies.

THEOREM 2. *Let $\Sigma = 2\mu$ in the BCP (6). Let $\pi = \{\pi^n, n \in \mathbb{N}_+\}$ be an arbitrary sequence of admissible control policies. Then,*

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \hat{J}^{\pi,n}(t) \geq \gamma^*.$$

The proof of Theorem 2 is presented in the OA A.4.

The following theorem states that either a sequence of turning-off policies or a sequence of idling policies achieves the asymptotic lower bound presented in Theorem 2. Let $\lfloor \cdot \rfloor$ denote the floor operator.

THEOREM 3. *Let $\Sigma = 2\mu$ in the BCP (6).*

1. *Suppose that $B(\gamma_p) > c$ and thus $\gamma^* \in (\gamma_r, \gamma_p)$ by (18). Let $\pi = \{\pi^n, n \in \mathbb{N}_+\}$ be a sequence of turning-off policies (recall Definition 2) such that the parameters of the turning-off policy are $(\lfloor \sqrt{n} q_1(\gamma^*) \rfloor, \lfloor \sqrt{n} q_2(\gamma^*) \rfloor)$ in the nth system for all $n \in \mathbb{N}_+$. Then,*

$$\lim_{n\to\infty} \lim_{t\to\infty} \hat{J}^{\pi,n}(t) = \gamma^*.$$

2. *Suppose that $B(\gamma_p) \leq c$ and thus $\gamma^* = \gamma_p$ by (18). Let $\tilde{\pi} = \{\tilde{\pi}^n, n \in \mathbb{N}_+\}$ be a sequence of idling policies (recall Definition 2) such that the parameters of the idling policy are $(\lfloor \sqrt{n} q^* \rfloor - K, \lfloor \sqrt{n} q^* \rfloor)$ in the nth system for all $n \in \mathbb{N}_+$, where $K \in \mathbb{N}_+$ is an arbitrary constant. Then,*

$$\lim_{n\to\infty} \lim_{t\to\infty} \hat{J}^{\tilde{\pi},n}(t) = \gamma^*.$$

The proof of Theorem 3 is presented in the OA A.5. Theorems 2 and 3 imply the asymptotic optimality of the turning-off or idling policies depending on the comparison $B(\gamma_p)$ vs. c.

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7. Numerical Experiments

We consider a pre-limit system, where n is fixed, and test the performance of our proposed heuristic control policy numerically. The manufacturing and the set-up times are exponentially distributed in all experiments. We present numerical results where the inter-arrival times are exponentially and Erlang-2 distributed in Sections 7.1 and 7.2, respectively.

The type and the parameters of our heuristic control policy depend on the parameters of the BCP (6) and the latter parameters appear in the asymptotic regime (recall Theorem 3 and Assumptions 1 and 3). Because the asymptotic regime is an abstract notion and does not exist in real life, we approximate the parameters of the BCP (6). Specifically, we assume that the parameters $(\mu, \lambda^n, \tau, c_a, c_s, h, b, c^n, p^n, r^n)$ are known, but n is not necessarily known, and c_a and c_s are the coefficient of variations of the inter-arrival and manufacturing times, respectively. By considering Assumptions 1 and 3, we let

$$\theta = 1, \quad n = 1/\left(\mu - \lambda^n\right)^2, \quad \Sigma = \mu\left(c_a^2 + c_s^2\right), \tag{22a}$$

$$c = \frac{c^n}{n^{3/2}\tau}, \quad p = \frac{p^n}{n}, \quad r = \frac{r^n}{n}. \tag{22b}$$

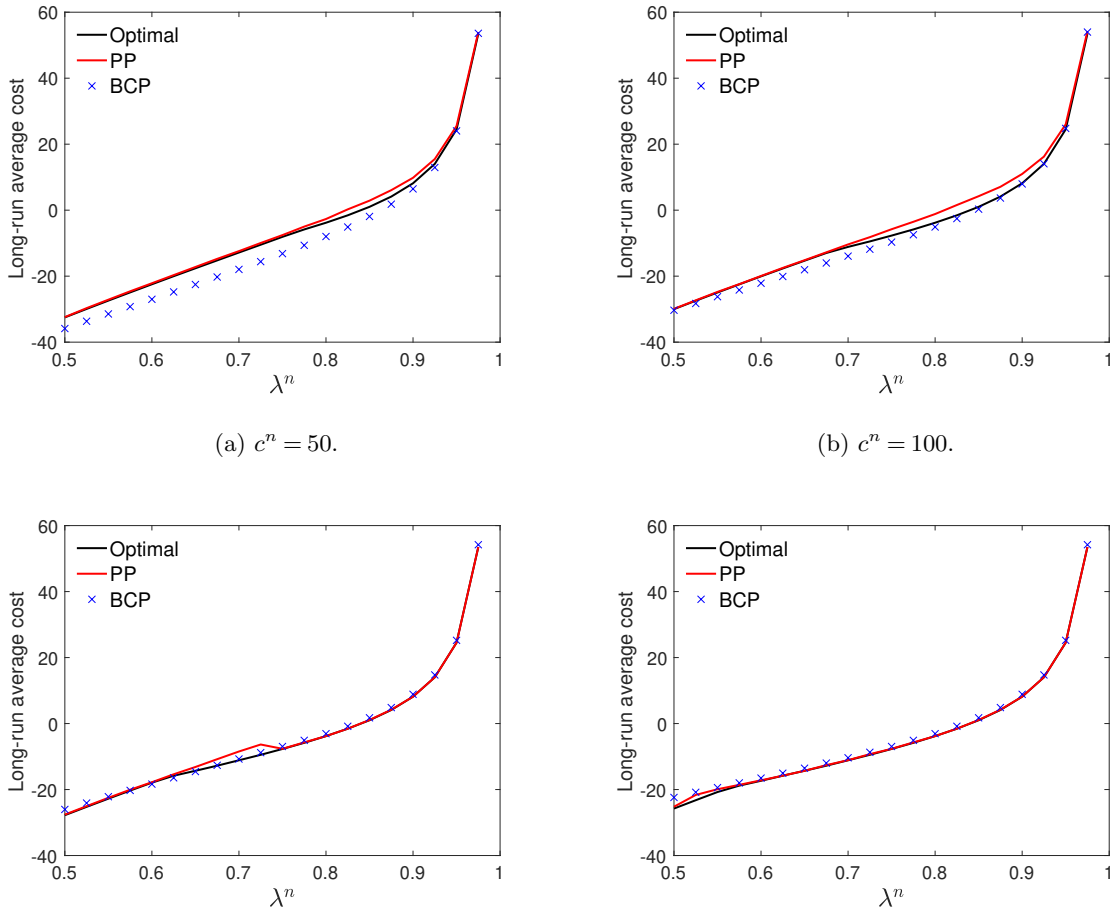
Then, we solve the BCP (6) under the parameters in (22). By considering Theorem 3, if $B(\gamma_p) > c$, then our proposed heuristic control policy is a turning-off policy with the parameters $(\lfloor \sqrt{n} q_1(\gamma^*) \rfloor, \lfloor \sqrt{n} q_2(\gamma^*) \rfloor)$. Otherwise, if $B(\gamma_p) \leq c$, then our proposed heuristic control policy is an idling policy with the parameters $(\lfloor \sqrt{n} q^* \rfloor - 1, \lfloor \sqrt{n} q^* \rfloor)$, where we choose $K = 1$ in the statement of Theorem 3 for simplicity.

By Theorem 1, we use $\sqrt{n}\gamma^*$ as a BCP approximation for the optimal long-run average cost. We compute the optimal long-run average cost by formulating the system as a Markov Decision Process (MDP) and then solving the MDP model by the linear programming (LP) approach (see Tan et al. (2022) for details about the LP model). The MDP model considers only the non-preemptive policies, that is, an ongoing manufacturing or set-up process cannot be interrupted. We consider stationary Markov policies in the MDP model.

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7.1. Experiments with exponentially distributed inter-arrival times

We let $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, $\lambda^n \in \{0.5, 0.525, 0.55, \ldots, 0.975\}$, and $c^n \in \{50, 100, 150, 200\}$. Therefore, we consider 80 different instances. Figure 3 presents the main results, where "PP" denotes the long-run average cost under our proposed heuristic control policy and "BCP" denotes the $\sqrt{n}\gamma^*$ value, that is, it is the BCP approximation for the optimal long-run average cost.



(a) $c^n = 50$. (b) $c^n = 100$.

(c) $c^n = 150$. (d) $c^n = 200$.

Figure 3 (Color online) Numerical results under exponentially distributed inter-arrival times where $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, $\lambda^n \in \{0.5, 0.525, 0.55, \ldots, 0.975\}$, and $c^n \in \{50, 100, 150, 200\}$.

According to the results, both the proposed policy and the BCP approximation for the optimal long-run average cost perform really well even when the system load (λ^n/μ) is small. The detailed

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numerical results are presented in the OA A.7.1. The optimal policy is either a turning-off or an idling policy in all instances. This is similar to what we see in the numerical experiments of Tan et al. (2022) done in Markovian systems. On the one hand, the optimal policy is a turning-off policy at 28 of the 80 instances and it is of turning-off type only when the system load is sufficiently small. On the other hand, the proposed policy is a turning-off policy at 51 of the 80 instances and those instances are mostly the ones in which $c^n \in \{50, 100\}$, that is, the instances in which the set-up cost is relatively small. Although the type of the proposed policy does not match with the type of the optimal policy at 27 of the 80 instances, its performance is close to optimal in all instances.

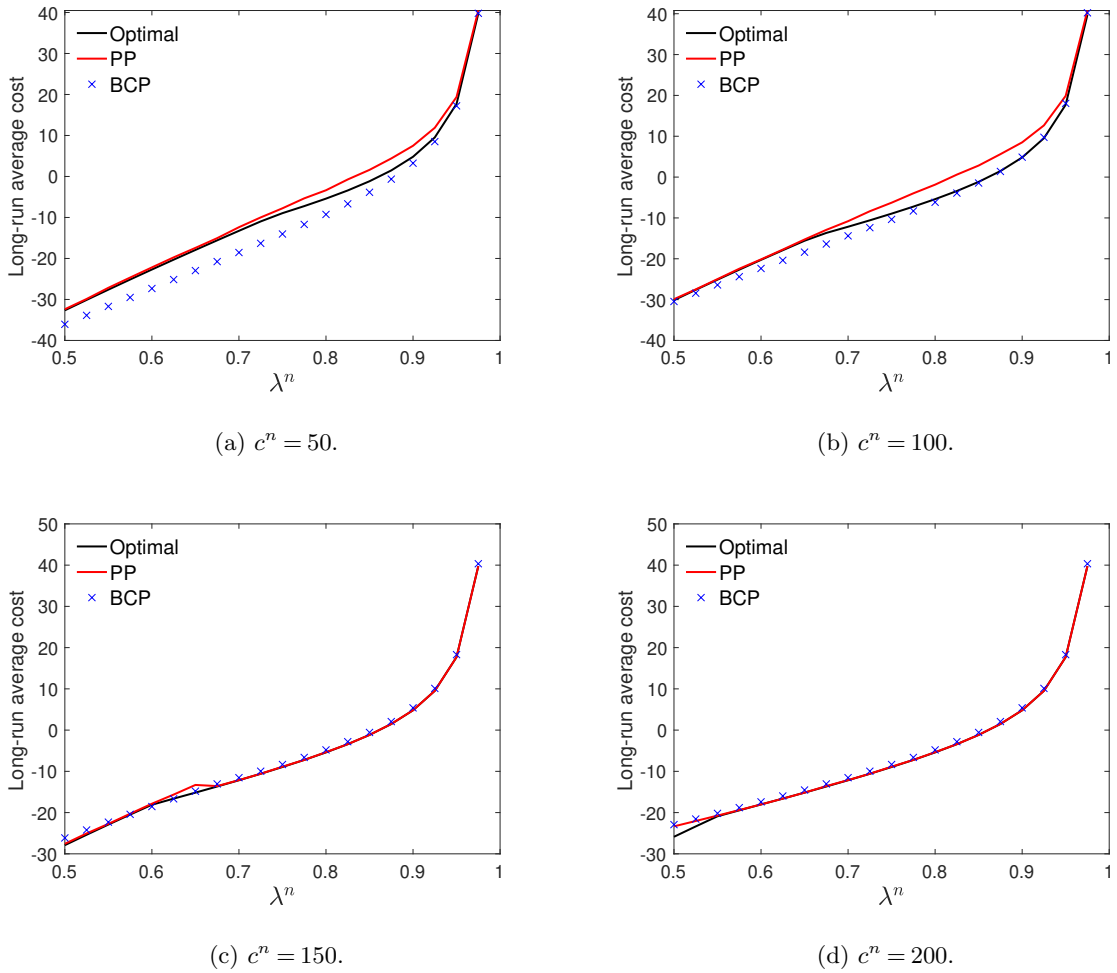
7.2. Experiments with Erlang-2 distributed inter-arrival times

We repeat the experiments in Section 7.1 with the following modification: the inter-arrival times have Erlang-2 distribution (with mean $1/\lambda^n$) rather than exponential distribution. By assuming that the system controller can observe the phase (the first or the second phase) of the next arrival, we formulate the system as an MDP and compute the optimal long-run average cost by solving the MDP. The structure of the optimal control policy under Erlang-2 distributed inter-arrival times is, in general, more complicated than the one under exponentially distributed inter-arrival times. First, under the optimal control policy, the threshold levels at which the server state changes may depend on the phase of the next arrival. For example, at instance 11, where $\lambda^n = 0.75$ and $c^n = 50$, the optimal control policy is of idling type such that the server idles if the inventory level reaches 3 while the next arrival is in phase 1 or if the inventory level reaches 4 while the next arrival is in phase 2. The detailed numerical results are presented in the OA A.7.2. Second, at 24 out of 80 instances, the optimal control policy is of hybrid type (recall Definition 2), where the server may idle after the set-up process is over. Figure 4 presents the main results, where we see that both the proposed policy and the BCP approximation for the optimal long-run average cost perform really well even under small system loads.

8. Future Research Directions

In practical manufacturing systems, there are often multiple machines (in series or in parallel) rather than a single machine. There are studies focusing only on energy consumption in serial production

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(a) $c^n = 50$. (b) $c^n = 100$.

(c) $c^n = 150$. (d) $c^n = 200$.

Figure 4 (Color online) Numerical results under Erlang-2 distributed inter-arrival times.

systems (Jia et al. (2016), Pei et al. (2022)). There are also studies focusing on energy consumption and inventory control jointly in serial production systems by using simulation-based techniques (Frigerio et al. (2023)). Therefore, a relevant future research topic is studying energy consumption and inventory control jointly in systems in the form of networks of machines. Significant energy saving can be done just by turning off the non-bottleneck machines at appropriate times.

An interesting future research direction is extending our single-item setting with a multi-item one. As explained in Section 2, that extension is not trivial because it is not possible to obtain a single-dimensional EWF of the multi-dimensional BCP by the conventional techniques. For interested readers, we present the multi-dimensional BCP in the OA A.6. A potential approach is to solve the BCP numerically and derive heuristic control policies from the optimal solution.

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Acknowledgments

The authors thank Oktay Karabağ for his help in the numerical experiments. This work was supported by TUBITAK (grant number 221M393) and European Unions Horizon 2020 Research and Innovation Programme [IN4ACT project under grant agreement No. 810318].

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ONLINE APPENDIX

This online appendix is associated with the manuscript titled "Asymptotically Optimal Energy Consumption and Inventory Control in a Make-to-Stock Manufacturing System". We present the proof of Lemma 1 in Section A.1. We present some additional results about the idling, turning-off, and hybrid policies in Section A.2. Then, we present the proofs related to the BCP (6), specifically the proofs of Lemmas 2 and 3 and Theorem 1, in Section A.3. We present the proofs of Theorems 2 and 3 in Sections A.4 and A.5, respectively. We discuss the challenges of analyzing a multi-class manufacturing system in Section A.6. Finally, we present detailed numerical results in Section A.7.

In the nth system, all the random variables and stochastic processes are defined in the complete probability space $(\Omega, \mathcal{F}^n(\infty), \mathbf{P})$ and $\mathbf{E}$ denotes expectation with respect to $\mathbf{P}$.

A.1. Proof of Lemma 1

We present the proofs of the first and second parts in Sections A.1.1 and A.1.2, respectively.

A.1.1. Proof of Part 1

Consider the idling policy with the parameters (q_1^n, q_2^n). Observe that the CTMC in Figure 1a is irreducible. We will show that the state (q_2^n, idle) is positive recurrent by proving that the expected time to reach that state starting from itself is finite. That average time has two components:

1. The server will be kept idle until the inventory level decreases to q_1^n, which takes on average $(q_2^n - q_1^n)/\lambda^n$ amount of time.
2. When the inventory level decreases to q_1^n, the server will produce nonstop until the inventory level reaches q_2^n. When the server is busy, the state evolution is very similar to the one of an $M/M/1$ queue. Recall that in an $M/M/1$ queue with arrival rate $\tilde{\lambda}$ and service rate $\tilde{\mu}$ such that $\tilde{\lambda} < \tilde{\mu}$, average time it takes to visit the state $k-1$ starting from the state k (that is, the mean first passage time from state k to stake $k-1$) is equal to $1/(\tilde{\mu} - \tilde{\lambda})$ for all $k \in \mathbb{N}_+$. Hence, starting from the inventory level q_1^n, expected time to reach the inventory level q_2^n is

$$\frac{q_2^n - q_1^n}{\mu - \lambda^n} = \frac{\sqrt{n}}{\theta}(q_2^n - q_1^n), \tag{A.1}$$

where the equality follows from Assumption 1.

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Therefore, expected time to reach the state (q_2^n, idle) starting from itself is equal to

$$(q_2^n - q_1^n)\left(\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}\right) < \infty, \tag{A.2}$$

implying that the state (q_2^n, idle) is positive recurrent. Because the CTMC is irreducible, all states are positive recurrent.

Next, let us consider the turning-off policy with the parameters (q_1^n, q_2^n). Observe that the CTMC in Figure 1b is irreducible. We will show that the state $(q_2^n, \text{turned-off})$ is positive recurrent by proving that the expected time to reach that state starting from itself is finite. That average time has three components:

1. The server will be kept turned off until the inventory level decreases to q_1^n, which takes on average $(q_2^n - q_1^n)/\lambda^n$ amount of time.
2. When the inventory level decreases to q_1^n, the server will be turned on and enter the set-up stage, which takes on average $1/\tau$ amount of time.
3. After the set-up stage, the server will produce nonstop until the inventory level reaches q_2^n. Let D^n be the random variable denoting the cumulative number of demands arriving in the set-up stage. Then,

$$\mathbf{P}(D^n = i) = \left(\frac{\lambda^n}{\tau + \lambda^n}\right)^i \frac{\tau}{\tau + \lambda^n}, \qquad \forall i \in \mathbb{N}, \tag{A.3}$$

and thus $\mathbf{E}[D^n] = \lambda^n/\tau$. Recall that when the server is busy, the state evolution is very similar to the one of an $M/M/1$ queue. Given that the inventory level decreases to $q_1^n - i$ for some $i \in \mathbb{N}$ at the end of the set-up stage, expected time to reach the inventory level q_2^n is

$$\frac{q_2^n - q_1^n + i}{\mu - \lambda^n} = \frac{\sqrt{n}}{\theta}(q_2^n - q_1^n + i),$$

where the equality follows from Assumption 1. Therefore, after the set-up process, expected time to reach the inventory level q_2^n is equal to

$$\sum_{i=0}^{\infty} \frac{\sqrt{n}}{\theta}(q_2^n - q_1^n + i)\,\mathbf{P}(D^n = i) = \frac{\sqrt{n}}{\theta}\left(q_2^n - q_1^n + \frac{\lambda^n}{\tau}\right). \tag{A.4}$$

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Consequently, expected time to reach the state $(q_2^n, \text{turned-off})$ starting from itself is equal to

$$\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_1^n + \frac{\lambda^n}{\tau}\right) < \infty, \tag{A.5}$$

implying that the state $(q_2^n, \text{turned-off})$ is positive recurrent. Because the CTMC is irreducible, all states are positive recurrent.

Finally, let us consider the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n). Observe that the CTMC in Figure 1c is irreducible. We will show that the state $(q_2^n, \text{turned-off})$ is positive recurrent by proving that the expected time to reach that state starting from itself is finite. That average time has three components:

1. The server will be kept turned off until the inventory level decreases to q_1^n, which takes on average $(q_2^n - q_1^n)/\lambda^n$ amount of time.
2. When the inventory level decreases to q_1^n, the server will be turned on and enter the set-up stage, which takes on average $1/\tau$ amount of time.
3. Recall that D^n denotes the cumulative number of demands arriving in the set-up stage (recall (A.3)). When the set-up is over,
   (a) if $D^n \leq q_1^n - q_0^n - 1$, then the server idles until the inventory level decreases to q_0^n. When the inventory level hits q_0^n, the server produces nonstop until the inventory level reaches q_2^n again.
   (b) If $D^n > q_1^n - q_0^n - 1$, the server produces nonstop until the inventory level reaches q_2^n.

Therefore, the expected time the server is not busy in a renewal cycle is equal to

$$\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \sum_{i=0}^{q_1^n - q_0^n - 1} \frac{q_1^n - i - q_0^n}{\lambda^n} \mathbf{P}(D^n = i). \tag{A.6}$$

We have the following trivial lower and upper bounds on the term in (A.6).

$$\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} \leq \frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \sum_{i=0}^{q_1^n - q_0^n - 1} \frac{q_1^n - i - q_0^n}{\lambda^n} \mathbf{P}(D^n = i) \leq \frac{q_2^n - q_0^n}{\lambda^n} + \frac{1}{\tau}. \tag{A.7}$$

Next, the expected time the server is busy in a renewal cycle is equal to

$$\sum_{i=0}^{q_1^n - q_0^n - 1} \frac{q_2^n - q_0^n}{\mu - \lambda^n} \mathbf{P}(D^n = i) + \sum_{i=q_1^n - q_0^n}^{\infty} \frac{q_2^n - q_1^n + i}{\mu - \lambda^n} \mathbf{P}(D^n = i)$$

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$$= \frac{\sqrt{n}}{\theta}\left(\sum_{i=0}^{q_1^n-q_0^n-1}(q_2^n-q_0^n)\,\mathbf{P}(D^n=i)+\sum_{i=q_1^n-q_0^n}^{\infty}(q_2^n-q_1^n+i)\,\mathbf{P}(D^n=i)\right)$$
$$= \frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n+\sum_{i=q_1^n-q_0^n}^{\infty}(q_0^n-q_1^n+i)\,\mathbf{P}(D^n=i)\right), \tag{A.8}$$

where the first equality follows from Assumption 1. We have the following trivial lower and upper bounds on the term in (A.8).

$$\frac{\sqrt{n}}{\theta}(q_2^n-q_0^n) \le \frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n+\sum_{i=q_1^n-q_0^n}^{\infty}(q_0^n-q_1^n+i)\,\mathbf{P}(D^n=i)\right) \tag{A.9}$$
$$\le \frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n+\sum_{i=0}^{\infty} i\mathbf{P}(D^n=i)\right)$$
$$= \frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n+\frac{\lambda^n}{\tau}\right). \tag{A.10}$$

By (A.7) and (A.9), expected time to return to the state $(q_2^n, \text{turned-off})$ starting from itself is greater than or equal to

$$\frac{q_2^n-q_1^n}{\lambda^n}+\frac{1}{\tau}+\frac{\sqrt{n}}{\theta}(q_2^n-q_0^n). \tag{A.11}$$

By (A.7) and (A.10), expected time to return to the state $(q_2^n, \text{turned-off})$ starting from itself is less than or equal to

$$\frac{q_2^n-q_0^n}{\lambda^n}+\frac{1}{\tau}+\frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n+\frac{\lambda^n}{\tau}\right)<\infty. \tag{A.12}$$

The upper bound in (A.12) implies that the state $(q_2^n, \text{turned-off})$ is positive recurrent. Because the CTMC is irreducible, all states are positive recurrent.

A.1.2. Proof of Part 2

The following lemma reduces the set of admissible policies without sacrificing the optimal performance and will be helpful in the proof of the second part of Lemma 1.

LEMMA A.1.1. *For all $n \in \mathbb{N}_+$, we can exclude the following class of policies from the set of admissible policies without loss of optimality:*

1. *The admissible policies under which the server is turned off when the server is not busy.*
2. *The admissible policies under which the server is turned off at more than one inventory levels.*

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3. *The admissible policies under which there are transitions from a state with busy server to a state with idle server at more than one inventory levels.*
4. *The admissible policies under which the server is turned off and there is a transition from a state with busy server to a state with idle server.*

Proof. The inventory level increases only when the server is busy and decreases when the server is idle, turned off, or in set-up. Under an admissible policy, after the server is turned off, it must be turned on at some point in the future. Otherwise, the inventory level will decrease to negative infinity and thus the state process cannot have a stationary distribution.

Part 1. By (1f), the server must be busy or idle just before it is turned off. Let π^n be an admissible policy under which the server becomes idle at the inventory level $q_3^n \in \mathbb{Z}$ and is turned off when the inventory level decreases to $q_2^n \in \mathbb{Z}$ while the server is idle. Then, $q_3^n > q_2^n$. Let $q_1^n \in \mathbb{Z}$ denote the inventory level at which the server is turned on again and enters the set-up stage (and thus $q_2^n > q_1^n$). Before the server becomes idle at the inventory level q_3^n, it was either busy or must have completed a set-up process. Therefore, we will consider two cases.

Suppose that the server was busy just before it becomes idle at the inventory level q_3^n. Consider the admissible policy denoted by $\tilde{\pi}^n$ which takes the same actions as π^n does except that the server is turned off at the inventory level q_3^n instead of q_2^n. Then, during the time interval on which the inventory level decreases from q_3^n to q_2^n, the server is turned off under $\tilde{\pi}^n$ and is idle under π^n. Under both policies, the server is turned on again when the inventory level decreases to q_1^n. Because the power saving under the turned-off state is more than the one under the idle state (recall Assumption 3), the cost under $\tilde{\pi}^n$ is less than the one under π^n under all sample paths.

Suppose that the server completed a set-up process just before it becomes idle at the inventory level q_3^n. Let q_4^n denote the inventory level at which the server is turned on again and enters the set-up stage just before the inventory level decreases to q_3^n. Then, $q_4^n > q_3^n$. Consider the admissible policy denoted by $\tilde{\pi}^n$ which takes the same actions as π^n does except that the server is kept turned off at the inventory level q_4^n and is turned on when the inventory level decreases to q_1^n. Then, during

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the time interval when the inventory level decreases from q_4^n to q_2^n, the server is turned off under $\tilde{\pi}^n$. However, under π^n, the server is in set-up stage when the inventory level decreases from q_4^n to q_3^n and is idle when the inventory level decreases from q_3^n to q_2^n. Because the power saving under the turned-off state is greater than the one under the idle state and the set-up stage is costly, the cost under $\tilde{\pi}^n$ is less than the one under π^n under all sample paths. Under $\tilde{\pi}^n$, if the server is turned off when the server is not busy, by the arguments above one can construct another admissible control policy with lower cost under all sample paths.

Consequently, for any admissible policy under which the server is turned off when the server is not busy, there exists another admissible policy under which the server is turned off only when the server is busy and which has a lower cost under all sample paths.

Part 2. Let π^n be an admissible policy under which the server is turned off at two different inventory levels. By the first part of this lemma, we can assume that the server is turned off only when it is busy. Let the server is turned off at the inventory levels $a^n \in \mathbb{Z}$ and $b^n \in \mathbb{Z}$ such that $a^n < b^n$. Without loss of generality, we can assume that the states $(a^n, \text{turned off})$ and $(b^n, \text{turned off})$ are visited infinitely many times. Otherwise, if one of those states is visited finitely many times, then that state becomes transient, we can ignore it, and thus the desired result trivially follows. After the first time the inventory level reaches a_n, the server will be turned off and the inventory level will decrease. Because the inventory level increases only when the server is busy and the server is always turned off when it reaches a_n, the inventory level will neither exceed a^n nor rise to b^n anymore. Therefore, the state $(b^n, \text{turned off})$ must be transient, which is a contradiction.

Part 3. Let π^n be an admissible policy under which the busy server is idled when the inventory level reaches a^n or b^n such that $a^n, b^n \in \mathbb{Z}$ and $a^n < b^n$. Without loss of generality, we can assume that the states (a^n, idle) and (b^n, idle) are visited infinitely many times. Otherwise, if one of those states is visited finitely many times, then that state becomes transient, we can ignore it, and thus the desired result trivially follows. After the first time the system state becomes (a^n, idle), the production will stop and thus the inventory level will never reach b_n again. Therefore, the state (b^n, idle) becomes transient, which is a contradiction.

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Part 4. Let π^n be an admissible policy under which the server is turned off when the inventory level reaches $a^n \in \mathbb{Z}$ and the busy server is idled when the inventory level reaches $b^n \in \mathbb{Z}$. By the first part of this lemma, we can assume that the server is turned off only when it is busy. By Definition 1, $a^n \neq b^n$. The rest of the proof is similar to the ones of the second and the third parts of this lemma. First, suppose that $a^n < b^n$. After the inventory level reaches a^n, the server will be turned off and thus the inventory level will never reach b^n. Therefore, the state (b^n, idle) becomes transient, and we can ignore it. Similarly, if $a^n > b^n$, after the first time the inventory level reaches b^n, the production will stop and thus the inventory level will neither exceed b^n nor reach a^n. Therefore, the state $(a^n, \text{turned off})$ becomes transient, and we can ignore it. □

Next, we will prove the second part of Lemma 1 by Lemma A.1.1. First, let π^n be an admissible policy under which the server is never turned off. The server must be idled at some states, otherwise both the inventory level and the long-run average cost will tend to infinity. By the third part of Lemma A.1.1, there exists an admissible policy of idling type with less than or equal long-run average cost than π^n has. Therefore, if the server is never turned off, we can focus on idling policies without sacrificing the optimal performance.

Second, let π^n be an admissible policy under which the server is never idled. The server must be turned-off at some states, otherwise both the inventory level and the long-run average cost will tend to infinity. By the second part of Lemma A.1.1, there exists an admissible policy of turning-off type with less than or equal long-run average cost than π^n has. Therefore, if the server is never idled, we can focus on turning off policies without sacrificing the optimal performance.

Third, let π^n be an admissible policy under which the server is turned off at some states and idled at some other states. By Definition 1 and Lemma A.1.1, there exists an admissible policy of idling, turning-off, or hybrid type with less than or equal long-run average cost than π^n has. Therefore, we can focus on idling, turning-off, and hybrid policies without loss of optimality.

A.2. Additional Results about the Idling, Turning-off, and Hybrid Policies

We represent the state process as a regenerative process and present the induced renewal process in Section A.2.1. Then, we present a result about the busy time process in Section A.2.2. We

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present some auxiliary results in Section A.2.3. We present necessary growth conditions for the policy parameters in order to have a finite asymptotic cost in Section A.2.4.

A.2.1. Regenerative process representation and the induced renewal process

We focus on idling, hybrid, and turning-off policies by Lemma 1. Recall that the state process $\{\mathfrak{X}^n(t), t \in \mathbb{R}_+\}$ starts from the induced limiting distribution (see Assumption 2) and thus it is a stationary process. Observe that the state process is also a regenerative process (see Sigman and Wolff (1993)). Specifically, by recalling Figure 1, let us consider the following special states for all $n \in \mathbb{N}_+$ and $(q_0^n, q_1^n, q_2^n) \in \mathbb{Z}^3$ such that $q_0^n < q_1^n < q_2^n$.

- Under the idling policy with the parameters (q_1^n, q_2^n), consider the state (q_2^n, idle).
- Under the turning-off policy with the parameters (q_1^n, q_2^n), consider the state $(q_2^n, \text{turned-off})$.
- Under the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n), consider the state $(q_2^n, \text{turned-off})$.

The idling, hybrid, and turning-off policies regenerate themselves after they reach the states mentioned above. Furthermore, the idling, hybrid, and turning-off policies induce *delayed renewal processes* where the renewal cycles are defined as the time it takes for the state process to reach the special state defined above starting from the special state (see chapter 3.5 of Ross (1996)). On the one hand, because the state process does not start from the special state with probability one (recall Assumption 2), the induced renewal processes are delayed. On the other hand, because the system state is a stationary process, the induced delayed renewal process is actually the *equilibrium renewal process* (see page 131 of Ross (1996)).

For notational convenience, we call the first (and the only non-i.i.d) renewal cycle as the zeroth cycle and thus the first i.i.d. renewal cycle is called the first renewal cycle. We let N^n denote the associated counting process for the delayed renewal process, that is, $N^n(t)$ denotes the cumulative number of renewal cycles completed up to time t for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. We let L_k^n denote the length of the kth renewal cycle for all $i \in \mathbb{N}$. Clearly, $\{L_k^n, k \in \mathbb{N}_+\}$ is an i.i.d. sequence of random variables. Furthermore, under the idling and the turning-off policies, $\mathbf{E}[L_1^n]$ is given in (A.2) and (A.5), respectively. Under the hybrid policy, a lower bound and an upper bound on $\mathbf{E}[L_1^n]$ are given

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in (A.11) and (A.12), respectively. Because the induced delayed renewal process is the equilibrium renewal process, the distribution of the zeroth renewal cycle is equal to the excess distribution of the first renewal cycle (see page 131 and theorem 3.5.2 of Ross (1996)) and thus

$$\mathbf{P}\left(L_0^n \leq x\right) = \frac{1}{\mathbf{E}\left[L_1^n\right]} \int_0^x \mathbf{P}\left(L_1^n > y\right) \mathrm{d}y, \qquad \forall x \in \mathbb{R}_+,\, n \in \mathbb{N}_+. \tag{A.13}$$

The following lemma provides an exponentially decaying tail bound on the length of the renewal cycles.

LEMMA A.2.1. *Fix an arbitrary $n \in \mathbb{N}_+$ and consider an arbitrary idling, turning-off, or hybrid policy. There exists a pair $(C_{11}^n, C_{12}^n) \in \mathbb{R}_{++}^2$ such that*

$$\mathbf{P}\left(L_0^n > x\right) \vee \mathbf{P}\left(L_1^n > x\right) \leq C_{11}^n \mathrm{e}^{-C_{12}^n x}, \qquad \forall x \in \mathbb{R}_+.$$

Consequently, all moments of L_0^n and L_1^n exist for all $n \in \mathbb{N}_+$.

Proof. Fix an arbitrary $n \in \mathbb{N}_+$. For all $m \in \mathbb{Z}$, let $T_{m(m+1)}^n$ denote the time it takes for the inventory level to reach $m+1$ starting from the inventory level m when the server is busy. Observe that $\left\{T_{m(m+1)}^n, m \in \mathbb{Z}\right\}$ is an i.i.d. sequence of random variables. Let f_{busy}^n denote the probability density function of T_{01}^n. By the busy-period analysis of an $M/M/1$ queue (see corollary III.8.7 of Asmussen (2003)),

$$f_{\text{busy}}^n(x) = \frac{1}{x\sqrt{\rho^n}} \mathrm{e}^{-(\lambda^n+\mu)x} I_1(2x\sqrt{\lambda^n \mu}), \qquad \forall x \in \mathbb{R}_{++},$$

where $I_1(\cdot)$ is a modified Bessel function of the first kind. By algebra, for all $x \in \left[0, \left(\sqrt{\mu} - \sqrt{\lambda^n}\right)^2\right)$,

$$\mathbf{E}\left[\mathrm{e}^{xT_{01}^n}\right] = \int_0^\infty \mathrm{e}^{xu} f_{\text{busy}}^n(u) \mathrm{d}u = \frac{1}{2\lambda^n}\left(\lambda^n + \mu - x - \sqrt{\left(\lambda^n + \mu - x\right)^2 - 4\lambda^n \mu}\right) < \infty, \tag{A.14}$$

implying that the moment generating function (MGF) of T_{01}^n exists.

Next, we will prove that the MGF of L_1^n exists. First, let us consider the idling policy with the parameters (q_1^n, q_2^n). Let T_{idle}^n denote the length of the time interval on which the server idles in the first renewal cycle. Then, T_{idle}^n has Erlang distribution with the parameters $q_2^n - q_1^n$ and λ^n. Recall that if E has Erlang distribution with the parameters $k \in \mathbb{N}_+$ and $\alpha \in \mathbb{R}_{++}$, then

$$\mathbf{E}\left[e^{xE}\right] = \left(1 - \frac{x}{\alpha}\right)^{-k}, \qquad \forall x \in [0, \alpha). \tag{A.15}$$

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Therefore, for all $x \in \left[0, \left(\sqrt{\mu} - \sqrt{\lambda^n}\right)^2 \wedge \lambda^n\right)$,

$$\begin{aligned}\mathbf{E}\left[\mathrm{e}^{xL_1^n}\right] &= \mathbf{E}\left[\exp\left\{xT_{\text{idle}}^n + x\sum_{m=0}^{q_2^n-q_1^n-1} T_{(q_1^n+m)(q_1^n+m+1)}^n\right\}\right] \\ &= \mathbf{E}\left[\mathrm{e}^{xT_{\text{idle}}^n}\right]\mathbf{E}\left[\mathrm{e}^{xT_{01}^n}\right]^{q_2^n-q_1^n} \\ &< \infty,\end{aligned}$$

where the second equality follows from the fact that the lengths of the idle and busy periods in the first renewal cycle are mutually independent of each other due to the Markov property and the strict inequality follows from (A.14) and (A.15). Consequently, under the idling policy, the MGF of L_1^n exists.

Next, let us consider the turning-off policy with the parameters (q_1^n, q_2^n). Let T_{off}^n denote the length of the time interval on which the server is turned off in the first renewal cycle. Then, T_{off}^n has Erlang distribution with the parameters $q_2^n - q_1^n$ and λ^n. Recall from (A.3) that D^n denotes the cumulative number of demands arrived in a set-up process. Recall that κ_j denotes the length of the jth set-up process, and thus $\kappa_j \sim \exp(\tau)$ for all $j \in \mathbb{N}_+$. By algebra, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}_+$,

$$\left(\kappa_j \middle| D^n = i\right) \sim \text{Erlang distribution with the parameters } i+1 \text{ and } \tau + \lambda^n. \tag{A.16}$$

For all $x \in [0, \tau + \lambda^n)$, let

$$C_{13}^n(x) := \left(1 - \frac{x}{\lambda^n}\right)^{-(q_2^n - q_1^n)} \frac{\tau}{\tau + \lambda^n - x}\left(\frac{\lambda^n + \mu - x - \sqrt{(\lambda^n + \mu - x)^2 - 4\lambda^n\mu}}{2\lambda^n}\right)^{q_2^n - q_1^n}. \tag{A.17}$$

Observe that $C_{13}^n(x)$ is finite and well-defined for all $x \in \left[0, \left(\sqrt{\mu} - \sqrt{\lambda^n}\right)^2 \wedge \lambda^n\right)$. Then,

$$\begin{aligned}\mathbf{E}\left[\mathrm{e}^{xL_1^n}\right] &= \sum_{i=0}^{\infty} \mathbf{E}\left[\mathrm{e}^{xL_1^n}\middle| D^n = i\right]\mathbf{P}\left(D^n = i\right) \\ &= \sum_{i=0}^{\infty} \mathbf{E}\left[\exp\left\{xT_{\text{off}}^n + x\kappa_1 + x\sum_{m=0}^{q_2^n-q_1^n+i-1} T_{(q_1^n-i+m)(q_1^n-i+m+1)}^n\right\}\middle| D^n = i\right]\mathbf{P}\left(D^n = i\right) \\ &= \sum_{i=0}^{\infty} \mathbf{E}\left[\mathrm{e}^{xT_{\text{off}}^n}\right]\mathbf{E}\left[\mathrm{e}^{x\kappa_1}\middle| D^n = i\right]\mathbf{E}\left[\mathrm{e}^{xT_{01}^n}\right]^{q_2^n-q_1^n+i}\mathbf{P}\left(D^n = i\right)\end{aligned} \tag{A.18}$$

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$$= \mathbf{E}\left[\mathrm{e}^{xT^n_{\mathrm{off}}}\right] \sum_{i=0}^{\infty} \left(1 - \frac{x}{\tau+\lambda^n}\right)^{-(i+1)} \left(\frac{\lambda^n+\mu-x-\sqrt{(\lambda^n+\mu-x)^2-4\lambda^n\mu}}{2\lambda^n}\right)^{q_2^n-q_1^n+i} \times \mathbf{P}\left(D^n=i\right)$$

$$= C^n_{13}(x) \sum_{i=0}^{\infty} \left(\frac{\lambda^n+\mu-x-\sqrt{(\lambda^n+\mu-x)^2-4\lambda^n\mu}}{2\left(\tau+\lambda^n-x\right)}\right)^{i}, \tag{A.19}$$

where the third equality follows from the fact that conditional on the event $\{D^n=i\}$, the random variables T^n_{off}, κ_1, and $T^n_{m(m+1)}$ for all $m\in\mathbb{N}_+$ are mutually independent of each other due to the Markov property; the fourth equality follows from (A.14), (A.15), and (A.16); and the last equality follows from (A.3) and (A.17). Let

$$P^n(x) := \frac{\lambda^n+\mu-x-\sqrt{(\lambda^n+\mu-x)^2-4\lambda^n\mu}}{2\left(\tau+\lambda^n-x\right)}, \qquad \forall x\in\left[0,\left(\sqrt{\mu}-\sqrt{\lambda^n}\right)^2\wedge\lambda^n\right).$$

Observe that P^n is continuous in x and $P^n(0)=\lambda^n/(\tau+\lambda^n)<1$. Therefore, there exists an $\epsilon^n>0$ such that $P^n(x)<1$ for all $x\in[0,\epsilon^n)$. Hence, for all $x\in\left[0,\left(\sqrt{\mu}-\sqrt{\lambda^n}\right)^2\wedge\lambda^n\wedge\epsilon^n\right)$, the sum in (A.19) becomes a convergent geometric series. Consequently, under the turning-off policy, the MGF of L^n_1 exists.

Let us consider the hybrid policy with the parameters (q^n_0,q^n_1,q^n_2). For all $i\in\{0,1,\ldots,q^n_1-q^n_0-1\}$, we have $\left(T^n_{\mathrm{idle}}\middle|D^n=i\right)$ has Erlang distribution with the parameters $q^n_1-q^n_0-i$ and λ^n. Therefore, we have

$$\begin{aligned}
\mathbf{E}\left[\mathrm{e}^{xL^n_1}\right] &= \sum_{i=0}^{\infty}\mathbf{E}\left[\mathrm{e}^{xL^n_1}\middle|D^n=i\right]\mathbf{P}\left(D^n=i\right)\\
&= \sum_{i=0}^{q^n_1-q^n_0-1}\mathbf{E}\left[\exp\left\{xT^n_{\mathrm{off}}+x\kappa_1+xT^n_{\mathrm{idle}}+x\sum_{m=0}^{q^n_2-q^n_0-1}T^n_{(q^n_0+m)(q^n_0+m+1)}\right\}\middle|D^n=i\right]\mathbf{P}\left(D^n=i\right)\\
&\quad+\sum_{i=q^n_1-q^n_0}^{\infty}\mathbf{E}\left[\exp\left\{xT^n_{\mathrm{off}}+x\kappa_1+x\sum_{m=0}^{q^n_2-q^n_1+i-1}T^n_{(q^n_1-i+m)(q^n_1-i+m+1)}\right\}\middle|D^n=i\right]\mathbf{P}\left(D^n=i\right)\\
&= \sum_{i=0}^{q^n_1-q^n_0-1}\mathbf{E}\left[\mathrm{e}^{xT^n_{\mathrm{off}}}\right]\mathbf{E}\left[\mathrm{e}^{x\kappa_1}\middle|D^n=i\right]\mathbf{E}\left[\mathrm{e}^{xT^n_{\mathrm{idle}}}\middle|D^n=i\right]\mathbf{E}\left[\mathrm{e}^{xT^n_{01}}\right]^{q^n_2-q^n_0}\mathbf{P}\left(D^n=i\right) \qquad \text{(A.20)}\\
&\quad+\sum_{i=q^n_1-q^n_0}^{\infty}\mathbf{E}\left[\mathrm{e}^{xT^n_{\mathrm{off}}}\right]\mathbf{E}\left[\mathrm{e}^{x\kappa_1}\middle|D^n=i\right]\mathbf{E}\left[\mathrm{e}^{xT^n_{01}}\right]^{q^n_2-q^n_1+i}\mathbf{P}\left(D^n=i\right), \qquad \text{(A.21)}
\end{aligned}$$

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where the third equality follows from the fact that conditional on the event $\{D^n = i\}$, the random variables T^n_{off}, κ_1, T^n_{idle}, and $T^n_{m(m+1)}$ for all $m \in \mathbb{Z}$ are mutually independent of each other due to the Markov property. For all $x \in \left[0, \left(\sqrt{\mu} - \sqrt{\lambda^n}\right)^2 \wedge \lambda^n\right)$, the sum in (A.20) is finite by (A.14), (A.15), and the fact that finitely many terms are summed. The sum in (A.21) is less than or equal to the sum in (A.18) and thus it is finite for all $x \in \left[0, \left(\sqrt{\mu} - \sqrt{\lambda^n}\right)^2 \wedge \lambda^n \wedge \epsilon^n\right)$. Consequently, under the hybrid policy, the MGF of L^n_1 exists.

Because the MGF of L^n_1 exists, there exists a pair $(C^n_{13}, C^n_{14}) \in \mathbb{R}^2_{++}$ such that

$$\mathbf{P}\left(L^n_1 > x\right) \leq C^n_{13} \mathrm{e}^{-C^n_{14} x}, \qquad \forall x \in \mathbb{R}_+,$$

see page 186 of Resnick (2005) for a proof. Finally, by (A.13), for all $x \in \mathbb{R}_+$,

$$\mathbf{P}\left(L^n_0 > x\right) = \frac{1}{\mathbf{E}\left[L^n_1\right]} \int_x^\infty \mathbf{P}\left(L^n_1 > y\right) \mathrm{d}y \leq \frac{1}{\mathbf{E}\left[L^n_1\right]} \int_x^\infty C^n_{13} \mathrm{e}^{-C^n_{14} y} \mathrm{d}y = \frac{C^n_{13}}{\mathbf{E}\left[L^n_1\right] C^n_{14}} \mathrm{e}^{-C^n_{14} x},$$

which completes the proof. □

A.2.2. Busy time analysis

The following lemma presents the convergence of the fluid-scaled busy time process.

LEMMA A.2.2. *Let $\pi = \{\pi^n, n \in \mathbb{N}_+\}$ be an arbitrary sequence of admissible policies such that each policy is of idling, turning-off, or hybrid type. Then,*

$$\lim_{n \to \infty} \lim_{t \to \infty} \frac{1}{t} \mathbf{E}\left[\bar{T}^{\pi,n}(t)\right] = 1,$$

Proof. We will use the renewal-reward theorem (see Section A.2.1 for the definition of the renewal cycles). For a reference, see theorem 3.6.1 of Ross (1996) and the remark in page 135 of Ross (1996) for an extension of the theorem statement to reward being earned gradually during the renewal cycle. Theorem 3.6.1 of Ross (1996) continues to hold for delayed renewal processes if both the expected length of the zeroth cycle and the expected reward earned in the zeroth cycle are finite.

We will omit π from the notation for convenience. Fix an arbitrary $n \in \mathbb{N}_+$. Consider the idling policy with the parameters (q^n_1, q^n_2). Expected time the server is busy in the zeroth renewal cycle

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is less than or equal to $\mathbf{E}[L_0^n]$, which is finite by Lemma A.2.1. Expected time the server is busy in the first renewal cycle is given in (A.1) and the expected length of the first renewal cycle is given in (A.2). By the renewal-reward theorem,

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[T^n(t)\right]=\frac{\frac{\sqrt{n}}{\theta}}{\frac{1}{\lambda^n}+\frac{\sqrt{n}}{\theta}}. \tag{A.22}$$

Next, let us consider the turning-off policy with the parameters (q_1^n, q_2^n). Expected time the server is busy in the zeroth renewal cycle is finite by Lemma A.2.1. Expected time the server is busy in the first renewal cycle is given in (A.4) and the expected length of the first renewal cycle is given in (A.5). Therefore, by the renewal-reward theorem,

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[T^n(t)\right]=\frac{\frac{\sqrt{n}}{\theta}\left(q_2^n-q_1^n+\frac{\lambda^n}{\tau}\right)}{\frac{q_2^n-q_1^n}{\lambda^n}+\frac{1}{\tau}+\frac{\sqrt{n}}{\theta}\left(q_2^n-q_1^n+\frac{\lambda^n}{\tau}\right)}. \tag{A.23}$$

Lastly, let us consider the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n). Expected time the server is busy in the zeroth renewal cycle is finite by Lemma A.2.1. An upper bound on the expected time the server is not busy in the first renewal cycle is given in the RHS of (A.7). A lower bound on the expected time the server is busy in the first renewal cycle is given in the left-hand-side (LHS) of (A.9). Furthermore, an upper bound on $\mathbf{E}[L_1^n]$ is given in (A.12). Because $T^n(t)\le t$ for all $t\in\mathbb{R}_+$ and by the renewal-reward theorem,

$$\frac{\frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n\right)}{\frac{q_2^n-q_0^n}{\lambda^n}+\frac{1}{\tau}+\frac{\sqrt{n}}{\theta}\left(q_2^n-q_0^n\right)}\le\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[T^n(t)\right]\le 1. \tag{A.24}$$

The desired convergence result follows from (A.22), (A.23), (A.24), the fact that $q_2^n-q_1^n\ge 1$, $q_2^n-q_0^n\ge 2$, and $\lim_{t\to\infty}\mathbf{E}\left[\bar{T}^n(t)\right]/t=\lim_{t\to\infty}\mathbf{E}\left[T^n(t)\right]/t$ for all $n\in\mathbb{N}_+$. □

A.2.3. Auxiliary results about the idling, turning-off, and hybrid policies

On the one hand, by Definition 2 (and see Figure 1), for a fixed $n\in\mathbb{N}_+$, under the idling, turning-off, and hybrid policies, the inventory level never exceeds q_2^n and thus it is bounded above. On the other hand, the inventory level is not bounded below. Recall that in Section A.2.1, we show that under the idling, turning-off, and hybrid policies, the state process is a regenerative process

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and we present the induced equilibrium renewal process. Let M_k^n be the random variable denoting the minimum inventory level in the kth renewal cycle for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$. Observe that $\{M_k^n, k \in \mathbb{N}_+\}$ is an i.i.d. sequence of random variables. The following auxiliary result states that all moments of M_1^n exist for all sufficiently large n.

LEMMA A.2.3. *There exists an $n_0 \in \mathbb{N}_+$ such that n_0 depends only on (θ, τ) and if $n \geq n_0$, then $\mathbf{E}\left[|M_1^n|^x\right] < \infty$ for all $x \in \mathbb{N}_+$.*

Proof. Let $n_0 = n_0(\theta, \tau) \in \mathbb{N}_+$ be such that if $n \geq n_0$, then $\tau > \theta/\sqrt{n}$. Because $\tau > 0$, n_0 exists. Under the idling, turning-off, and hybrid policies (recall Definition 2), when the server is busy, the state evolution is equivalent to the one of a random walk with upward jump rate μ and downward jump rate λ^n. By equation (4.4) of Seneta (1967), the conditional distribution of M_1^n with respect to the inventory level at which the server becomes busy, denoted by $IB^n \in \mathbb{Z}$, is as follows.

$$\mathbf{P}\left(M_1^n \geq k \mid IB^n = i\right) = \frac{\sum_{j=q_2^n-i}^{q_2^n-k} \left(\frac{1}{\rho^n}\right)^j}{\sum_{j=0}^{q_2^n-k} \left(\frac{1}{\rho^n}\right)^j}, \qquad \forall k \leq i < q_2^n.$$

Therefore, by algebra,

$$\mathbf{P}\left(M_1^n = k \mid IB^n = i\right) = \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k} \left(\frac{1}{\rho^n} - 1\right) \left(\left(\frac{1}{\rho^n}\right)^{q_2^n-i} - 1\right)}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k} - 1\right) \left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k+1} - 1\right)}, \qquad \forall k \leq i < q_2^n.$$

For all $x, n \in \mathbb{N}_+$ and $i \in \mathbb{Z}$ such that $i < q_2^n$, we have

$$\begin{aligned}
\mathbf{E}\left[|M_1^n|^x \mid IB^n = i\right] &= \sum_{k \leq i} |k|^x \mathbf{P}\left(M_1^n = k \mid IB^n = i\right) \\
&= \sum_{k \leq i} |k|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k} \left(\frac{1}{\rho^n} - 1\right) \left(\left(\frac{1}{\rho^n}\right)^{q_2^n-i} - 1\right)}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k} - 1\right) \left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k+1} - 1\right)} \\
&\leq \left(\frac{1}{\rho^n}\right)^{q_2^n-i+1} \sum_{k \leq i} |k|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k}}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k} - 1\right)^2} \\
&\leq \left(\frac{1}{\rho^n}\right)^{q_2^n-i+1} \sum_{k < q_2^n} |k|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k}}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k} - 1\right)^2},
\end{aligned} \tag{A.25}$$

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where the first inequality follows from algebra and the last inequality follows from the fact that $i < q_2^n$. We will use the ratio test (see chapter 9.2.4 of Bartle and Sherbert (2011)) to show that the sum in (A.25) is finite. Specifically,

$$\frac{|k-1|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k+1}}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k+1}-1\right)^2}}{|k|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k}}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k}-1\right)^2}} = \left|1-\frac{1}{k}\right|^x \frac{1}{\rho^n}\left(\frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k}-1}{\left(\frac{1}{\rho^n}\right)^{q_2^n-k+1}-1}\right)^2 \to \rho^n \quad \text{as } k \to -\infty.$$

Because $\rho^n < 1$ for all $n \in \mathbb{N}_+$ and by corollary 9.2.5 of Bartle and Sherbert (2011), we have

$$\xi(x,n) := \sum_{k<q_2^n} |k|^x \frac{\left(\frac{1}{\rho^n}\right)^{q_2^n-k}}{\left(\left(\frac{1}{\rho^n}\right)^{q_2^n-k}-1\right)^2} < \infty, \qquad \forall x, n \in \mathbb{N}_+. \tag{A.26}$$

By (A.25) and (A.26), for all $x, n \in \mathbb{N}_+$ and $i \in \mathbb{Z}$ such that $i < q_2^n$,

$$\mathbf{E}\left[|M_1^n|^x \mid IB^n = i\right] \le \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-i+1}. \tag{A.27}$$

Next, we will consider the idling, turning-off, and hybrid policies one by one.

The idling policy. Consider the idling policy with the parameters (q_1^n, q_2^n) (recall Definition 2). In the first renewal cycle, the server becomes busy as soon as the inventory level decreases to q_1^n, and thus $IB^n = q_1^n$. By (A.26) and (A.27),

$$\mathbf{E}\left[|M_1^n|^x\right] = \mathbf{E}\left[|M_1^n|^x \mid IB^n = q_1^n\right] \le \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1} < \infty, \qquad \forall x, n \in \mathbb{N}_+.$$

The turning-off policy. Consider the turning-off policy with the parameters (q_1^n, q_2^n) (recall Definition 2). In the first renewal cycle, as soon as the inventory level decreases to q_1^n, the server is turned on and enters the set-up process. Recall that D^n is the random variable denoting the cumulative number of demands arriving in the set-up process. Then, $IB^n = q_1^n - D^n$ for all $n \in \mathbb{N}_+$. If $n \ge n_0$, for all $x \in \mathbb{N}_+$,

$$\begin{aligned}\mathbf{E}\left[|M_1^n|^x\right] &= \sum_{d=0}^{\infty} \mathbf{E}\left[|M_1^n|^x \mid IB^n = q_1^n - d\right] \mathbf{P}\left(D^n = d\right) \\ &\le \sum_{d=0}^{\infty} \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+d+1} \mathbf{P}\left(D^n = d\right)\end{aligned}$$

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$$
\begin{aligned}
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1}\sum_{d=0}^{\infty}\left(\frac{1}{\rho^n}\right)^d \mathbf{P}\left(D^n=d\right)\\
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1}\sum_{d=0}^{\infty}\left(\frac{1}{\rho^n}\right)^d \left(\frac{\lambda^n}{\tau+\lambda^n}\right)^d \frac{\tau}{\tau+\lambda^n}\\
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1}\frac{\tau}{\tau+\lambda^n}\sum_{d=0}^{\infty}\left(\frac{\mu}{\tau+\mu-\frac{\theta}{\sqrt{n}}}\right)^d\\
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1}\frac{\tau}{\tau+\lambda^n}\frac{\tau+\mu-\frac{\theta}{\sqrt{n}}}{\tau-\frac{\theta}{\sqrt{n}}}\\
&< \infty,
\end{aligned}
$$

where the first inequality follows from (A.27), the third equality follows from (A.3), the fourth equality follows from Assumption 1, the last equality follows if $n \geq n_0$, and the strict inequality follows from (A.26).

The hybrid policy. Consider the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n) (recall Definition 2). In the first renewal cycle, as soon as the inventory level decreases to q_1^n, the server is turned on and enters the set-up process. When the set-up is over,

1. if $D^n \leq q_1^n - q_0^n - 1$, then the server idles until the inventory level decreases to q_0^n. When the inventory level hits q_0^n, the server will produce nonstop until the inventory level reaches q_2^n. Therefore, if $D^n \leq q_1^n - q_0^n - 1$, $IB^n = q_0^n$.
2. If $D^n > q_1^n - q_0^n - 1$, the server will produce nonstop until the inventory level reaches q_2^n. Therefore, if $D^n > q_1^n - q_0^n - 1$, then $IB^n = q_1^n - D^n$.

Hence, if $n \geq n_0$, for all $x \in \mathbb{N}_+$,

$$
\begin{aligned}
\mathbf{E}\left[|M_1^n|^x\right] &= \mathbf{E}\left[|M_1^n|^x \mid IB^n = q_0^n\right]\mathbf{P}\left(D^n \leq q_1^n - q_0^n - 1\right)\\
&\qquad + \sum_{d=q_1^n-q_0^n}^{\infty}\mathbf{E}\left[|M_1^n|^x \mid IB^n = q_1^n - d\right]\mathbf{P}\left(D^n = d\right)\\
&\leq \mathbf{E}\left[|M_1^n|^x \mid IB^n = q_0^n\right] + \sum_{d=q_1^n-q_0^n}^{\infty}\mathbf{E}\left[|M_1^n|^x \mid IB^n = q_1^n - d\right]\mathbf{P}\left(D^n = d\right)\\
&\leq \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_0^n+1} + \sum_{d=q_1^n-q_0^n}^{\infty}\xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+d+1}\mathbf{P}\left(D^n = d\right)
\end{aligned}
$$

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$$
\begin{aligned}
&= \xi(x,n)\left[\left(\frac{1}{\rho^n}\right)^{q_2^n-q_0^n+1} + \left(\frac{1}{\rho^n}\right)^{q_2^n-q_1^n+1} \sum_{d=q_1^n-q_0^n}^{\infty} \left(\frac{1}{\rho^n}\right)^d \mathbf{P}\left(D^n=d\right)\right] \\
&\leq \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_0^n+1}\left[1+\sum_{d=0}^{\infty}\left(\frac{1}{\rho^n}\right)^d\left(\frac{\lambda^n}{\tau+\lambda^n}\right)^d \frac{\tau}{\tau+\lambda^n}\right] \\
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_0^n+1}\left[1+\frac{\tau}{\tau+\lambda^n}\sum_{d=0}^{\infty}\left(\frac{\mu}{\tau+\mu-\frac{\theta}{\sqrt{n}}}\right)^d\right] \\
&= \xi(x,n)\left(\frac{1}{\rho^n}\right)^{q_2^n-q_0^n+1}\left[1+\frac{\tau}{\tau+\lambda^n}\frac{\tau+\mu-\frac{\theta}{\sqrt{n}}}{\tau-\frac{\theta}{\sqrt{n}}}\right] \\
&< \infty,
\end{aligned}
$$

where the second inequality follows from (A.27), the third inequality follows from the fact that $q_0^n < q_1^n$ and (A.3), the third equality follows from Assumption 1, the last equality follows if $n \geq n_0$, and the strict inequality follows from (A.26). □

The following lemma states that all moments of the inventory level process exist.

LEMMA A.2.4. *Fix an arbitrary $n \geq n_0$, where n_0 is defined in Lemma A.2.3. Under the idling, turning-off, and hybrid policies, $\mathbf{E}\left[|Q^n(t)|^x\right] = \mathbf{E}\left[|Q^n(0)|^x\right] < \infty$ for all $x \in \mathbb{N}_+$ and $t \in \mathbb{R}_+$.*

Proof. Let us fix an arbitrary $n \geq n_0$. By Assumption 2, $\mathfrak{X}^n(t) = (Q^n(t), Y^n(t))$ has the same distribution with $\mathfrak{X}^n(0) = (Q^n(0), Y^n(0))$ for all $t \in \mathbb{R}_+$. Therefore,

$$
\mathbf{P}\left(Q^n(0)=k\right) = \mathbf{P}\left(Q^n(t)=k\right), \qquad \forall k \in \mathbb{Z},\, t \in \mathbb{R}_+. \tag{A.28}
$$

Let $\mathbb{I}(\cdot)$ denote the indicator function. For all $t \in \mathbb{R}_{++}$ and $k \in \mathbb{Z}$, we have

$$
\mathbf{P}\left(Q^n(0)=k\right) = \frac{1}{t}\int_0^t \mathbf{P}\left(Q^n(t)=k\right)\mathrm{d}t = \frac{1}{t}\mathbf{E}\left[\int_0^t \mathbb{I}\left(Q^n(t)=k\right)\mathrm{d}t\right], \tag{A.29}
$$

where the first equality follows from (A.28) and the second equality follows from Tonelli's theorem. By Lemma A.2.1, for all $k \in \mathbb{Z}$,

$$
\mathbf{E}\left[\int_0^{L_0^n+L_1^n} \mathbb{I}\left(Q^n(t)=k\right)\mathrm{d}t\right] \leq \mathbf{E}\left[L_0^n\right] + \mathbf{E}\left[L_1^n\right] < \infty. \tag{A.30}
$$

By (A.29), (A.30), and the renewal-reward theorem, for all $k \in \mathbb{Z}$,

$$
\mathbf{P}\left(Q^n(0)=k\right) = \lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\int_0^t \mathbb{I}\left(Q^n(t)=k\right)\mathrm{d}t\right] = \frac{1}{\mathbf{E}\left[L_1^n\right]}\mathbf{E}\left[\int_{L_0^n}^{L_0^n+L_1^n} \mathbb{I}\left(Q^n(t)=k\right)\mathrm{d}t\right]. \tag{A.31}
$$

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Recall that M_1^n is the minimum inventory level in the first renewal cycle. Then,

$$|Q^n(t)|^x \le |q_2^n|^x + |M_1^n|^x, \qquad \forall x \in \mathbb{N}_+,\, t \in [L_0^n, L_0^n + L_1^n]. \tag{A.32}$$

Therefore, for all $x \in \mathbb{N}_+$,

$$\begin{aligned}
\mathbf{E}\left[|Q^n(0)|^x\right] &= \sum_{k \le q_2^n} |k|^x \mathbf{P}\left(Q^n(0) = k\right) \\
&= \frac{1}{\mathbf{E}\left[L_1^n\right]} \sum_{k \le q_2^n} |k|^x \mathbf{E}\left[\int_{L_0^n}^{L_0^n + L_1^n} \mathbb{I}\left(Q^n(t) = k\right) \mathrm{d}t\right] \\
&= \frac{1}{\mathbf{E}\left[L_1^n\right]} \mathbf{E}\left[\int_{L_0^n}^{L_0^n + L_1^n} \left(\sum_{k \le q_2^n} |k|^x \mathbb{I}\left(Q^n(t) = k\right)\right) \mathrm{d}t\right] \\
&= \frac{1}{\mathbf{E}\left[L_1^n\right]} \mathbf{E}\left[\int_{L_0^n}^{L_0^n + L_1^n} |Q^n(t)|^x \, \mathrm{d}t\right] \\
&\le \frac{1}{\mathbf{E}\left[L_1^n\right]} \mathbf{E}\left[\int_{L_0^n}^{L_0^n + L_1^n} \left(|q_2^n|^x + |M_1^n|^x\right) \mathrm{d}t\right] \\
&= \frac{1}{\mathbf{E}\left[L_1^n\right]} \mathbf{E}\left[\left(|q_2^n|^x + |M_1^n|^x\right) L_1^n\right] \\
&= |q_2^n|^x + \frac{1}{\mathbf{E}\left[L_1^n\right]} \mathbf{E}\left[|M_1^n|^x \, L_1^n\right] \\
&\le |q_2^n|^x + \frac{1}{\mathbf{E}\left[L_1^n\right]} \sqrt{\mathbf{E}\left[|M_1^n|^{2x}\right] \mathbf{E}\left[\left(L_1^n\right)^2\right]} \\
&< \infty,
\end{aligned} \tag{A.33}$$

where the second equality follows from (A.31), the third equality follows from Tonelli's theorem, the first inequality follows from (A.32), the second inequality follows from Hölder's inequality, and the strict inequality follows from Lemmas A.2.1 and A.2.3.

Finally, the desired result follows from (A.28) and (A.33). □

The following auxiliary result is about the existence of the long-run average cost under the idling, turning-off, and hybrid policies.

LEMMA A.2.5. *Fix an arbitrary $n \ge n_0$, where n_0 is defined in Lemma A.2.3. Under the idling, turning-off, and hybrid policies, $\lim_{t\to\infty} J^n(t)$ exists, that is, the long-run average cost exists.*

Proof. We will use the renewal-reward theorem.

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Idling policy. Consider the idling policy with the parameters (q_1^n, q_2^n). Expected length of the first renewal cycle is given in (A.2). Let C_k^n denote the cost earned in the kth renewal cycle for all $k \in \mathbb{N}$. Observe that $\{\mathsf{C}_k^n, k \in \mathbb{N}_+\}$ is an i.i.d. sequence of random variables. By Assumption 3, expected energy cost saving in the first renewal cycle is equal to

$$\frac{pn\,(q_2^n - q_1^n)}{\lambda^n}. \tag{A.34}$$

Observe also that the term in (A.34) is an upper bound on the expected energy cost saving in the zeroth renewal cycle because the system state starts that cycle with the induced limiting distribution.

Expected holding and backorder cost in the zeroth renewal cycle is less than or equal to

$$\begin{aligned}
(h \vee b)\mathbf{E}\left[\int_0^{L_0^n} |Q^n(t)|\,\mathrm{d}t\right] &= (h \vee b)\mathbf{E}\left[\int_0^{\infty} |Q^n(t)|\,\mathbb{I}\,(L_0^n > t)\,\mathrm{d}t\right] \\
&= (h \vee b)\int_0^{\infty} \mathbf{E}\left[|Q^n(t)|\,\mathbb{I}\,(L_0^n > t)\right]\mathrm{d}t \\
&\leq (h \vee b)\int_0^{\infty} \sqrt{\mathbf{E}\left[|Q^n(t)|^2\right]\mathbf{P}\left(\mathbb{I}\,(L_0^n > t)\right)}\mathrm{d}t \\
&= (h \vee b)\sqrt{\mathbf{E}\left[|Q^n(0)|^2\right]}\int_0^{\infty} \sqrt{\mathbf{P}\left(\mathbb{I}\,(L_0^n > t)\right)}\mathrm{d}t \\
&\leq (h \vee b)\sqrt{\mathbf{E}\left[|Q^n(0)|^2\right]}\int_0^{\infty} \sqrt{C_{11}^n}\mathrm{e}^{-0.5C_{12}^n t}\mathrm{d}t \\
&= 2(h \vee b)\sqrt{C_{11}^n\mathbf{E}\left[|Q^n(0)|^2\right]}\frac{1}{C_{12}^n} \\
&< \infty,
\end{aligned} \tag{A.35}$$

where the second equality follows from Tonelli's theorem, the first inequality follows from Hölder's inequality, the third equality follows from (A.28), the second inequality follows from Lemma A.2.1, and the strict inequality follows from Lemma A.2.4. Furthermore, expected holding and backorder cost in the first renewal cycle is less than or equal to

$$(h \vee b)\left(|q_2^n| + \mathbf{E}\left[|M_1^n|\right]\right), \tag{A.36}$$

where M_1^n is the minimum inventory level in the first renewal cycle.

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By (A.34), (A.35), and (A.36),

$$\mathbf{E}\left[|\mathsf{C}_0^n|\right] < \infty, \qquad -\frac{pn\left(q_2^n - q_1^n\right)}{\lambda^n} \leq \mathbf{E}\left[\mathsf{C}_1^n\right] \leq (h \vee b)\left(|q_2^n| + \mathbf{E}\left[|M_1^n|\right]\right).$$

Because $n \geq n_0$, and by Lemmas A.2.1 and A.2.3, and the renewal-reward theorem, $\lim_{t\to\infty} J^n(t)$ exists.

Turning-off policy. Consider the turning-off policy with the parameters (q_1^n, q_2^n). Expected length of the first renewal cycle is finite and given in (A.5). By Assumption 3, expected energy cost saving and expected set-up cost in the first renewal cycle are equal to

$$\frac{rn\left(q_2^n - q_1^n\right)}{\lambda^n} \quad \text{and} \quad cn^{3/2}, \tag{A.37}$$

respectively. Observe that the terms in (A.37) are upper bounds on the expected energy cost saving and expected set-up cost in the zeroth renewal cycle, respectively. Furthermore, expected holding and backorder cost in the zeroth renewal cycle is finite by (A.35) and the same cost in the first renewal cycle is less than or equal to the term in (A.36). Therefore, $\mathbf{E}\left[|\mathsf{C}_0^n|\right] < \infty$ and by (A.36) and (A.37),

$$-\frac{rn\left(q_2^n - q_1^n\right)}{\lambda^n} \leq \mathbf{E}\left[\mathsf{C}_1^n\right] \leq cn^{3/2} + (h \vee b)\left(|q_2^n| + \mathbf{E}\left[|M_1^n|\right]\right).$$

Because $n \geq n_0$, and by Lemmas A.2.1 and A.2.3, and the renewal-reward theorem, $\lim_{t\to\infty} J^n(t)$ exists.

Hybrid policy. Consider the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n). A lower bound and an upper bound on the expected length of the first renewal cycle are given in (A.11) and (A.12), respectively. Expected energy cost saving when the server is turned off and the expected set-up cost in the first renewal cycle are given in (A.37), respectively. The terms in (A.37) are upper bounds on the expected energy cost saving when the server is turned off and expected set-up cost in the zeroth renewal cycle, respectively. By Assumption 3, the expected energy cost saving in the first renewal cycle when the server is idle is equal to

$$pn \sum_{i=0}^{q_1^n - q_0^n - 1} \frac{q_1^n - i - q_0^n}{\lambda^n} \mathbf{P}(D^n = i) \leq pn \frac{q_1^n - q_0^n}{\lambda^n}, \tag{A.38}$$

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and the LHS of (A.38) is an upper bound on the expected energy cost saving when the server is idle in the zeroth renewal cycle.

Expected holding and backorder cost in the zeroth renewal cycle is finite by (A.35) and the same cost in the first renewal cycle is less than or equal to the term in (A.36). Therefore, $\mathbf{E}\left[|\mathsf{C}_0^n|\right] < \infty$ and by (A.36), (A.37), and (A.38),

$$-\frac{rn\left(q_2^n - q_1^n\right)}{\lambda^n} - \frac{pn\left(q_1^n - q_0^n\right)}{\lambda^n} \leq \mathbf{E}\left[\mathsf{C}_1^n\right] \leq cn^{3/2} + (h \vee b)\left(|q_2^n| + \mathbf{E}\left[|M_1^n|\right]\right).$$

Because $n \geq n_0$, and by Lemmas A.2.1 and A.2.3, and the renewal-reward theorem, $\lim_{t\to\infty} J^n(t)$ exists. □

A.2.4. Asymptotic analyses of the idling, turning-off, and the hybrid policies

We prove necessary growth conditions for the policy parameters associated with the idling, turning-off, and the hybrid policies under which the diffusion scaled long-run average cost stays finite as n tends to infinity. First, we present an auxiliary result

For all $n \in \mathbb{N}_+$ and $k \in \mathbb{Z}$, let $Z_{k,k+1}^n$ denote the expected total inventory related (holding and backorder) cost incurred until the inventory level reaches $k+1$ starting from the inventory level k when the server is busy.

LEMMA A.2.6. *For all $n \in \mathbb{N}_+$ and $k \in \mathbb{Z}$,*

$$Z_{k,k+1}^n = \begin{cases} \frac{b\mu n}{\theta^2} + \frac{b|k+1|\sqrt{n}}{\theta}, & \text{if } k < 0, \\ \frac{b\lambda^n n}{\theta^2}, & \text{if } k = 0, \\ \frac{n}{\theta^2}\left[\left(\rho^n\right)^k b\lambda^n + h\mu\left(\left(\rho^n\right)^{k+1} - (k+1)\rho^n + k\right)\right], & \text{if } k > 0. \end{cases}$$

Proof. Fix an arbitrary $n \in \mathbb{N}_+$. Recall that $T_{k(k+1)}^n$ denotes the time it takes for the inventory level to reach $k+1$ starting from the inventory level k when the server is busy. By the busy-period analysis of an $M/M/1$ queue (see proposition III.8.10 of Asmussen (2003)), we have

$$\mathbf{E}\left[T_{k(k+1)}^n\right] = \frac{1}{\mu - \lambda^n} = \frac{\sqrt{n}}{\theta}, \qquad \forall k \in \mathbb{Z}, \tag{A.39}$$

where the second equality follows from Assumption 1.

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Case 1: $k<0$. When the server is busy, the stochastic evolution of the system is the same for all $k\in\mathbb{Z}$. Because the number of backordered demands when the inventory level is k is $|k+1|$ more than the one when the inventory level is -1 and by (A.39), we have

$$Z^n_{k,k+1}=Z^n_{-1,0}+\frac{b|k+1|}{\mu-\lambda^n}=Z^n_{-1,0}+\frac{b|k+1|\sqrt{n}}{\theta},\qquad \forall k<0. \tag{A.40}$$

First, let us consider the case $k=-1$. We have

$$\begin{aligned} Z^n_{-1,0}&=\frac{b}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}Z^n_{-2,0}\\ &=\frac{b}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}\left(Z^n_{-2,-1}+Z^n_{-1,0}\right)\\ &=\frac{b}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}\left(2Z^n_{-1,0}+\frac{b\sqrt{n}}{\theta}\right), \end{aligned} \tag{A.41}$$

where the last equality follows from (A.40). The equality in (A.41) implies

$$Z^n_{-1,0}=\frac{b\mu n}{\theta^2}. \tag{A.42}$$

By (A.40) and (A.42),

$$Z^n_{k,k+1}=\frac{b\mu n}{\theta^2}+\frac{b|k+1|\sqrt{n}}{\theta},\qquad \forall k<0.$$

Case 2: $k=0$. We have

$$Z^n_{0,1}=0+\frac{\lambda^n}{\mu+\lambda^n}Z^n_{-1,1}=\frac{\lambda^n}{\mu+\lambda^n}\left(Z^n_{-1,0}+Z^n_{0,1}\right)=\frac{\lambda^n}{\mu+\lambda^n}\left(\frac{b\mu n}{\theta^2}+Z^n_{0,1}\right), \tag{A.43}$$

where the last equality follows from (A.42). The last equality in (A.43) implies

$$Z^n_{0,1}=\frac{b\lambda^n n}{\theta^2}. \tag{A.44}$$

Case 3: $k>0$. First, let us consider the case $k=1$. We have

$$\begin{aligned} Z^n_{1,2}&=\frac{h}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}Z^n_{0,2}\\ &=\frac{h}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}\left(Z^n_{0,1}+Z^n_{1,2}\right)\\ &=\frac{h}{\mu+\lambda^n}+\frac{\lambda^n}{\mu+\lambda^n}\left(\frac{b\lambda^n n}{\theta^2}+Z^n_{1,2}\right), \end{aligned} \tag{A.45}$$

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where the last equality follows from (A.44). The equality in (A.45) implies

$$Z_{1,2}^n = \frac{h}{\mu} + \rho^n \frac{b\lambda^n n}{\theta^2}. \tag{A.46}$$

Next,

$$\begin{aligned} Z_{2,3}^n &= \frac{2h}{\mu+\lambda^n} + \frac{\lambda^n}{\mu+\lambda^n}\left(Z_{1,2}^n + Z_{2,3}^n\right) \\ &= \frac{2h}{\mu+\lambda^n} + \frac{\lambda^n}{\mu+\lambda^n}\left(\frac{h}{\mu} + \rho^n \frac{b\lambda^n n}{\theta^2} + Z_{2,3}^n\right), \\ \implies Z_{2,3}^n &= \frac{2h}{\mu} + \frac{h}{\mu}\rho^n + \left(\rho^n\right)^2 \frac{b\lambda^n n}{\theta^2}, \end{aligned}$$

where the second equality follows from (A.46). Similarly,

$$Z_{3,4}^n = \frac{3h}{\mu} + \frac{2h}{\mu}\rho^n + \frac{h}{\mu}\left(\rho^n\right)^2 + \left(\rho^n\right)^3 \frac{b\lambda^n n}{\theta^2}.$$

Inductively, for all $k \geq 1$,

$$\begin{aligned} Z_{k,k+1}^n &= \left(\rho^n\right)^k \frac{b\lambda^n n}{\theta^2} + \frac{h}{\mu}\sum_{i=0}^{k-1}(k-i)\left(\rho^n\right)^i \\ &= \left(\rho^n\right)^k \frac{b\lambda^n n}{\theta^2} + \frac{h}{\mu}\left(k\frac{1-\left(\rho^n\right)^k}{1-\rho^n} - \rho^n \frac{1-k\left(\rho^n\right)^{k-1} + (k-1)\left(\rho^n\right)^k}{\left(1-\rho^n\right)^2}\right) \\ &= \left(\rho^n\right)^k \frac{b\lambda^n n}{\theta^2} + \frac{h}{\mu}\frac{\left(\rho^n\right)^{k+1} - (k+1)\rho^n + k}{\left(1-\rho^n\right)^2} \\ &= \frac{n}{\theta^2}\left[\left(\rho^n\right)^k b\lambda^n + h\mu\left(\left(\rho^n\right)^{k+1} - (k+1)\rho^n + k\right)\right], \end{aligned}$$

where the last equality follows from Assumption 1. □

The following equalities will be helpful in the proof of the upcoming result.

$$\sum_{k=q_1^n}^{q_2^n-1} |k+1| = \frac{1}{2}\left(q_2^n - q_1^n\right)\left|q_1^n + q_2^n + 1\right|, \qquad \forall q_1^n < q_2^n \leq 0, \tag{A.47}$$

$$\begin{aligned} \sum_{k=q_1^n}^{q_2^n-1}\left(\left(\rho^n\right)^{k+1} - (k+1)\rho^n + k\right) &= \left(\rho^n\right)^{q_1^n+1}\frac{1-\left(\rho^n\right)^{q_2^n-q_1^n}}{1-\rho^n} - \rho^n\left(q_2^n - q_1^n - 1\right) \\ &\quad + \frac{\theta}{2\mu\sqrt{n}}\left(q_2^n - q_1^n\right)\left(q_2^n + q_1^n - 1\right), \qquad \forall\, 1 \leq q_1^n < q_2^n. \end{aligned} \tag{A.48}$$

The following lemma is the main result of this subsection.

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LEMMA A.2.7. *Let* $\{q_0^n, q_1^n, q_2^n, n \in \mathbb{N}_+\}$ *be an arbitrary sequence in* $\mathbb{Z}^3$ *such that* $q_0^n < q_1^n < q_2^n$ *for all* $n \in \mathbb{N}_+$ *and*

$$\lim_{n\to\infty} \frac{q_i^n}{\sqrt{n}} = x_i \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \qquad \forall i \in \{0,1,2\}. \tag{A.49}$$

1. *Let* $\pi_i = \{\pi_i^n, n \in \mathbb{N}_+\}$ *be a sequence of policies such that each policy is of idling type with the parameters* $(q_1^n, q_2^n) \in \mathbb{Z}^2$ *for all* $n \in \mathbb{N}_+$. *If* $x_1 = \pm\infty$ *or* $x_2 = \pm\infty$, *then* $\lim_{n\to\infty} \lim_{t\to\infty} \hat{J}^{\pi_i,n}(t) = \infty$.
2. *Let* $\pi_{off} = \{\pi_{off}^n, n \in \mathbb{N}_+\}$ *be a sequence of policies such that each policy is of turning-off type with the parameters* $(q_1^n, q_2^n) \in \mathbb{Z}^2$ *for all* $n \in \mathbb{N}_+$. *If* $x_1 = \pm\infty$ *or* $x_2 = \pm\infty$ *or* $\{(x_1, x_2) \in \mathbb{R}^2$ *such that* $x_1 = x_2\}$, *then* $\lim_{n\to\infty} \lim_{t\to\infty} \hat{J}^{\pi_{off},n}(t) = \infty$.
3. *Let* $\pi_h = \{\pi_h^n, n \in \mathbb{N}_+\}$ *be a sequence of policies such that each policy is of hybrid type with the parameters* $(q_0^n, q_1^n, q_2^n) \in \mathbb{Z}^3$ *for all* $n \in \mathbb{N}_+$. *If* $x_0 = \pm\infty$ *or* $x_2 = \pm\infty$ *or* $\{(x_0, x_2) \in \mathbb{R}^2$ *such that* $x_0 = x_2\}$, *then* $\lim_{n\to\infty} \lim_{t\to\infty} \hat{J}^{\pi_h,n}(t) = \infty$.

Proof. We will omit the policy indexes π_i^n, π_{off}^n, and π_h^n for notational convenience. Let us fix an arbitrary $n \geq n_0$, where n_0 is defined in Lemma A.2.3.

The idling policy. Consider the idling policy with the parameters (q_1^n, q_2^n). We will use the renewal-reward theorem to derive a lower bound on the long-run average cost. Expected length of the first renewal cycle is given in (A.2).

Recall from the proof of Lemma A.2.5 that the expected cost in the zeroth renewal cycle is finite. We will compute a lower bound on the expected cost in the first renewal cycle. Recall that the expected energy cost saving in the first renewal cycle is given in (A.34). In order to compute a lower bound on the expected total holding and backorder cost in the first renewal cycle, we will only focus on the aforementioned cost incurred when the server is busy. The reason is that the total holding and backorder cost incurred when the server is idle grows slower in n than the one when the server is busy does. By recalling Lemma A.2.6, a lower bound on the expected total inventory holding and backorder cost in the first renewal cycle is

$$\sum_{k=q_1^n}^{q_2^n-1} Z_{k,k+1}^n \tag{A.50}$$

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By Lemma A.2.5, $\lim_{t\to\infty} J^n(t)$ exists. By (A.2), (A.34), (A.50), and the renewal-reward theorem,

$$\lim_{t\to\infty} J^n(t) \geq \frac{-\frac{pn\left(q_2^n-q_1^n\right)}{\lambda^n}+\sum_{k=q_1^n}^{q_2^n-1} Z_{k,k+1}^n}{\left(q_2^n-q_1^n\right)\left(\frac{1}{\lambda^n}+\frac{\sqrt{n}}{\theta}\right)}. \tag{A.51}$$

By (A.51) and because $\lim_{t\to\infty} \hat{J}^n(t) = \lim_{t\to\infty} J^n(t)/\sqrt{n}$, we have

$$\liminf_{n\to\infty} \lim_{t\to\infty} \hat{J}^n(t) \geq \liminf_{n\to\infty} \frac{-\frac{p\sqrt{n}\left(q_2^n-q_1^n\right)}{\lambda^n}+\frac{1}{\sqrt{n}}\sum_{k=q_1^n}^{q_2^n-1} Z_{k,k+1}^n}{\left(q_2^n-q_1^n\right)\left(\frac{1}{\lambda^n}+\frac{\sqrt{n}}{\theta}\right)}. \tag{A.52}$$

Observe that in the denominator of the term in the RHS of (A.52), the order of the term with the highest order is $\sqrt{n}\left(q_2^n-q_1^n\right)$. We will analyze the numerator case by case.

Case 1. Suppose that $x_2 = -\infty$ (recall (A.49)). Because $q_1^n < q_2^n$ for all $n \in \mathbb{N}_+$, it must be the case that $x_1 = -\infty$. The order of the denominator of the term in the RHS of (A.52) is $\sqrt{n}\left(q_2^n-q_1^n\right)$. By Lemma A.2.6 and the equation (A.47), in the numerator, the highest order term is the sum term with the order $\left(q_2^n-q_1^n\right)\left|q_1^n+q_2^n\right|$. Therefore, the order of the ratio in the RHS of (A.52) is $\left|q_1^n+q_2^n\right|/\sqrt{n}$. Because $x_1 = -\infty$, $x_2 = -\infty$, and by (A.49), the RHS of (A.52) is equal to $+\infty$.

Case 2. Suppose that $x_2 \in \mathbb{R}$. Because $q_1^n < q_2^n$ for all $n \in \mathbb{N}_+$, it must be the case that $x_1 = -\infty$. Therefore, the order of the denominator of the term in the RHS of (A.52) is $\sqrt{n}|q_1^n|$. By Lemma A.2.6 and the equation (A.47), the highest order term in the numerator of the term in the RHS of (A.52) is the one with the sum and its order is $(q_1^n)^2$. Therefore, the order of the ratio in the RHS of (A.52) is $|q_1^n|/\sqrt{n}$. Because $x_1 = -\infty$ and by (A.49), the RHS of (A.52) is equal to $+\infty$.

Case 3. Suppose that $x_2 = +\infty$. There are three sub-cases to consider.

Case 3.1. Suppose that $x_1 = +\infty$. The order of the denominator of the term in the RHS of (A.52) is $\sqrt{n}\left(q_2^n-q_1^n\right)$. By Lemma A.2.6 and the equation (A.48), in the numerator, the highest order term is the sum term with the order $\left(q_2^n-q_1^n\right)\left(q_1^n+q_2^n\right)$. Therefore, the order of the ratio in the RHS of (A.52) is $\left(q_1^n+q_2^n\right)/\sqrt{n}$. Because $x_1 = +\infty$, $x_2 = +\infty$, and by (A.49), the RHS of (A.52) is equal to $+\infty$.

Case 3.2. Suppose that $x_1 \in \mathbb{R}$. The order of the denominator of the term in the RHS of (A.52) is $\sqrt{n}q_2^n$. By Lemma A.2.6 and the equation (A.48), in the numerator, the highest order term is the

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sum term with the order $(q_2^n)^2$. Therefore, the order of the ratio in the RHS of (A.52) is $q_2^n/\sqrt{n}$. Because $x_2 = +\infty$ and by (A.49), the RHS of (A.52) is equal to $+\infty$.

Case 3.3. Suppose that $x_1 = -\infty$. The order of the denominator of the term in the RHS of (A.52) is $\sqrt{n}\,(q_2^n - q_1^n)$. By Lemma A.2.6 and the equations (A.47) and (A.48), in the numerator, the highest order term is the sum term with the order $(q_1^n)^2 + (q_2^n)^2$. Therefore, the order of the ratio in the RHS of (A.52) is $(q_2^n - q_1^n)/\sqrt{n}$. Because $x_1 = -\infty$, $x_2 = +\infty$, and by (A.49), the RHS of (A.52) is equal to $+\infty$.

Consequently, in all cases, the RHS of (A.52) is equal to $+\infty$.

The turning-off policy. Consider the turning-off policy with the parameters (q_1^n, q_2^n). We will use the renewal-reward theorem. Expected length of the first renewal cycle is given in (A.5).

Recall from the proof of Lemma A.2.5 that the expected cost in the zeroth renewal cycle is finite. We will compute a lower bound on the expected cost in the first renewal cycle. Recall that the expected energy cost saving and expected set-up cost in the first renewal cycle are given in (A.37). Recall that D^n is the cumulative number of demands arriving in the set-up stage. A lower bound on the expected total inventory holding and backorder cost in the first renewal cycle is

$$\sum_{i=0}^{\infty} \mathbf{P}(D^n = i) \left(\sum_{k=q_1^n - i}^{q_2^n - 1} Z_{k,k+1}^n \right) \geq \sum_{k=q_1^n}^{q_2^n - 1} Z_{k,k+1}^n, \tag{A.53}$$

where the inequality follows from the fact that $Z_{k,k+1}^n \geq 0$ for all $k \in \mathbb{Z}$ (recall Lemma A.2.6).

By Lemma A.2.5, $\lim_{t\to\infty} J^n(t)$ exists. By (A.5), (A.37), (A.53), and the renewal-reward theorem,

$$\lim_{t\to\infty} J^n(t) \geq \frac{cn^{3/2} - \frac{r n \left(q_2^n - q_1^n\right)}{\lambda^n} + \sum_{k=q_1^n}^{q_2^n - 1} Z_{k,k+1}^n}{\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_1^n + \frac{\lambda^n}{\tau}\right)}. \tag{A.54}$$

By (A.54) and because $\lim_{t\to\infty} \hat{J}^n(t) = \lim_{t\to\infty} J^n(t)/\sqrt{n}$, we have

$$\liminf_{n\to\infty} \lim_{t\to\infty} \hat{J}^n(t) \geq \liminf_{n\to\infty} \frac{cn - \frac{r\sqrt{n}\left(q_2^n - q_1^n\right)}{\lambda^n} + \frac{1}{\sqrt{n}} \sum_{k=q_1^n}^{q_2^n - 1} Z_{k,k+1}^n}{\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_1^n + \frac{\lambda^n}{\tau}\right)}. \tag{A.55}$$

Again, we will make a case-by-case analysis. First, suppose that $x_1 = \pm\infty$ or $x_2 = \pm\infty$. The proof is very similar to the one used for the idling policy: we will consider three cases. In all cases except

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the first one, the convergence of the term in the RHS of (A.55) to infinity follows with the same logic described for the convergence of the term in the RHS of (A.52). The proof for the first case is as follows.

Case 1. Suppose that $x_2 = -\infty$. Because $q_1^n < q_2^n$ for all $n \in \mathbb{N}_+$, it must be the case that $x_1 = -\infty$. The order of the denominator of the term in the RHS of (A.55) is $\sqrt{n}\,(q_2^n - q_1^n)$. By Lemma A.2.6 and the equation (A.47), in the numerator, the highest order term is either the set-up cost with the order n or the sum term with the order $(q_2^n - q_1^n)\,|q_1^n + q_2^n|$. In each case, the RHS of (A.55) is equal to $+\infty$.

Second, suppose that $x_1, x_2 \in \mathbb{R}$ such that $x_1 = x_2$. This assumption and (A.49) imply that $q_2^n - q_1^n = o(\sqrt{n})$, where $o(\cdot)$ is the little-o notation. The order of the denominator of the term in the RHS of (A.55) is $\sqrt{n}\,(q_2^n - q_1^n) = o(n)$. Because the set-up cost is $O(n)$ and in the numerator, the RHS of (A.55) is equal to $+\infty$.

The hybrid policy. Consider the hybrid policy with the parameters (q_0^n, q_1^n, q_2^n). We will use the renewal-reward theorem. A lower bound and an upper bound on the expected length of the first renewal cycle are given in (A.11) and (A.12), respectively.

Recall from the proof of Lemma A.2.5 that the expected cost in the zeroth renewal cycle is finite. We will compute a lower bound on the expected cost in the first renewal cycle. First, the expected energy cost saving when the server is turned off and the expected set-up cost in the first renewal cycle are given in (A.37). Second, the expected energy cost saving when the server is idle is given in the LHS of (A.38). Third, similar to how we derive (A.53), we will compute a lower bound on the expected total holding and backorder cost in the first renewal cycle. Again, we will only focus on the aforementioned cost incurred when the server is busy. A lower bound on the expected total inventory holding and backorder cost in the first renewal cycle is

$$\begin{aligned}
&\mathbf{P}(D^n < q_1^n - q_0^n) \sum_{k=q_0^n}^{q_2^n - 1} Z_{k,k+1}^n + \sum_{i=q_1^n - q_0^n}^{\infty} \left(\sum_{k=q_1^n - i}^{q_2^n - 1} Z_{k,k+1}^n \right) \mathbf{P}(D^n = i) \\
&\qquad \geq \mathbf{P}(D^n < q_1^n - q_0^n) \sum_{k=q_0^n}^{q_2^n - 1} Z_{k,k+1}^n + \sum_{i=q_1^n - q_0^n}^{\infty} \left(\sum_{k=q_0^n}^{q_2^n - 1} Z_{k,k+1}^n \right) \mathbf{P}(D^n = i)
\end{aligned}$$

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$$= \sum_{k=q_0^n}^{q_2^n-1} Z_{k,k+1}^n, \tag{A.56}$$

where the first inequality follows from the fact that $q_1^n - i \le q_0^n$ for all $i \ge q_1^n - q_0^n$ and $Z_{k,k+1}^n \ge 0$ for all $k \in \mathbb{Z}$ (recall Lemma A.2.6).

By Lemma A.2.5, $\lim_{t\to\infty} J^n(t)$ exists. By (A.12), (A.37), the RHS of (A.38), (A.56), the fact that $r > p$, and the renewal-reward theorem,

$$\lim_{t\to\infty} J^n(t) \ge \frac{cn^{3/2} - \frac{rn\left(q_2^n - q_0^n\right)}{\lambda^n} + \sum_{k=q_0^n}^{q_2^n-1} Z_{k,k+1}^n}{\frac{q_2^n - q_0^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_0^n + \frac{\lambda^n}{\tau}\right)}. \tag{A.57}$$

By (A.57) and because $\lim_{t\to\infty} \hat{J}^n(t) = \lim_{t\to\infty} J^n(t)/\sqrt{n}$, we have

$$\liminf_{n\to\infty} \lim_{t\to\infty} \hat{J}^n(t) \ge \liminf_{n\to\infty} \frac{cn - \frac{r\sqrt{n}\left(q_2^n - q_0^n\right)}{\lambda^n} + \frac{1}{\sqrt{n}} \sum_{k=q_0^n}^{q_2^n-1} Z_{k,k+1}^n}{\frac{q_2^n - q_0^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_0^n + \frac{\lambda^n}{\tau}\right)}. \tag{A.58}$$

Observe that the term in the RHS of (A.58) is the same of the one in the RHS of (A.55) except that q_1^n is replaced with q_0^n. Therefore, the rest of the proof is exactly the same of the one associated with the turning-off policy except that the pair (q_1^n, x_1) is replaced with (q_0^n, x_0). □

A.3. Proofs related to the BCP (6)

We prove Lemmas 2 and 3 in Sections A.3.1 and A.3.2, respectively. We present the Taylor's theorem for the relative value function in Section A.3.3. Finally, we present the proof of Theorem 1 in Section A.3.4.

A.3.1. Proof of Lemma 2

Let (Q, I, F) be an arbitrary feasible point for the BCP (6). Because Q is a semimartingale (see definition V.5.18 of Çınlar (2011)) and by using the extant second derivative Meyer-Ito formula (see theorem IV.71 of Protter (2005)) with the function f defined in (19) on (6b), for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
\mathbf{E}\left[f(Q(t))\right] - \mathbf{E}\left[f(Q(0))\right] = \frac{1}{2}\Sigma\mathbf{E}\left[\int_{0_+}^{t} f''(Q(s-))\mathrm{d}s\right] + \theta\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\mathrm{d}s\right] \\
+ \mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\left(-\mu\mathrm{d}I(s) - \mu\mathrm{d}F(s)\right)\right] \\
+ \mathbf{E}\left[\sum_{0<s\le t:\,|\Delta Q(s)|>0} \Big(\Delta f(Q(s)) - f'(Q(s-))\Delta Q(s)\Big)\right].
\end{aligned} \tag{A.59}$$

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If $\Delta I(t) > 0$ for some $t \in \mathbb{R}_+$, then $\Delta Q(t) = -\mu \Delta I(t)$ and $\Delta F(t) = 0$ by (6b) and (6d). Let us decompose the idle time process such that

$$I(t) = I^c(t) + \sum_{0<s\leq t:\, \Delta I(s)>0} \Delta I(s), \qquad \forall t \in \mathbb{R}_{++},$$

where I^c is the continuous part of I (see section 4.9 of Harrison (2013) for details). Observe that

$$\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\,(-\mu \mathrm{d}I(s))\right] = \mu \mathbf{E}\left[\int_{0_+}^{t} -f'(Q(s-))\mathrm{d}I^c(s)\right] + \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta I(s)>0} f'(Q(s-))\Delta Q(s)\right].$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\,(-\mu \mathrm{d}I(s))\right] + \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta I(s)>0} \Big(\Delta f(Q(s)) - f'(Q(s-))\Delta Q(s)\Big)\right] \\
&\qquad = \mu \mathbf{E}\left[\int_{0_+}^{t} -f'(Q(s-))\mathrm{d}I^c(s)\right] + \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta I(s)>0} \Delta f(Q(s))\right] \\
&\qquad \geq p\mathbf{E}\left[I^c(t)\right] + p\mathbf{E}\left[\sum_{0<s\leq t:\, \Delta I(s)>0} \Delta I(s)\right] \\
&\qquad = p\mathbf{E}\left[I(t)\right], \qquad \text{(A.60)}
\end{aligned}$$

where the inequality follows from (7b).

If $\Delta F(t) > 0$, then $\Delta Q(t) = -\mu \Delta F(t)$ and $\Delta I(t) = 0$ by (6b) and (6d). Because F is piecewise constant,

$$\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\,(-\mu \mathrm{d}F(s))\right] = \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta F(s)>0} f'(Q(s-))\Delta Q(s)\right]. \qquad \text{(A.61)}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\,(-\mu \mathrm{d}F(s))\right] + \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta F(s)>0} \Big(\Delta f(Q(s)) - f'(Q(s-))\Delta Q(s)\Big)\right] \\
&\qquad = \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta F(s)>0} \Delta f(Q(s))\right] \\
&\qquad \geq \mathbf{E}\left[\sum_{0<s\leq t:\, \Delta F(s)>0} (r\Delta F(s) - c)\right], \qquad \text{(A.62)}
\end{aligned}$$

where the inequality follows from (7c).

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Therefore, by (7a), (A.59), (A.60), and (A.62),

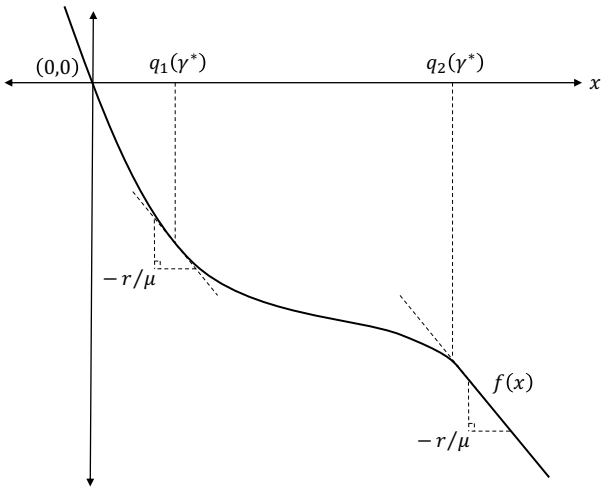
$$\mathbf{E}\left[f(Q(t))\right]-\mathbf{E}\left[f(Q(0))\right]\geq\gamma t-\mathbf{E}\left[\int_{0_+}^{t}\left(h\left(Q(s-)\right)^{+}+b\left(Q(s-)\right)^{-}\right)\mathrm{d}s\right]$$
$$+p\mathbf{E}\left[I(t)\right]+\mathbf{E}\left[\sum_{0<s\leq t:\,\Delta F(s)>0}\left(r\Delta F(s)-c\right)\right].$$

Finally, by (6e), (7d), and dividing both sides to t and letting $t\to\infty$,

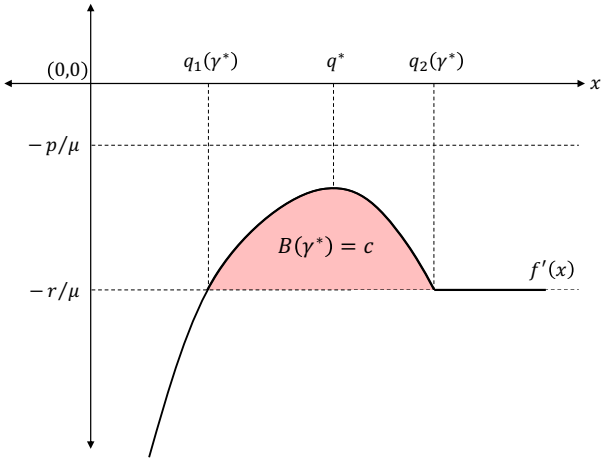
$$\liminf_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,\Delta F(s)>0}\left(c-r\Delta F(s)\right)-pI(t)+\int_{0+}^{t}\left(h\left(Q(s-)\right)^{+}+b\left(Q(s-)\right)^{-}\right)\mathrm{d}s\right]\geq\gamma.$$

A.3.2. Proof of Lemma 3

Figure A.3.1a presents an illustration of the relative value function f.



(a) The relative value function f defined in (19).



(b) f' defined in (A.63).

Figure A.3.1 (Color online) Illustration of f and f' for $\gamma^*\in(\gamma_r,\gamma_p)$.

By (11) and (19),

$$\lim_{x\uparrow q_2(\gamma^*)}f(x)=f_{\gamma^*}\left(q_2(\gamma^*)\right)=\lim_{x\downarrow q_2(\gamma^*)}f(x),$$

implying that f is continuous. By (10) and (19),

$$f'(x)=\begin{cases}g_{\gamma^*}(x), & \text{if } x<q_2(\gamma^*),\\ -r/\mu, & \text{if } x\geq q_2(\gamma^*).\end{cases}\tag{A.63}$$

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Figure A.3.1b presents an illustration of f'. By (14b), $g_{\gamma^*}(q_2(\gamma^*)) = -r/\mu$. Therefore,

$$\lim_{x \uparrow q_2(\gamma^*)} f'(x) = g_{\gamma^*}(q_2(\gamma^*)) = -r/\mu = \lim_{x \downarrow q_2(\gamma^*)} f'(x),$$

implying that g_{γ^*} is continuous and thus $f \in \mathbb{C}_1$. By (A.63), let us define

$$f''(x) := \begin{cases} g'_{\gamma^*}(x), & \text{if } x < q_2(\gamma^*), \\ 0, & \text{if } x \geq q_2(\gamma^*). \end{cases} \tag{A.64}$$

By (12c) and (15), we have $g'_{\gamma^*}(q_2(\gamma^*)) < 0$. Because g'_{γ^*} is continuous, f'' is piecewise continuous with a single jump at the point $q_2(\gamma^*)$.

Because $f'(x) = f'_{\gamma^*}(x) = g_{\gamma^*}(x)$ for all $x \leq q_2(\gamma^*)$ and g_{γ^*} is a solution to the ODE (8) under γ^*, the pair (γ^*, f) satisfies the condition (20a). Because $g'_{\gamma^*}(x) < 0$ for all $x \geq q_2(\gamma^*)$ by (12c) and (15), we have $f''(x) = 0 \geq g'_{\gamma^*}(x)$ for all $x \geq q_2(\gamma^*)$. Furthermore, because $g_{\gamma^*}(q_2(\gamma^*)) = -r/\mu$ by (14b) and because $g'_{\gamma^*}(x) < 0$ for all $x > q_2(\gamma^*)$ by (12c) and (15), we have $f'(x) = -r/\mu \geq g_{\gamma^*}(x)$ for all $x \geq q_2(\gamma^*)$ by (A.63). Therefore, for all $x > q_2(\gamma^*)$,

$$\frac{1}{2}\Sigma f''(x) + \theta f'(x) + h(x)^+ + b(x)^- \geq \frac{1}{2}\Sigma g'_{\gamma^*}(x) + \theta g_{\gamma^*}(x) + h(x)^+ + b(x)^- = \gamma^*,$$

where the equality follows from the fact that g_{γ^*} is a solution to the ODE (8) under γ^*. Consequently, the pair (γ^*, f) satisfies the condition (7a).

Next, by (A.63) and the fact that $r > p$, $f'(x) = -r/\mu < -p/\mu$ for all $x > q_2(\gamma^*)$. If $x \leq q_2(\gamma^*)$,

$$f'(x) = g_{\gamma^*}(x) \leq \max_{u \in \mathbb{R}} g_{\gamma^*}(u) = \frac{\gamma^* - hq^*}{\theta} \leq \frac{\gamma_p - hq^*}{\theta} = -p/\mu,$$

where the first equality follows from (A.63), the second equality follows from (12d), the second inequality follows from (18), and the last equality follows from (13b). Consequently, the pair (γ^*, f) satisfies the condition (7b).

Let us consider the condition (7c), which holds trivially when $x = y$. Therefore, suppose that $x < y$. Recall that $g_{\gamma^*}(q_1(\gamma^*)) = -r/\mu$ by (14a). By (12c), (15), and (A.63), $f'(x) = g_{\gamma^*}(x) \leq -r/\mu$ for all $x \leq q_1(\gamma^*)$. Therefore,

$$\text{if } x < y \leq q_1(\gamma^*), \text{ then } f(x) - f(y) \geq \frac{r}{\mu}(y - x). \tag{A.65}$$

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Similarly, by (A.63), $f'(x) = -r/\mu$ for all $x \geq q_2(\gamma^*)$. Therefore,

$$\text{if } q_2(\gamma^*) \leq x < y, \text{ then } f(x) - f(y) = \frac{r}{\mu}(y - x). \tag{A.66}$$

Next, let us consider the case in which $q_1(\gamma^*) \leq x < y \leq q_2(\gamma^*)$. By (A.63),

$$f(x) - f(y) = -\int_x^y g_{\gamma^*}(u)\mathrm{d}u.$$

Observe that $f(x) - f(y)$ is the blue area depicted in Figure A.3.2, that is, $f(x) - f(y) = a_1$, where a_1 is the area of the blue region in Figure A.3.2. Similarly, $r(y-x)/\mu = a_1 + a_2 + a_3$ in Figure A.3.2. By (18) and recalling Figure 2b, $c \geq B(\gamma^*) = \sum_{m=2}^{5} a_m$ in Figure A.3.2. Therefore, $r(y-x)/\mu - c \leq a_1 = f(x) - f(y)$. Consequently,

$$\text{if } q_1(\gamma^*) \leq x < y \leq q_2(\gamma^*), \text{ then } f(x) - f(y) \geq \frac{r}{\mu}(y - x) - c. \tag{A.67}$$

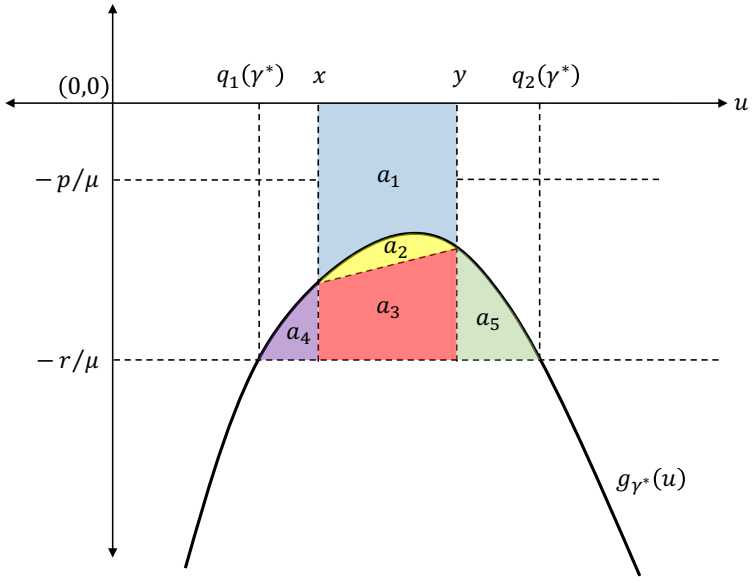


Figure A.3.2 (Color online) Illustration of g_{γ^*} where $\{a_1, a_2, a_3, a_4, a_5\}$ denote the areas depicted with five different colors.

Let us consider the case in which $x \leq q_1(\gamma^*) < y \leq q_2(\gamma^*)$.

$$f(x) - f(y) = (f(x) - f(q_1(\gamma^*))) + (f(q_1(\gamma^*)) - f(y))$$

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$$\geq \frac{r}{\mu}(q_1(\gamma^*) - x) + \frac{r}{\mu}(y - q_1(\gamma^*)) - c$$
$$= \frac{r}{\mu}(y - x) - c,$$

where the inequality follows from (A.65) and (A.67). The proofs for the remaining two cases ($x \leq q_1(\gamma^*) < q_2(\gamma^*) < y$ and $q_1(\gamma^*) < x < q_2(\gamma^*) < y$) are very similar and follow from (A.65), (A.66), and (A.67), thus we skip those proofs. Consequently, the pair (γ^*, f) satisfies the condition (7c).

Let us consider the condition (7d). Let

$$C_1 := \frac{|\gamma^*|}{\theta} + (b+h)\frac{\Sigma}{2\theta^2}\left(1 + \frac{\Sigma}{2\theta}\right) + \frac{r}{\mu}\left(1 + q_2(\gamma^*)\right) + \left|f(q_2(\gamma^*))\right|,$$
$$C_2 := \frac{r}{\mu} + \frac{|\gamma^*|}{\theta} + (b+h)\left(\frac{\Sigma}{2\theta^2} + \frac{1}{2\theta}\right),$$

then condition (7d) holds by (11), (14b), (18), and (19).

Finally, let us consider the condition (20b). By (A.63),

$$f(q_1(\gamma^*)) - f(q_2(\gamma^*)) = -\int_{q_1(\gamma^*)}^{q_2(\gamma^*)} g_{\gamma^*}(x)\mathrm{d}x. \tag{A.68}$$

If $\gamma^* \in (\gamma_r, \gamma_p)$, then $B(\gamma^*) = c$ by (18). Then,

$$\begin{aligned} c &= B(\gamma^*) \\ &= \frac{r}{\mu}\left(q_2(\gamma^*) - q_1(\gamma^*)\right) + \int_{q_1(\gamma^*)}^{q_2(\gamma^*)} g_\gamma(x)\mathrm{d}x \\ &= \frac{r}{\mu}\left(q_2(\gamma^*) - q_1(\gamma^*)\right) + f(q_2(\gamma^*)) - f(q_1(\gamma^*)), \end{aligned} \tag{A.69}$$

where the second equality follows from (16) and the last equality follows from (A.68). Observe that (A.69) implies the condition (20b). □

By (A.64),

$$f'''(x) = \begin{cases} 0, & \text{if } x < 0, \\ -\frac{2(h+b)}{\Sigma}\mathrm{e}^{-\frac{2\theta}{\Sigma}x}, & \text{if } 0 \leq x < q_2(\gamma^*), \\ 0, & \text{if } x \geq q_2(\gamma^*). \end{cases} \tag{A.70}$$

Therefore, f''' is bounded and piecewise smooth. Because f'' is also bounded (recall (A.64)), there exists a $C_3 > 0$ such that

$$|f''(x)| \vee |f'''(x)| \leq C_3, \qquad \forall x \in \mathbb{R}. \tag{A.71}$$

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A.3.3. Taylor's theorem for the relative value function

LEMMA A.3.1. *Consider $f \in \mathbb{C}_1$ defined in (19) and recall f'' and f''' defined in (A.64) and (A.70), respectively. There exists a constant $C_4 > 0$ such that the following inequalities hold.*

1. *For all $(x,y) \in \mathbb{R}^2$ such that $x \neq y$, if $q_2(\gamma^*) \notin (x \wedge y, x \vee y]$, then*

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| \leq C_4 |y-x|^3.$$

2. *For all $(x,y) \in \mathbb{R}^2$ such that $x \neq y$, if $q_2(\gamma^*) \in (x \wedge y, x \vee y]$, then*

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| \leq C_4 \left((y-x)^2 + |y-x|^3 \right).$$

Proof. Observe that the relative value function f is piecewise smooth on the intervals $(-\infty, 0)$, $[0, q_2(\gamma^*))$, and $[q_2(\gamma^*), \infty)$. Specifically, f is a quadratic function on the interval $(-\infty, 0)$ and is linear on the interval $[q_2(\gamma^*), \infty)$. We start by proving the first part of the lemma. There are four cases to consider

Case 1.1: $x \vee y < 0$ or $x \wedge y \geq q_2(\gamma^*)$. By Taylor's theorem and (A.70),

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2} f''(x)(y-x)^2.$$

Case 1.2: $x, y \in [0, q_2(\gamma^*))$. By Taylor's theorem,

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2} f''(x)(y-x)^2 + \frac{1}{6} f'''(a_6)(y-x)^3,$$

where $a_6 \in [x \wedge y, x \vee y]$ is a constant. The desired result follows from (A.71).

Case 1.3: $x < 0$ and $y \in [0, q_2(\gamma^*))$. By Taylor's theorem,

$$f(y) = f(0) + f'(0)y + \frac{1}{2} f''(0)y^2 + \frac{1}{6} f'''(a_7)y^3, \tag{A.72}$$

where $a_7 \in [0, y]$ is a constant. By (A.70) and the fact that $f \in \mathbb{C}_1$ and f'' is constant on $(-\infty, 0]$,

$$f(0) = f(x) - f'(x)x + \frac{1}{2} f''(x)x^2, \tag{A.73a}$$

$$f'(0) = f'(x) - f''(x)x, \tag{A.73b}$$

$$f''(0) = f''(x). \tag{A.73c}$$

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By plugging the equalities in (A.73) into (A.72), we obtain

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 + \frac{1}{6}f'''(a_7)y^3.$$

Finally, the desired result follows from (A.71).

Case 1.4: $y < 0$ *and* $x \in [0, q_2(\gamma^*))$. Because f'' is continuous at 0 and by (A.70) and Taylor's theorem,

$$f(y) = f(0) + f'(0)y + \frac{1}{2}f''(0)y^2. \tag{A.74}$$

Furthermore, by Taylor's theorem

$$f(0) = f(x) - f'(x)x + \frac{1}{2}f''(x)x^2 - \frac{1}{6}f'''(a_8)x^3, \tag{A.75a}$$

$$f'(0) = f'(x) - f''(x)x + \frac{1}{2}f'''(a_9)x^2, \tag{A.75b}$$

$$f''(0) = f''(x) - f'''(a_{10})x, \tag{A.75c}$$

where a_8, a_9, a_{10} are constants on the interval $[0, x]$. By plugging the equalities in (A.75) into the RHS of the equality (A.74), we obtain

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 - \frac{1}{6}f'''(a_8)x^3 + \frac{1}{2}f'''(a_9)x^2y - \frac{1}{2}f'''(a_{10})xy^2.$$

Finally, the desired result follows from (A.71) and the fact that $|x^3| \vee |x^2y| \vee |xy^2| \leq |y-x|^3$.

Next, we will prove the second part of the lemma. There are four cases to consider.

Case 2.1: $x \in [0, q_2(\gamma^*))$ *and* $y \geq q_2(\gamma^*)$. By Taylor's theorem and because $f''(q_2(\gamma^*)) = 0$,

$$f(y) = f(q_2(\gamma^*)) + f'(q_2(\gamma^*))(y - q_2(\gamma^*)). \tag{A.76}$$

Let us fix an arbitrary $\epsilon \in (0, q_2(\gamma^*)]$. By Taylor's theorem and (A.71),

$$\left| f(q_2(\gamma^*) - \epsilon) - f(x) - f'(x)(q_2(\gamma^*) - \epsilon - x) - \frac{1}{2}f''(x)(q_2(\gamma^*) - \epsilon - x)^2 \right| \leq \frac{C_3}{6}\left(q_2(\gamma^*) - \epsilon - x\right)^3,$$

$$|f'(q_2(\gamma^*) - \epsilon) - f'(x) - f''(x)(q_2(\gamma^*) - \epsilon - x)| \leq \frac{C_3}{2}\left(q_2(\gamma^*) - \epsilon - x\right)^2.$$

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By letting $\epsilon \downarrow 0$ in the inequalities above and because $f \in \mathbb{C}_1$, we have

$$\left| f(q_2(\gamma^*)) - f(x) - f'(x)(q_2(\gamma^*) - x) - \frac{1}{2} f''(x)(q_2(\gamma^*) - x)^2 \right| \leq \frac{C_3}{6} \left(q_2(\gamma^*) - x\right)^3, \tag{A.77a}$$

$$|f'(q_2(\gamma^*)) - f'(x) - f''(x)(q_2(\gamma^*) - x)| \leq \frac{C_3}{2} \left(q_2(\gamma^*) - x\right)^2. \tag{A.77b}$$

By (A.76) and (A.77),

$$\begin{aligned} f(y) &\leq f(x) + f'(x)(q_2(\gamma^*) - x) + \frac{1}{2} f''(x)(q_2(\gamma^*) - x)^2 + \frac{C_3}{6} \left(q_2(\gamma^*) - x\right)^3 \\ &\quad + f'(x)(y - q_2(\gamma^*)) + f''(x)(q_2(\gamma^*) - x)(y - q_2(\gamma^*)) + \frac{C_3}{2} \left(q_2(\gamma^*) - x\right)^2 (y - q_2(\gamma^*)) \\ &= f(x) + f'(x)(y - x) + \frac{1}{2} f''(x)(y - x)^2 - \frac{1}{2} f''(x)(y - q_2(\gamma^*))^2 + \frac{C_3}{6} \left(q_2(\gamma^*) - x\right)^3 \\ &\quad + \frac{C_3}{2} \left(q_2(\gamma^*) - x\right)^2 (y - q_2(\gamma^*)) \\ &\leq f(x) + f'(x)(y - x) + \frac{1}{2} f''(x)(y - x)^2 + \frac{2}{3} C_3 \left((y - x)^2 + (y - x)^3\right). \end{aligned}$$

Similarly, we can derive that

$$f(y) \geq f(x) + f'(x)(y - x) + \frac{1}{2} f''(x)(y - x)^2 - \frac{2}{3} C_3 \left((y - x)^2 + (y - x)^3\right).$$

Consequently, we have

$$\left| f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} f''(x)(y - x)^2 \right| \leq \frac{2}{3} C_3 \left((y - x)^2 + (y - x)^3\right).$$

Case 2.2: $x \geq q_2(\gamma^*)$ *and* $y \in [0, q_2(\gamma^*))$. Let us fix an arbitrary $\epsilon \in (0, q_2(\gamma^*) - y]$. By Taylor's theorem,

$$f(y) = f(q_2(\gamma^*) - \epsilon) + f'(q_2(\gamma^*) - \epsilon)(y - q_2(\gamma^*) + \epsilon) + \frac{1}{2} f''(a_{11}(\epsilon))(y - q_2(\gamma^*) + \epsilon)^2,$$

where $a_{11}(\epsilon) \in [y, q_2(\gamma^*) - \epsilon]$ is a constant. Because $f \in \mathbb{C}_1$ and by (A.71) and letting $\epsilon \downarrow 0$ in the equality above, we obtain

$$|f(y) - f(q_2(\gamma^*)) - f'(q_2(\gamma^*))(y - q_2(\gamma^*))| \leq \frac{1}{2} C_3 (y - q_2(\gamma^*))^2. \tag{A.78}$$

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Furthermore, because f is linear on the interval $[q_2(\gamma^*), \infty)$,

$$f(q_2(\gamma^*)) = f(x) + f'(x)(q_2(\gamma^*) - x), \tag{A.79a}$$

$$f'(q_2(\gamma^*)) = f'(x), \tag{A.79b}$$

$$f''(q_2(\gamma^*)) = f''(x) = 0. \tag{A.79c}$$

By plugging the equalities in (A.79) into the LHS of the inequality (A.78) and because $(y - q_2(\gamma^*))^2 \leq (y-x)^2$, we obtain

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| \leq \frac{1}{2} C_3 (y-x)^2.$$

Case 2.3: $x < 0$ *and* $y \geq q_2(\gamma^*)$. By Case 2.1,

$$\left| f(y) - f(0) - f'(0)y - \frac{1}{2} f''(0) y^2 \right| \leq \frac{2}{3} C_3 \left(y^2 + y^3 \right). \tag{A.80}$$

The desired result follows from plugging the equalities in (A.73) into the LHS of the inequality (A.80).

Case 2.4: $x \geq q_2(\gamma^*)$ *and* $y < 0$. Let us fix an arbitrary $\epsilon \in (0, q_2(\gamma^*)]$. By Taylor's theorem,

$$f'(0) = f'(q_2(\gamma^*) - \epsilon) - f''(q_2(\gamma^*) - \epsilon)(q_2(\gamma^*) - \epsilon) + \frac{1}{2} f'''(a_{12}(\epsilon))(q_2(\gamma^*) - \epsilon)^2,$$

where $a_{12}(\epsilon) \in [0, q_2(\gamma^*) - \epsilon]$ is a constant. Therefore, by (A.71),

$$|f'(0) - f'(q_2(\gamma^*) - \epsilon)| \leq C_3 \left((q_2(\gamma^*) - \epsilon) + (q_2(\gamma^*) - \epsilon)^2 \right).$$

Because $f'(x) = f'(q_2(\gamma^*))$, $x \geq q_2(\gamma^*)$, and f' is continuous, as $\epsilon \downarrow 0$, we obtain

$$|f'(0) - f'(x)| \leq C_3 (x + x^2). \tag{A.81}$$

Because $f''(x) = 0$ and by Case 2.2,

$$|f(0) - f(x) + f'(x)x| \leq \frac{1}{2} C_3 x^2. \tag{A.82}$$

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By (A.74), (A.81), (A.82), and because $f''(x)=0$, we have

$$\begin{aligned} f(y) &\le f(x) - f'(x)x + \frac{1}{2}C_3x^2 + f'(x)y - C_3(xy+x^2y) + \frac{1}{2}f''(0)y^2 \\ &= f(x) + f'(x)(y-x) + \frac{1}{2}C_3x^2 - C_3(xy+x^2y) + \frac{1}{2}f''(0)y^2. \end{aligned}$$

Because $f''(x)=0$, $x^2 \vee y^2 \vee |xy| \le (y-x)^2$, and $x^2|y| \le |y-x|^3$, and by (A.71),

$$f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(x-y)^2 \le 2C_3\left((y-x)^2 + |y-x|^3\right).$$

Similarly, we can derive the associated lower bound and thus the desired result follows. □

A.3.4. Proof of Theorem 1

First, we will present a preliminary result, which will be useful in the proof of the first part of the theorem. Then, we will present the proof of the second part of the theorem for convenience. Finally, we will present the proof of the first part of the theorem.

A path-wise lower bound on the inventory process under the control band policy. Let (q_1,q_2) be an arbitrary pair in $\mathbb{R}^2$ such that $q_1<q_2$. Let x be an arbitrary function in $\mathbb{C}$ such that $x(0)=q_1$ and $\mathbb{C}$ is the space of continuous functions. Let $(z,\ell)\in\mathbb{C}^2$ be the unique process pair which satisfies the conditions

C1. $z(t)=x(t)-\ell(t)\le q_1$ for all $t\in\mathbb{R}_+$,

C2. $\ell(0)=0$ and ℓ is nondecreasing,

C3. $\int_0^\infty (q_1-z(t))\mathrm{d}\ell(t)=0$.

Then, $(z,\ell)\in\mathbb{C}^2$ is the conventional one-sided regulator mapping (see chapter 2.2 of Harrison (2013)) where the barrier level q_1 is above (rather than below) the process z. Whenever the process z hits the barrier level q_1, the regulator process ℓ increases to keep the process z below the barrier level with minimal effort. Observe that z is the barrier policy with the upper barrier level q_1.

Next, let $\tilde{q}\in\mathbb{D}$ denote the control-band policy associated with x with the parameters (q_1,q_2). Specifically, $\tilde{q}$ follows the process x until it reaches q_2. At that time, the process $\tilde{q}$ makes a downward jump to the level q_1 and keeps following the process x. Let $N(t)$ denote the cumulative number

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of times that the process $\tilde{q}$ makes downward jumps on the interval $[0,t]$ for all $t \in \mathbb{R}_+$. Because $x(0) = q_1 < q_2$, we have $N(0) = 0$. Furthermore, N is a nondecreasing pure-jump process and

$$\tilde{q}(t) = x(t) - (q_2 - q_1)N(t), \qquad \forall t \in \mathbb{R}_+.$$

LEMMA A.3.2. *We have $\tilde{q}(t) \geq z(t)$ for all $t \in \mathbb{R}_+$, where $\tilde{q}$ and z are defined above.*

Proof. We will use the proof by contradiction technique. Suppose that the lemma statement is incorrect and let

$$t_1 := \inf\{t \geq 0 : \tilde{q}(t) < z(t)\}.$$

There are two cases to consider. First, suppose that $\tilde{q}$ makes a downward jump at time t_1. Then, $\tilde{q}(t_1-) = q_2$ and $\tilde{q}(t_1) = q_1$. Because $z(t) \leq q_1$ for all $t \in \mathbb{R}_+$, it must be the case that $z(t_1) = q_1$ and $\tilde{q}(t_1+) < z(t_1+)$. Then,

$$\begin{aligned} & x(t_1) - (q_2 - q_1)N(t_1) = \tilde{q}(t_1) = q_1 = z(t_1) = x(t_1) - \ell(t_1), \\ \implies & (q_2 - q_1)N(t_1) = \ell(t_1). \end{aligned} \tag{A.83}$$

Because N is a pure jump process and $x \in \mathbb{C}$, $N(t_1+) = N(t_1)$. Because ℓ is nondecreasing,

$$\begin{aligned} \tilde{q}(t_1+) = x(t_1+) - (q_2 - q_1)N(t_1+) = x(t_1+) - (q_2 - q_1)N(t_1) &= x(t_1+) - \ell(t_1) \\ &\geq x(t_1+) - \ell(t_1+) \\ &= z(t_1+), \end{aligned} \tag{A.84}$$

which is a contradiction.

Second, suppose that $\tilde{q}$ is continuous at time t_1. Because N is a pure jump process, $N(t_1+) = N(t_1)$. Because $z \in \mathbb{C}$, it must be the case that $\tilde{q}(t_1) = z(t_1)$ and $\tilde{q}(t_1+) < z(t_1+)$. Then, the equality in (A.83) as well as the inequality in (A.84) hold, which creates a contradiction. □

Next, we will present the proof of the second part of Theorem 1.

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Case 2: $B(\gamma_p) \le c$. We have $\gamma^* = \gamma_p$ by (18). Consider the barrier policy with the barrier level q^* and the initial inventory level $Q(0) = q^*$. We will prove that the objective function value under the aforementioned barrier policy is equal to γ^*, which will give us the desired result by Lemmas 2 and 3. Because $F(t) = 0$ for all $t \in \mathbb{R}_+$, we have

$$\mu \mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\mathrm{d}F(s)\right] = \mathbf{E}\left[\sum_{0<s\le t:\, \Delta F(s)>0} (c - r\Delta F(s))\right] = 0, \qquad \forall t \in \mathbb{R}_{++}.$$

Furthermore, because $Q(0) = q^*$ and by the definition of the regulator component of the one-dimensional reflection mapping (recall (21)), the process I is continuous, that is, $\Delta I(t) = 0$ for all $t \in \mathbb{R}_{++}$. Therefore, under the barrier policy, the inventory process Q is continuous by (6b) and the equality (A.59) becomes equivalent to

$$\begin{aligned}\mathbf{E}\left[f(Q(t))\right] - \mathbf{E}\left[f(Q(0))\right] = \frac{1}{2}\Sigma \mathbf{E}\left[\int_{0_+}^{t} f''(Q(s-))\mathrm{d}s\right] + \theta \mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\mathrm{d}s\right] \\ - \mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\mu \mathrm{d}I(s)\right]. \qquad \text{(A.85)}\end{aligned}$$

Then,

$$-\mathbf{E}\left[\int_{0_+}^{t} f'(Q(s-))\mu \mathrm{d}I(s)\right] = -\mathbf{E}\left[\int_{0_+}^{t} f'(q^*)\mu \mathrm{d}I(s)\right] = -f'(q^*)\mu \mathbf{E}\left[I(t)\right] = p\mathbf{E}\left[I(t)\right], \qquad \text{(A.86)}$$

where the first equality follows from the fact that the idle time process increases only when the inventory process hits the barrier level q^* and the last equality follows from the fact that $f'(q^*) = g_{\gamma_p}(q^*) = -p/\mu$ by (12) and (13b).

Because the inventory level never exceeds q^* and $q^* < q_2(\gamma^*)$ by (15), and by (20a), (A.85), and (A.86), we have

$$\mathbf{E}\left[f(Q(t))\right] - \mathbf{E}\left[f(Q(0))\right] = \gamma^* t - \mathbf{E}\left[\int_{0_+}^{t} \left(h\left(Q(s-)\right)^+ + b\left(Q(s-)\right)^-\right)\mathrm{d}s - pI(t)\right]. \qquad \text{(A.87)}$$

Next, let us consider the LHS of the equality (A.87). Because $Q(0) = q^*$, we have

$$\frac{1}{t}\mathbf{E}\left[f(Q(0))\right] = \frac{1}{t}\mathbf{E}\left[f(q^*)\right] \to 0 \quad \text{as } t \to \infty. \qquad \text{(A.88)}$$

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By the Jensen's inequality and (7d), for all $t \in \mathbb{R}_+$,

$$\frac{1}{t}\left|\mathbf{E}\left[f(Q(t))\right]\right| \leq \frac{1}{t}\mathbf{E}\left[\left|f(Q(t))\right|\right] \leq \frac{1}{t}\left(C_1 + C_2\mathbf{E}\left[\left(Q(t)\right)^2\right]\right). \tag{A.89}$$

Observe that the inventory process Q behaves like a Brownian motion with drift θ and variance Σ which is reflected from above at the barrier level q^*. Let W be a one-dimensional reflected Brownian motion (RBM) starting from 0 with drift $(-\theta)$ and variance Σ (see section 1.10 of Harrison (2013)). Then, $Q \overset{d}{=} q^* - W$, where $\overset{d}{=}$ denotes "equal in distribution". Therefore,

$$\mathbf{E}\left[\left(Q(t)\right)^2\right] = \mathbf{E}\left[\left(q^* - W(t)\right)^2\right] = \left(q^*\right)^2 - 2q^*\mathbf{E}\left[W(t)\right] + \mathbf{E}\left[\left(W(t)\right)^2\right], \qquad \forall t \in \mathbb{R}_+. \tag{A.90}$$

By equation (1.51) of Harrison (2013),

$$\mathbf{P}\left(W(t) \leq w\right) = \phi\left(\frac{w+\theta t}{\sqrt{\Sigma t}}\right) - \mathrm{e}^{\frac{-2\theta w}{\Sigma}}\phi\left(\frac{-w+\theta t}{\sqrt{\Sigma t}}\right), \qquad \forall w \in \mathbb{R}_+,\, t \in \mathbb{R}_{++}. \tag{A.91}$$

By (A.91) and algebra, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
\mathbf{E}\left[W(t)\right] &= \left(\frac{\Sigma}{2\theta} + \frac{\theta t}{2}\right)\mathrm{Erf}\left[\theta\sqrt{\frac{t}{2\Sigma}}\right] + \mathrm{e}^{-\frac{\theta^2 t}{2\Sigma}}\sqrt{\frac{\Sigma t}{2\pi}} - \frac{\theta t}{2}, \\
\mathbf{E}\left[\left(W(t)\right)^2\right] &= \Sigma t + \frac{\theta^2 t^2}{2} - \frac{\sqrt{\Sigma t}\left(\Sigma + \theta^2 t\right)}{\theta\sqrt{2\pi}}\mathrm{e}^{-\frac{\theta^2 t}{2\Sigma}} - \left(-\frac{\Sigma^2}{2\theta^2} + \frac{\theta^2 t^2}{2} + \Sigma t\right)\mathrm{Erf}\left[\theta\sqrt{\frac{t}{2\Sigma}}\right],
\end{aligned}$$

where Erf[·] is the Gauss error function, that is,

$$\mathrm{Erf}[x] = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-s^2}\mathrm{d}s, \qquad \forall x \in \mathbb{R}.$$

Because $\lim_{x\to\infty}\mathrm{Erf}[x] = 1$, we have

$$\frac{1}{t}\mathbf{E}\left[W(t)\right] = \left(\frac{\Sigma}{2\theta t} + \frac{\theta}{2}\right)\mathrm{Erf}\left[\theta\sqrt{\frac{t}{2\Sigma}}\right] + \mathrm{e}^{-\frac{\theta^2 t}{2\Sigma}}\sqrt{\frac{\Sigma}{2\pi t}} - \frac{\theta}{2} \to 0 \quad \text{as } t \to \infty. \tag{A.92}$$

Furthermore, because $\lim_{x\to\infty}\left[x^2\left(1 - \mathrm{Erf}[x]\right)\right] = 0$,

$$\begin{aligned}
\frac{1}{t}\mathbf{E}\left[\left(W(t)\right)^2\right] &= \Sigma + \frac{\theta^2 t}{2} - \frac{\sqrt{\Sigma}\left(\Sigma + \theta^2 t\right)}{\theta\sqrt{2\pi t}}\mathrm{e}^{-\frac{\theta^2 t}{2\Sigma}} - \left(-\frac{\Sigma^2}{2\theta^2 t} + \frac{\theta^2 t}{2} + \Sigma\right)\mathrm{Erf}\left[\theta\sqrt{\frac{t}{2\Sigma}}\right] \\
&\to 0 \quad \text{as } t \to \infty.
\end{aligned} \tag{A.93}$$

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Therefore, by (A.90), (A.92), and (A.93), we have

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[(Q(t))^2\right]=0. \tag{A.94}$$

By (A.88), (A.89), (A.94), dividing both sides of the equality (A.87) to t, and letting $t\to\infty$,

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[-pI(t)+\int_{0_+}^{t}\left(h\left(Q(s-)\right)^{+}+b\left(Q(s-)\right)^{-}\right)\mathrm{d}s\right]=\gamma^*.$$

Finally, observe that the barrier policy is a feasible solution to the BCP (6).

Case 1: $B(\gamma_p)>c$. We have $\gamma^*\in(\gamma_r,\gamma_p)$ by (18). Consider the control-band policy with the parameters $(q_1(\gamma^*),q_2(\gamma^*))$ and the initial inventory level $Q(0)=q_1(\gamma^*)$. We will prove that the objective function value under the aforementioned control-band policy is equal to γ^*. Because γ^* is a lower bound on the optimal objective function value of the BCP (6) by Lemmas 2 and 3, we will obtain the desired result. Because $I(t)=0$ for all $t\in\mathbb{R}_+$, we have

$$\mu\mathbf{E}\left[\int_{0_+}^{t}f'(Q(s-))\mathrm{d}I(s)\right]=p\mathbf{E}\left[I(t)\right]=0,\qquad\forall t\in\mathbb{R}_{++}.$$

Therefore, under the control-band policy, the equality (A.59) becomes equivalent to

$$\begin{aligned}\mathbf{E}\left[f(Q(t))\right]-\mathbf{E}\left[f(Q(0))\right]=\frac{1}{2}\Sigma\mathbf{E}\left[\int_{0_+}^{t}f''(Q(s-))\mathrm{d}s\right]+\theta\mathbf{E}\left[\int_{0_+}^{t}f'(Q(s-))\mathrm{d}s\right]\\-\mathbf{E}\left[\int_{0_+}^{t}f'(Q(s-))\mu\mathrm{d}F(s)\right]\\+\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\Big(\Delta f(Q(s))-f'(Q(s-))\Delta Q(s)\Big)\right].\end{aligned} \tag{A.95}$$

Then,

$$\begin{aligned}-\mathbf{E}\left[\int_{0_+}^{t}f'(Q(s-))\mu\mathrm{d}F(s)\right]+\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\Big(\Delta f(Q(s))-f'(Q(s-))\Delta Q(s)\Big)\right]\\=\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\Delta f(Q(s))\right]\\=\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\Big(f(q_1(\gamma^*))-f(q_2(\gamma^*))\Big)\right]\\=\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\left(\frac{r}{\mu}\left(q_2(\gamma^*)-q_1(\gamma^*)\right)-c\right)\right]\\=\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\left(r\Delta F(s)-c\right)\right],\end{aligned} \tag{A.96}$$

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where the first equality follows from (A.61), the second equality follows from the definition of the control-band policy, the third equality follows from (20b), and the last equality follows from the definition of the control-band policy.

Because the inventory level never exceeds $q_2(\gamma^*)$, and by (20a), (A.95), and (A.96), we have

$$\begin{aligned}&\mathbf{E}\left[f(Q(t))\right]-\mathbf{E}\left[f(Q(0))\right]\\&\quad=\gamma^* t-\mathbf{E}\left[\int_{0_+}^{t}\left(h\left(Q(s-)\right)^+ + b\left(Q(s-)\right)^-\right)\mathrm{d}s+\sum_{0<s\le t:\,\Delta F(s)>0}\left(c-r\Delta F(s)\right)\right].\end{aligned}\tag{A.97}$$

Next, let us consider the LHS of the equality (A.97). Because $Q(0)=q_1(\gamma^*)$, we have

$$\frac{1}{t}\mathbf{E}\left[f(Q(0))\right]=\frac{1}{t}\mathbf{E}\left[f(q_1(\gamma^*))\right]\to 0\quad\text{as } t\to\infty.\tag{A.98}$$

Let Z be the barrier policy with the upper barrier level $q_1(\gamma^*)$ and the initial inventory level $Z(0)=q_1(\gamma^*)$. By Lemma A.3.2, we have $Q(t)\ge Z(t)$ for all $t\in\mathbb{R}_+$. Because $Q(t)\le q_2(\gamma^*)$ for all $t\in\mathbb{R}_+$,

$$\left(Q(t)\right)^2\le\left(q_2(\gamma^*)\right)^2+\left(Z(t)\right)^2,\qquad\forall t\in\mathbb{R}_+.\tag{A.99}$$

Therefore,

$$\begin{aligned}\frac{1}{t}\left|\mathbf{E}\left[f(Q(t))\right]\right|&\le\frac{1}{t}\mathbf{E}\left[\left|f(Q(t))\right|\right]\\&\le\frac{1}{t}\left(C_1+C_2\mathbf{E}\left[\left(Q(t)\right)^2\right]\right)\\&\le\frac{1}{t}\left(C_1+C_2\left(q_2(\gamma^*)\right)^2+C_2\mathbf{E}\left[\left(Z(t)\right)^2\right]\right)\to 0\quad\text{as } t\to\infty,\end{aligned}\tag{A.100}$$

where the first inequality follows from the Jensen's inequality, the second inequality follows from (7d), the last inequality follows from (A.99), and the convergence result follows from (A.94), where we consider the barrier policy with a different upper barrier level.

By (A.98), (A.100), dividing both sides of the equality (A.97) to t, and letting $t\to\infty$,

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,\Delta F(s)>0}\left(c-r\Delta F(s)\right)+\int_{0_+}^{t}\left(h\left(Q(s-)\right)^+ + b\left(Q(s-)\right)^-\right)\mathrm{d}s\right]=\gamma^*.$$

Finally, observe that the control-band policy is a feasible solution to the BCP (6).

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A.4. Proof of Theorem 2

Let us fix an arbitrary sequence of admissible policies denoted by $\pi = \{\pi^n, n \in \mathbb{N}_+\}$ such that without loss of generality, π^n is of idling, turning-off, or hybrid type for all $n \in \mathbb{N}_+$ (recall the second part of Lemma 1). Let π_i be an arbitrary subsequence of π consisting of idling policies, π_{off} be an arbitrary subsequence of π consisting of turning-off policies, and π_h be an arbitrary subsequence of π consisting of hybrid policies. Without loss of generality, by Lemma A.2.7, we assume that

$$\max\left\{\limsup_{n\to\infty}\left(|\hat{q}_1^{\pi_i,n}| \vee |\hat{q}_2^{\pi_i,n}|\right), \limsup_{n\to\infty}\left(|\hat{q}_1^{\pi_{\text{off}},n}| \vee |\hat{q}_2^{\pi_{\text{off}},n}|\right), \limsup_{n\to\infty}\left(|\hat{q}_0^{\pi_h,n}| \vee |\hat{q}_2^{\pi_h,n}|\right)\right\} < \infty, \tag{A.101a}$$

$$\min\left\{\liminf_{n\to\infty}\left(\hat{q}_2^{\pi_{\text{off}},n} - \hat{q}_1^{\pi_{\text{off}},n}\right), \liminf_{n\to\infty}\left(\hat{q}_2^{\pi_h,n} - \hat{q}_0^{\pi_h,n}\right)\right\} > 0, \tag{A.101b}$$

The reason is that the long-run average cost under a subsequence violating (A.101) tends to infinity as $n \to \infty$ by Lemma A.2.7. We omit the policy notation π for convenience henceforth.

By recalling (1), let us define the natural filtration $\mathbb{F}^n := \{\mathcal{F}^n(t), t \in \mathbb{R}_+\}$ where

$$\mathcal{F}^n(t) := \sigma\left\{\mathfrak{X}^n(0), A^n(s), T^n(s), I^n(s), F^n(s), U^n(s), S \circ T^n(s),\, 0 \le s \le t\right\}, \quad \forall t \in \mathbb{R}_+,\, n \in \mathbb{N}_+,$$

such that $\mathcal{F}^n(t)$ is augmented by all $\mathbf{P}$-null sets in the complete probability space $(\Omega, \mathcal{F}^n(\infty), \mathbf{P})$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. Therefore, $\mathcal{F}^n(t)$ is the information learned up to time t. Let $e \in \mathbb{D}$ be such that $e(t) := t$ for all $t \in \mathbb{R}_+$. Then, $A^n - \lambda^n e$ is a martingale with respect to the filtration $\mathbb{F}^n$ (see theorem I.24 of Protter (2005)). Similarly, $S \circ T^n - \mu T^n$ is a martingale with respect to the filtration $\mathbb{F}^n$ (see Pang et al. (2007)). Let $\hat{\mathbb{F}}^n := \{\mathcal{F}^n(nt), t \in \mathbb{R}_+\}$ for all $n \in \mathbb{N}_+$. Then, the diffusion-scaled process $\hat{X}^n$ (recall (3)) is a martingale with respect to the filtration $\hat{\mathbb{F}}^n$. Therefore, by (3) and (4), $\hat{Q}^n$ is a semimartingale with respect to the filtration $\hat{\mathbb{F}}^n$ (see definition V.5.18 of Çınlar (2011)). Furthermore, $\hat{X}^n$ is a square-integrable càdlàg martingale and a local martingale (see pages 12 and 37 of Protter (2005), respectively), and $\hat{Q}^n$ is a quadratic pure jump semimartingale for all $n \in \mathbb{N}_+$ by theorem II.26 of Protter (2005).

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By using the extant second derivative Meyer-Ito formula (see theorem IV.71 of Protter (2005)) with the function f defined in (19) and the quadratic pure jump semimartingale $\hat{Q}^n$ (see page 71 of Protter (2005)), for all $t \in \mathbb{R}_{++}$ and $n \in \mathbb{N}_+$,

$$f\big(\hat{Q}^n(t)\big) - f\big(\hat{Q}^n(0)\big) = \int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{Q}^n(s) + \sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} \Big(\Delta f\big(\hat{Q}^n(s)\big) - f'\big(\hat{Q}^n(s-)\big)\Delta\hat{Q}^n(s)\Big). \tag{A.102}$$

For all $t \in \mathbb{R}_{++}$ and $n \geq n_0$, where n_0 is defined in Lemma A.2.3,

$$\mathbf{E}\left[\left|f\big(\hat{Q}^n(t)\big)\right|\right] = \mathbf{E}\left[\left|f\big(\hat{Q}^n(0)\big)\right|\right] \leq C_1 + C_2\mathbf{E}\left[\left(\hat{Q}^n(0)\right)^2\right] = C_1 + \frac{C_2}{n}\mathbf{E}\left[\left(Q^n(0)\right)^2\right] < \infty,$$

where the first equality follows from (A.28), the first inequality follows from (7d), and the strict inequality follows from Lemma A.2.4. Therefore, by (A.28), for all $t \in \mathbb{R}_{++}$ and $n \geq n_0$,

$$\mathbf{E}\left[f\big(\hat{Q}^n(t)\big) - f\big(\hat{Q}^n(0)\big)\right] = 0. \tag{A.103}$$

By (4), for all $t \in \mathbb{R}_{++}$ and $n \in \mathbb{N}_+$,

$$\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{Q}^n(s)\right] = \mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{X}^n(s)\right] + \theta\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] - \mu\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{I}^n(s) + \mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right)\right]. \tag{A.104}$$

The following lemma states that we can ignore the first expectation in the RHS of (A.104).

Lemma A.4.1. *We have*

$$\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{X}^n(s)\right] = 0, \qquad \forall n \in \mathbb{N}_+,\, t \in \mathbb{R}_{++}.$$

The proof of Lemma A.4.1 is presented in Section A.4.1.

By Taylor's theorem (recall Lemma A.3.1), for all $t \in \mathbb{R}_{++}$ and $n \in \mathbb{N}_+$,

$$\Delta f\big(\hat{Q}^n(t)\big) = f'\big(\hat{Q}^n(t-)\big)\Delta\hat{Q}^n(t) + \frac{1}{2}f''\big(\hat{Q}^n(t-)\big)\left(\Delta\hat{Q}^n(t)\right)^2 + R^n(t), \tag{A.105}$$

where $R^n : \mathbb{R}_+ \to \mathbb{R}$ denotes the remainder term.

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By (A.102), (A.103), (A.104), (A.105), and Lemma A.4.1, for all $t \in \mathbb{R}_{++}$ and $n \geq n_0$,

$$\begin{aligned}
0 = {} & \frac{\theta}{t}\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] - \frac{\mu}{t}\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{I}^n(s)+\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right] \\
& + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2\right] + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} R^n(s)\right] \\
= {} & \frac{\mu}{t}\mathbf{E}\left[\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] + \frac{\theta}{t}\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] \\
& - \frac{\mu}{t}\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{I}^n(s)+\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right] \\
& + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] \\
& + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} R^n(s)\right]. \qquad \text{(A.106)}
\end{aligned}$$

By (A.106), (7a), (7b), Lemma 3, and the fact that $\hat{I}^n(0) = 0$, for all $t \in \mathbb{R}_{++}$ and $n \geq n_0$,

$$\begin{aligned}
0 \geq {} & \gamma^* - \frac{1}{t}\mathbf{E}\left[\int_{0+}^{t}\left(h\left(\hat{Q}^n(s-)\right)^+ + b\left(\hat{Q}^n(s-)\right)^-\right)\mathrm{d}s\right] + \frac{p}{t}\mathbf{E}\left[\hat{I}^n(t)\right] \\
& - \frac{\mu}{t}\mathbf{E}\left[\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right] \\
& + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] \\
& + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} R^n(s)\right]. \qquad \text{(A.107)}
\end{aligned}$$

By (A.107), for all $t \in \mathbb{R}_{++}$ and $n \geq n_0$,

$$\begin{aligned}
& \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - p\hat{I}^n(t) + \int_{0+}^{t}\left(h\left(\hat{Q}^n(s-)\right)^+ + b\left(\hat{Q}^n(s-)\right)^-\right)\mathrm{d}s\right] \\
& \quad \geq \gamma^* + \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - \mu\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right] \qquad \text{(A.108a)} \\
& \qquad + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] \qquad \text{(A.108b)} \\
& \qquad + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\,|\Delta\hat{Q}^n(s)|>0} R^n(s)\right]. \qquad \text{(A.108c)}
\end{aligned}$$

The following lemmas state that the terms in (A.108b) and (A.108c) converge to zero in the limit and the second term in (A.108a) is nonnegative in the limit.

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LEMMA A.4.2. *We have*

$$\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}|R^n(s)|\right]=0.$$

The proof of Lemma A.4.2 is presented in Section A.4.2.

LEMMA A.4.3. *We have*

$$\lim_{n\to\infty}\limsup_{t\to\infty}\left|\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}\frac{1}{2}f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2-\mu\int_{0+}^{t}f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right|=0.$$

The proof of Lemma A.4.3 is presented in Section A.4.3.

LEMMA A.4.4. *We have*

$$\liminf_{n\to\infty}\liminf_{t\to\infty}\frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t)-r\hat{F}^n(t)-\mu\int_{0+}^{t}f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right]\ge 0.$$

The proof of Lemma A.4.4 is presented in Section A.4.4.

Finally, by (A.108) and Lemmas A.4.2, A.4.3, and A.4.4, we obtain the desired result.

A.4.1. Proof of Lemma A.4.1

Recall that $\hat{X}^n$ is a square-integrable càdlàg martingale and a local martingale for all $n\in\mathbb{N}_+$. Furthermore, $\{f'(\hat{Q}^n(t-)),t\in\mathbb{R}_{++}\}$ is an adapted càglàd (that is, left continuous with right limits) process. Therefore, the stochastic integral $\int_{0+}^{t}f'(\hat{Q}^n(s-))\mathrm{d}\hat{X}^n(s)$ is a locally square-integrable local martingale by theorem II.20 of Protter (2005). If we can prove that

$$\mathbf{E}\left[\sup_{0<s\le t}\left|\int_{0+}^{s}f'\big(\hat{Q}^n(u-)\big)\mathrm{d}\hat{X}^n(u)\right|\right]<\infty,\qquad\forall n\in\mathbb{N}_+,\,t\in\mathbb{R}_{++},\tag{A.109}$$

then the stochastic integral $\int_{0+}^{t}f'(\hat{Q}^n(s-))\mathrm{d}\hat{X}^n(s)$ becomes a martingale by theorem I.51 of Protter (2005) and the desired result follows. Therefore, we will prove (A.109) henceforth.

First, by (9) and (A.63), there exists strictly positive constants C_5 and C_6 such that

$$|f'(x)|\le C_5+C_6|x|,\qquad\forall x\in\mathbb{R}.\tag{A.110}$$

Let $\tilde{A}^n(t):=A^n(nt)/\sqrt{n}$ and $\tilde{S}^n(t):=S(nt)/\sqrt{n}$ for all $t\in\mathbb{R}_+$ and $n\in\mathbb{N}_+$. Observe that both $\tilde{A}^n$ and $\tilde{S}^n$ are nonnegative and nondecreasing processes for all $n\in\mathbb{N}_+$. Furthermore, $\tilde{A}^n(t)-\sqrt{n}\lambda^n t=\hat{A}^n(t)$ and $\tilde{S}^n\circ\bar{T}^n(t)-\sqrt{n}\mu\bar{T}^n(t)=\hat{S}^n\circ\bar{T}^n(t)$ for all $t\in\mathbb{R}_+$ and $n\in\mathbb{N}_+$.

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By (3) and the fact that $\bar{T}^n(t) \le t$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$, the expectation in (A.109) is less than or equal to

$$\mathbf{E}\left[\sup_{0<s\le t}\int_{0+}^{s}\left|f'\left(\hat{Q}^n(u-)\right)\right|\mathrm{d}\left(\tilde{A}^n(u)+\sqrt{n}(\lambda^n+\mu)u+\tilde{S}^n(u)\right)\right]$$

$$=\mathbf{E}\left[\sup_{0<s\le t}\int_{0+}^{s}\left|f'\left(\hat{Q}^n(u-)\right)\right|\mathrm{d}\tilde{A}^n(u)\right] \tag{A.111a}$$

$$+\mathbf{E}\left[\sup_{0<s\le t}\int_{0+}^{s}\left|f'\left(\hat{Q}^n(u-)\right)\right|\sqrt{n}(\lambda^n+\mu)\mathrm{d}u\right] \tag{A.111b}$$

$$+\mathbf{E}\left[\sup_{0<s\le t}\int_{0+}^{s}\left|f'\left(\hat{Q}^n(u-)\right)\right|\mathrm{d}\tilde{S}^n(u)\right]. \tag{A.111c}$$

By (3) and (4) and the fact that $\bar{T}^n(t) \le t$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$\sup_{0\le s\le t}\left|\hat{Q}^n(s)\right| \le \left|\hat{Q}^n(0)\right| + \tilde{A}^n(t) + \sqrt{n}(\lambda^n+\mu)t + \tilde{S}^n(t) + (\theta+\mu\sqrt{n})t. \tag{A.112}$$

By (A.110), the sum in (A.111a) is less than or equal to

$$\mathbf{E}\left[\sup_{0<s\le t}\int_{0+}^{s}\left(C_5+C_6\left|\hat{Q}^n(u-)\right|\right)\mathrm{d}\tilde{A}^n(u)\right]. \tag{A.113}$$

By (A.112), there exists a constant $C_7 > 0$ such that the sum in (A.113) is less than or equal to

$$C_7\mathbf{E}\left[\tilde{A}^n(t)+\left|\hat{Q}^n(0)\right|\tilde{A}^n(t)+\left(\tilde{A}^n(t)\right)^2+\left(\sqrt{n}\left(\lambda^n+2\mu\right)+\theta\right)t\tilde{A}^n(t)+\tilde{S}^n(t)\tilde{A}^n(t)\right]<\infty,$$

where the strict inequality follows from the fact that $\tilde{A}^n$, $\tilde{S}^n$, and $\hat{Q}^n(0)$ are mutually independent of each other, all moments of the Poisson distribution exist, and all moments of $\hat{Q}^n(0)$ exist by Lemma A.2.4. Similarly, we can prove that the sums in (A.111b) and (A.111c) are both finite, which implies (A.109).

A.4.2. Proof of Lemma A.4.2

Let $\mathcal{J}^n(t)$ denote the set of jump epochs of $\hat{Q}^n$ on the interval $(0,t]$ and let $|\mathcal{J}^n(t)|$ denote the cardinality of $\mathcal{J}^n(t)$. By (3), (4), and because $T^n(t) \le t$, we have

$$\mathbf{E}\left[|J^n(t)|\right] \le \mathbf{E}\left[A^n(nt)+S(nt)\right] = n(\lambda^n+\mu)t \le 2n\mu t, \quad \forall t \in \mathbb{R}_{++},\, n \in \mathbb{N}_+, \tag{A.114}$$

where the last inequality follows from Assumption 1.

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Let

$$\mathcal{J}_1^n(t) := \left\{ s \in \mathcal{J}^n(t) : \hat{Q}^n(s-) \wedge \hat{Q}^n(s) < q_2(\gamma^*) \leq \hat{Q}^n(s-) \vee \hat{Q}^n(s) \right\},$$

that is, $\mathcal{J}_1^n(t)$ is set of jump epochs of $\hat{Q}^n$ on the interval $(0,t]$ at which $\hat{Q}^n$ jumps over the number $q_2(\gamma^*)$. By definition, $\mathcal{J}_1^n(t) \subset \mathcal{J}^n(t)$.

We will derive an upper bound on the expectation of $|\mathcal{J}_1^n(t)|$. Let $\lceil \cdot \rceil$ denote the ceiling operator. By the definition of diffusion scaling, $|\mathcal{J}_1^n(t)|$ is exactly the number of transitions of the inventory process Q^n between the numbers $\lceil \sqrt{n} q_2(\gamma^*) \rceil - 1$ and $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ on the interval $(0, nt]$. Observe that the control policy which stochastically maximizes the aforementioned transitions is the idling policy with the parameters $\lceil \sqrt{n} q_2(\gamma^*) \rceil - 1$ and $\lceil \sqrt{n} q_2(\gamma^*) \rceil$. The CTMC transition diagram of the aforementioned idling policy is presented in Figure A.4.1, where the red arrows represent the transitions whose number we want to maximize. Because the idling policy never allows the inventory level to exceed $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ and makes the server produce nonstop when the inventory level falls below $\lceil \sqrt{n} q_2(\gamma^*) \rceil$, it stochastically maximizes the number of transitions between the states $\lceil \sqrt{n} q_2(\gamma^*) \rceil - 1$ and $\lceil \sqrt{n} q_2(\gamma^*) \rceil$. Let us also assume that $Q^n(0) = \lceil \sqrt{n} q_2(\gamma^*) \rceil$ to stochastically minimize the time until the first transition.

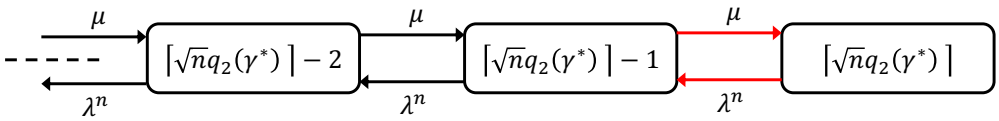


Figure A.4.1 (Color online) CTMC transition diagram of a particular idling policy.

Consider the expected time between revisiting the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$, which has two components. First, the state process spends on average $1/\lambda^n$ amount of time in the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$. Second, there is the average time to return back to the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$. Because the CTMC in Figure A.4.1 is equivalent to the one of an $M/M/1$ queue with the arrival rate λ^n and service rate μ, the expected time to return back to the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ is the expected busy period in the aforementioned $M/M/1$ queue, which is equal to $1/(\mu - \lambda^n) = \sqrt{n}/\theta$. Let $V^n(t)$ denotes the cumulative number of visits to the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ up to time $t \in \mathbb{R}_+$. Because the length of the time intervals between

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revisiting the state $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ are i.i.d. and by the elementary renewal theorem (see for example theorem 3.3.4 of Ross (1996)),

$$\lim_{t\to\infty} \frac{\mathbf{E}\left[V^n(nt)\right]}{nt} = \frac{1}{\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}}.$$

Therefore, there exists a $t_0(n) \in \mathbb{R}_+$ such that if $t \geq t_0(n)$, then

$$\mathbf{E}\left[V^n(nt)\right] \leq \frac{2nt}{\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}}.$$

Because each visit implies at most two transitions between the states $\lceil \sqrt{n} q_2(\gamma^*) \rceil - 1$ and $\lceil \sqrt{n} q_2(\gamma^*) \rceil$ and the initial state is assumed to be $\lceil \sqrt{n} q_2(\gamma^*) \rceil$, if $t \geq t_0(n)$,

$$\mathbf{E}\left[|J_1^n(t)|\right] \leq 2\left(\frac{2nt}{\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}} + 1\right). \tag{A.115}$$

If $t \geq t_0(n)$,

$$\begin{aligned}
\frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} |R^n(s)|\right] &\leq \frac{1}{t}\mathbf{E}\left[\sum_{s\in\mathcal{J}^n(t)\setminus\mathcal{J}_1^n(t)} \frac{C_4}{n\sqrt{n}}\right] + \frac{1}{t}\mathbf{E}\left[\sum_{s\in\mathcal{J}_1^n(t)} C_4\left(\frac{1}{n} + \frac{1}{n\sqrt{n}}\right)\right] \\
&\leq \frac{C_4}{tn\sqrt{n}}\mathbf{E}\left[|\mathcal{J}^n(t)\setminus\mathcal{J}_1^n(t)|\right] + \frac{2C_4}{nt}\mathbf{E}\left[|\mathcal{J}_1^n(t)|\right] \\
&\leq \frac{2\mu C_4}{\sqrt{n}} + \frac{8C_4}{\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}} + \frac{4C_4}{nt},
\end{aligned} \tag{A.116}$$

where the first inequality follows from Lemma A.3.1 and the fact that the magnitude of the jumps of $\hat{Q}^n$ is $1/\sqrt{n}$, and the third inequality follows from (A.114) and (A.115). Finally, the desired convergence result follows from (A.116).

A.4.3. Proof of Lemma A.4.3

Let $\breve{A}^n(t) := A^n(nt)$ and $\breve{S}^n(t) := S(nt)$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. Then, $\breve{A}^n(t) - n\lambda^n t$ and $\breve{S}^n \circ \bar{T}^n(t) - n\mu\bar{T}^n(t)$ are martingales with respect to the filtration $\hat{\mathbb{F}}^n$. For all $t \in \mathbb{R}_{++}$,

$$\begin{aligned}
&\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\left(\Delta\hat{Q}^n(s)\right)^2\right] \\
&\quad = \mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{X}^n(s)|>0} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\left(\Delta\hat{X}^n(s)\right)^2\right] \\
&\quad = \frac{1}{n}\mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\mathrm{d}\breve{A}^n(s) + \int_{0+}^{t} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\mathrm{d}\breve{S}^n \circ \bar{T}^n(s)\right]
\end{aligned}$$

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$$
\begin{aligned}
&= \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\lambda^n \mathrm{d}s + \mu \mathrm{d}\bar{T}^n(s)\right)\right] \\
&= \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\mu - \frac{\theta}{\sqrt{n}}\right)\mathrm{d}s\right] + \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\mu \mathrm{d}s\right] \\
&\qquad\qquad + \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\mu\left(\mathrm{d}\bar{T}^n(s) - \mathrm{d}s\right)\right] \\
&= \mu \mathbf{E}\left[\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right] - \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\frac{\theta}{\sqrt{n}}\mathrm{d}s\right] \\
&\qquad\qquad + \mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\mu\left(\mathrm{d}\bar{T}^n(s) - \mathrm{d}s\right)\right], \qquad \text{(A.117)}
\end{aligned}
$$

where the first and the second equalities follow from the fact that $\hat{Q}^n$ has a jump only when $\hat{X}^n$ has a jump and the magnitude of each jump is $1/\sqrt{n}$ (recall (1c), (3), and (4)); the third equality follows from the martingale property of the associated stochastic integrals (see theorem II.20 of Protter (2005) and recall the proof of Lemma A.4.1); and the fourth equality follows from Assumption 1.

By (A.117), we have

$$
\begin{aligned}
&\left|\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right| \\
&\qquad \le \frac{1}{t}\left|\mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\frac{\theta}{\sqrt{n}}\mathrm{d}s\right]\right| + \frac{1}{2t}\left|\mathbf{E}\left[\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mu\left(\mathrm{d}\bar{T}^n(s) - \mathrm{d}s\right)\right]\right|. \qquad \text{(A.118)}
\end{aligned}
$$

Let us consider the first term in (A.118).

$$
\frac{1}{t}\left|\mathbf{E}\left[\int_{0+}^{t} \frac{1}{2} f''\big(\hat{Q}^n(s-)\big)\frac{\theta}{\sqrt{n}}\mathrm{d}s\right]\right| \le \frac{1}{2t}\mathbf{E}\left[\int_{0+}^{t}\left|f''\big(\hat{Q}^n(s-)\big)\right|\frac{\theta}{\sqrt{n}}\mathrm{d}s\right] \le \frac{C_3\theta}{2\sqrt{n}}, \qquad \text{(A.119)}
$$

where the second inequality follows from (A.71).

Next, let us consider the second term in (A.118).

$$
\begin{aligned}
\frac{1}{2t}\left|\mathbf{E}\left[\int_{0+}^{t} f''\big(\hat{Q}^n(s-)\big)\mu\left(\mathrm{d}\bar{T}^n(s) - \mathrm{d}s\right)\right]\right| &\le \frac{\mu}{2t}\mathbf{E}\left[\int_{0+}^{t}\left|f''\big(\hat{Q}^n(s-)\big)\right|\left(\mathrm{d}s - \mathrm{d}\bar{T}^n(s)\right)\right] \\
&\le \frac{\mu}{2t}\mathbf{E}\left[\int_{0+}^{t} C_3\left(\mathrm{d}s - \mathrm{d}\bar{T}^n(s)\right)\right] \\
&= \frac{\mu C_3}{2t}\left(t - \mathbf{E}\left[\bar{T}^n(t)\right]\right) \\
&= \frac{\mu C_3}{2}\left(1 - \frac{1}{t}\mathbf{E}\left[\bar{T}^n(t)\right]\right), \qquad \text{(A.120)}
\end{aligned}
$$

where the first inequality follows from the fact that $\mathrm{d}\bar{T}^n(s) \in \{0, \mathrm{d}s\}$ (recall (1h)) and thus $\mathrm{d}\bar{T}^n(s) \le \mathrm{d}s$ for all $s \in \mathbb{R}_+$ and the second inequality follows from (A.71).

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By (A.119) and (A.120), the term in (A.118) is less than or equal to

$$\frac{C_3\theta}{2\sqrt{n}} + \frac{\mu C_3}{2}\left(1 - \frac{1}{t}\mathbf{E}\left[\bar{T}^n(t)\right]\right).$$

Finally, the desired convergence result follows from Lemma A.2.2.

A.4.4. Proof of Lemma A.4.4

Fix an arbitrary $n \in \mathbb{N}_+$. If π^n is of idling type, then the term inside the expectation in the statement of Lemma A.4.4 is equal to 0 trivially. Therefore, without loss of generality, let us assume that π^n is of turning-off type with the parameters (q_1^n, q_2^n) or hybrid type with the parameters (q_0^n, q_1^n, q_2^n) for all $n \in \mathbb{N}_+$. We let $\hat{q}_i^n := q_i^n/\sqrt{n}$ for all $i \in \{0,1,2\}$ and $n \in \mathbb{N}_+$.

We will use the renewal-reward theorem. Recall Section A.2.1 for the definition of the renewal cycles. Let $\{\alpha_k^n, \beta_k^n, \eta_k^n, k \in \mathbb{N}\}$ be nonnegative random variables such that for all $k \in \mathbb{N}$, $n\alpha_k^n$ denotes the starting time of the kth renewal cycle; $n\beta_k^n$ denotes the time when the server is turned on in the kth renewal cycle, and $n\eta_k^n$ denotes the time when the set-up process is completed in the kth renewal cycle. Observe that $\alpha_0^n := 0$. If $Y^n(0) = \text{set-up}$, then $\beta_0^n := 0$. If $Y^n(0) = \text{busy}$, then $\beta_0^n = \eta_0^n := 0$. Therefore,

$$\begin{aligned} &Q^n(n\alpha_k^n) = q_2^n, \qquad Q^n(n\beta_k^n) = q_1^n, \qquad \forall k, n \in \mathbb{N}_+, \\ &\hat{Q}^n(\alpha_k^n) = \hat{q}_2^n, \qquad \hat{Q}^n(\beta_k^n) = \hat{q}_1^n, \qquad \forall k, n \in \mathbb{N}_+. \end{aligned} \tag{A.121}$$

Recall the renewal process N^n defined in Section A.2.1, which denotes the cumulative number of renewal cycles completed up to time t for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$. Then,

$$\begin{aligned} &\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right) \\ &\quad= \sum_{k=0}^{N^n(nt)-1}\left(\int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{F}^n(s) + \int_{\beta_k^n}^{\eta_k^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{U}^n(s)\right) \\ &\qquad\quad + \int_{\alpha_{N^n(nt)}^n \wedge t}^{\beta_{N^n(nt)}^n \wedge t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{F}^n(s) + \int_{\beta_{N^n(nt)}^n \wedge t}^{\eta_{N^n(nt)}^n \wedge t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{U}^n(s) \\ &\quad= \sum_{k=0}^{N^n(nt)-1}\left(\int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s + \int_{\beta_k^n}^{\eta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s\right) \\ &\qquad\quad + \int_{\alpha_{N^n(nt)}^n \wedge t}^{\beta_{N^n(nt)}^n \wedge t} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s + \int_{\beta_{N^n(nt)}^n \wedge t}^{\eta_{N^n(nt)}^n \wedge t} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s, \end{aligned} \tag{A.122}$$

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where the second equality follows from (1h) and the definition of diffusion scaling.

Recall that L_k^n denotes the length of the kth renewal cycle for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$. Under the turning-off policy, $\mathbf{E}[L_1^n]$ is given in (A.5). Under the hybrid policy, a lower bound and an upper bound on $\mathbf{E}[L_1^n]$ are given in (A.11) and (A.12), respectively. Let us define the following cumulative reward functions that are earned gradually over time: For all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$R_1^n(t) := c\tau\sqrt{n}U^n(t) - rF^n(t), \tag{A.123a}$$

$$R_2^n(t) := \sqrt{n}N^n(t)\left(f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right)\right) + \sqrt{n}\left(f\left(Q^n\left(n\alpha^n_{N^n(t)} \wedge t\right)/\sqrt{n}\right) - f\left(Q^n\left(n\beta^n_{N^n(t)} \wedge t\right)/\sqrt{n}\right)\right), \tag{A.123b}$$

$$R_3^n(t) := \frac{r}{\mu}N^n(t)\left(q_1^n - q_2^n\right) + c\sqrt{n}\left(N^n(t)+1\right) + \frac{r}{\mu}\left(Q^n\left(n\beta^n_{N^n(t)} \wedge t\right) - Q^n\left(n\alpha^n_{N^n(t)} \wedge t\right)\right), \tag{A.123c}$$

$$R_4^n(t) := n\mu \sum_{k=0}^{N^n(t)-1} \int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s + n\mu \int_{\alpha^n_{N^n(t)} \wedge (t/n)}^{\beta^n_{N^n(t)} \wedge (t/n)} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s, \tag{A.123d}$$

$$R_5^n(t) := n\mu \sum_{k=0}^{N^n(t)-1} \int_{\beta_k^n}^{\eta_k^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s + n\mu \int_{\beta^n_{N^n(t)} \wedge (t/n)}^{\eta^n_{N^n(t)} \wedge (t/n)} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s. \tag{A.123e}$$

In order to be able to apply the renewal-reward theorem, we need to prove that the expected reward earned in the zeroth renewal cycle is finite under each reward function in (A.123). Recall that expected length of the zeroth renewal cycle is finite by Lemma A.2.1. By Assumption 2,

$$\left|\mathbf{E}\left[R_1^n(L_0^n)\right]\right| \leq c\tau\sqrt{n}\frac{1}{\tau} + r\frac{q_2^n - q_1^n}{\lambda^n} = c\sqrt{n} + r\frac{q_2^n - q_1^n}{\lambda^n} < \infty,$$

where the first inequality follows from the fact that the expected set-up time in the zeroth renewal cycle is bounded above by $1/\tau$ and the expected time the server is turned-off in the zeroth renewal cycle is bounded above by $\left(q_2^n - q_1^n\right)/\lambda^n$ due to the Markovian structure of the system state.

Second,

$$\left|\mathbf{E}\left[R_2^n(L_0^n)\right]\right| \leq -\sqrt{n}\left(f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right)\right) < \infty,$$

$$\left|\mathbf{E}\left[R_3^n(L_0^n)\right]\right| \leq \frac{r}{\mu}\left(q_2^n - q_1^n\right) + 2c\sqrt{n} < \infty,$$

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where the inequalities follow from the fact that f is non-increasing (recall Figure A.3.1a) and $q_1^n < q_2^n$ for all $n \in \mathbb{N}_+$.

Finally, observe that

$$|\mathbf{E}\left[R_4^n(L_0^n)\right]| = \left|\mathbf{E}\left[n\mu \int_{\alpha_0^n}^{\beta_0^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right| \leq \left|\mathbf{E}\left[n\mu \int_{\alpha_1^n}^{\beta_1^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right| < \infty,$$
$$|\mathbf{E}\left[R_5^n(L_0^n)\right]| = \left|\mathbf{E}\left[n\mu \int_{\beta_0^n}^{\eta_0^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right| \leq \left|\mathbf{E}\left[n\mu \int_{\beta_1^n}^{\eta_1^n} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\right| < \infty,$$

where the less than or equal signs follow from the fact that $f'(x) \leq 0$ for all $x \in \mathbb{R}_+$ and the total time the server is turned-off and in set-up in the zeroth renewal cycle are stochastically less than the ones in the first renewal cycle due to the Markovian structure of the system state and Assumption 2; and the strict inequalities will be proven later in (A.135) and (A.146), respectively.

The diffusion-scaled cumulative reward functions associated with the ones in (A.123) are as follows. For all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$\hat{R}_1^n(t) = c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t),$$
$$\hat{R}_2^n(t) = \sqrt{n}\hat{N}^n(t)\left(f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right)\right) + f\left(\hat{Q}^n\left(\alpha^n_{\sqrt{n}\hat{N}^n(t)} \wedge t\right)\right) - f\left(\hat{Q}^n\left(\beta^n_{\sqrt{n}\hat{N}^n(t)} \wedge t\right)\right),$$
$$\hat{R}_3^n(t) = \frac{r}{\mu}\sqrt{n}\hat{N}^n(t)\left(\hat{q}_1^n - \hat{q}_2^n\right) + c\left(\sqrt{n}\hat{N}^n(t) + 1\right) + \frac{r}{\mu}\left(\hat{Q}^n\left(\beta^n_{\sqrt{n}\hat{N}^n(t)} \wedge t\right) - \hat{Q}^n\left(\alpha^n_{\sqrt{n}\hat{N}^n(t)} \wedge t\right)\right),$$
$$\hat{R}_4^n(t) = \mu \sum_{k=0}^{\sqrt{n}\hat{N}^n(t)-1} \int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s + \mu \int_{\alpha^n_{\sqrt{n}\hat{N}^n(t)} \wedge t}^{\beta^n_{\sqrt{n}\hat{N}^n(t)} \wedge t} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s,$$
$$\hat{R}_5^n(t) = \mu \sum_{k=0}^{\sqrt{n}\hat{N}^n(t)-1} \int_{\beta_k^n}^{\eta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s + \mu \int_{\beta^n_{\sqrt{n}\hat{N}^n(t)} \wedge t}^{\eta^n_{\sqrt{n}\hat{N}^n(t)} \wedge t} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s.$$

Recall the term in the statement of Lemma A.4.4.

$$\begin{aligned}
&\frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - \mu\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right)\right] \\
&\qquad = \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_4^n(t) - \hat{R}_5^n(t)\right] \\
&\qquad = \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_2^n(t) + \hat{R}_2^n(t) - \hat{R}_4^n(t) - \hat{R}_5^n(t)\right] \qquad \text{(A.124)} \\
&\qquad \geq \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_3^n(t) + \hat{R}_2^n(t) - \hat{R}_4^n(t) - \hat{R}_5^n(t)\right], \qquad \text{(A.125)}
\end{aligned}$$

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where the first equality follows from (A.122) and (A.123), and the inequality follows from the fact that $\hat{R}_2^n(t) \leq \hat{R}_3^n(t)$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$ by (7c). We will prove that the iterated limit infimum of the term in (A.125) is nonnegative.

For all $n \in \mathbb{N}_+$, under the turning-off or hybrid policy, in the first renewal cycle, the expected time the server is turned-off is equal to $(q_2^n - q_1^n)/\lambda^n$ and the expected time the server spends in set-up is equal to $1/\tau$. Therefore, by the renewal-reward theorem, for all $n \in \mathbb{N}_+$,

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t)\right] &= \sqrt{n} \lim_{t\to\infty} \frac{\mathbf{E}\left[R_1^n(nt)\right]}{nt} = \frac{cn - r\sqrt{n}\frac{q_2^n - q_1^n}{\lambda^n}}{\mathbf{E}[L_1^n]}, \\
\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_3^n(t)\right] &= \sqrt{n} \lim_{t\to\infty} \frac{\mathbf{E}\left[R_3^n(nt)\right]}{nt} = \frac{\sqrt{n}\frac{r}{\mu}\left(q_1^n - q_2^n\right) + cn}{\mathbf{E}[L_1^n]}.
\end{aligned} \tag{A.126}$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_3^n(t)\right] &= \lim_{n\to\infty} \frac{r\sqrt{n}\left(q_2^n - q_1^n\right)\left(\frac{1}{\mu} - \frac{1}{\lambda^n}\right)}{\mathbf{E}[L_1^n]} \\
&= \lim_{n\to\infty} \frac{r\left(q_2^n - q_1^n\right)\frac{-\theta}{\mu\left(\mu - \frac{\theta}{\sqrt{n}}\right)}}{\mathbf{E}[L_1^n]},
\end{aligned} \tag{A.127}$$

where the second equality follows from Assumption 1. There are two cases to consider. First, suppose that at each n, the considered policy is of turning-off type. By (A.5), the term in (A.127) is equal to

$$\lim_{n\to\infty} \frac{r\left(q_2^n - q_1^n\right)\frac{-\theta}{\mu\left(\mu - \frac{\theta}{\sqrt{n}}\right)}}{\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_1^n + \frac{\lambda^n}{\tau}\right)} = 0. \tag{A.128}$$

Second, suppose that at each n, the considered policy is of hybrid type. Because $\theta > 0$ (recall Assumption 1), the term in (A.127) is nonpositive. By (A.11), the term in (A.127) is bounded below by

$$\lim_{n\to\infty} \frac{r\left(q_2^n - q_1^n\right)\frac{-\theta}{\mu\left(\mu - \frac{\theta}{\sqrt{n}}\right)}}{\frac{q_2^n - q_1^n}{\lambda^n} + \frac{1}{\tau} + \frac{\sqrt{n}}{\theta}\left(q_2^n - q_0^n\right)} = 0, \tag{A.129}$$

where the convergence result follows from the fact that $q_0^n < q_1^n < q_2^n$ for all $n \in \mathbb{N}_+$. By the sandwich theorem, the term in (A.127) is equal to 0. Therefore, by (A.127), (A.128), and (A.129),

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_3^n(t)\right] = 0. \tag{A.130}$$

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By (A.123b), (A.123d), and the renewal-reward theorem, for all $n \in \mathbb{N}_+$,

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_2^n(t)\right] = \sqrt{n}\lim_{t\to\infty}\frac{\mathbf{E}\left[R_2^n(nt)\right]}{nt} = \frac{n\left(f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right)\right)}{\mathbf{E}[L_1^n]}, \tag{A.131}$$
$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_4^n(t)\right] = \sqrt{n}\lim_{t\to\infty}\frac{\mathbf{E}\left[R_4^n(nt)\right]}{nt} = \frac{n}{\mathbf{E}[L_1^n]}\,\mu\mathbf{E}\left[\int_{\alpha_1^n}^{\beta_1^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s\right].$$

Therefore, for all $n \in \mathbb{N}_+$,

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_2^n(t) - \hat{R}_4^n(t)\right] = \frac{n}{\mathbf{E}[L_1^n]}\left(f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right) - \mu\mathbf{E}\left[\int_{\alpha_1^n}^{\beta_1^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s\right]\right). \tag{A.132}$$

By (4) and (A.121),

$$\begin{aligned}\hat{Q}^n(t) &= \hat{Q}^n(\alpha_k^n) - \left(\tilde{A}^n(t) - \tilde{A}^n(\alpha_k^n)\right)\\ &= \hat{q}_2^n - \left(\tilde{A}^n(t) - \tilde{A}^n(\alpha_k^n)\right), \qquad \forall t \in [\alpha_k^n, \eta_k^n),\ k \in \mathbb{N}_+,\end{aligned} \tag{A.133}$$

where $\tilde{A}^n(t) = A^n(nt)/\sqrt{n}$ for all $t \in \mathbb{R}_+$. Therefore, when the server is idle, turned-off, or under set-up, $\hat{Q}^n$ is a piecewise constant process with downward jumps at the demand arrival epochs.

Let $\{u_k^n, k \in \mathbb{N}_+\}$ denote the inter-arrival times for the demand in the nth system. Then $\{u_k^n, k \in \mathbb{N}_+\}$ is an i.i.d. sequence of random variables such that $u_1^n \sim \exp(\lambda^n)$. By (A.121) and (A.133), we have the following path-wise equality for all $k, n \in \mathbb{N}_+$:

$$\int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s = f'\left(\hat{q}_2^n\right)\frac{\sum_{j=1}^{A^n\left(n\alpha_k^n\right)+1} u_j^n - n\alpha_k^n}{\sqrt{n}} \tag{A.134a}$$
$$+ \sum_{i=1}^{q_2^n - q_1^n - 1} f'\left(\hat{q}_2^n - \frac{i}{\sqrt{n}}\right)\frac{u^n_{A^n\left(n\alpha_k^n\right)+1+i}}{\sqrt{n}}, \tag{A.134b}$$

where the term in the RHS of (A.134a) is associated with the residual demand inter-arrival time ending just after $n\alpha_k^n$. Because the system state regenerates at the beginning of the renewal cycle k (that is, at time $n\alpha_k^n$ for all $k \in \mathbb{N}_+$ and recall Section A.2.1), and by (A.134) and the memoryless property of the exponential distribution,

$$\mathbf{E}\left[\int_{\alpha_k^n}^{\beta_k^n} f'\big(\hat{Q}^n(s-)\big)\sqrt{n}\mathrm{d}s\right] = \sum_{i=0}^{q_2^n - q_1^n - 1} f'\left(\hat{q}_2^n - \frac{i}{\sqrt{n}}\right)\frac{1}{\lambda^n\sqrt{n}}, \qquad \forall k, n \in \mathbb{N}_+. \tag{A.135}$$

We have the following path-wise equality for all $n \in \mathbb{N}_+$:

$$f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right) = \int_{\hat{q}_1^n}^{\hat{q}_2^n} f'(u)\mathrm{d}u = \lim_{m\to\infty}\sum_{i=0}^{\lfloor (q_2^n - q_1^n)\sqrt{m}/\sqrt{n}\rfloor - 1} f'\left(\hat{q}_2^n - \frac{i}{\sqrt{m}}\right)\frac{1}{\sqrt{m}}, \tag{A.136}$$

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where the first equality follows from the fact that $f \in \mathbb{C}_1$ and the fundamental theorem of calculus; and the second equality follows from the fact that f' is continuous and thus the associated integral is actually a Riemann integral, which is the limit of all Riemann sums (see chapters 7.1 and 7.2 of Bartle and Sherbert (2011)). Because $f \in \mathbb{C}_1$ and by (A.101a) and (A.136), the following convergence result holds:

$$\lim_{n\to\infty} \left| f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right) - \sum_{i=0}^{q_2^n - q_1^n - 1} f'\left(\hat{q}_2^n - \frac{i}{\sqrt{n}}\right) \frac{1}{\sqrt{n}} \right| = 0. \tag{A.137}$$

Therefore, by (A.101a), (A.135), (A.137), and Assumption 1, for all $k \in \mathbb{N}_+$, we have

$$\lim_{n\to\infty} \left| f\left(\hat{q}_2^n\right) - f\left(\hat{q}_1^n\right) - \mu \mathbf{E}\left[\int_{\alpha_k^n}^{\beta_k^n} f'\left(\hat{Q}^n(s-)\right) \sqrt{n} \mathrm{d}s \right] \right| = 0. \tag{A.138}$$

By (A.5), (A.11), (A.12), and (A.101b),

$$\limsup_{n\to\infty} \frac{n}{\mathbf{E}[L_1^n]} < \infty. \tag{A.139}$$

Consequently, by (A.132), (A.138), and (A.139),

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t} \mathbf{E}\left[\hat{R}_2^n(t) - \hat{R}_4^n(t) \right] = 0. \tag{A.140}$$

By (A.123e), and the renewal-reward theorem, for all $n \in \mathbb{N}_+$,

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{E}\left[\hat{R}_5^n(t) \right] = \sqrt{n} \lim_{t\to\infty} \frac{\mathbf{E}\left[R_5^n(nt)\right]}{nt} = \frac{n}{\mathbf{E}[L_1^n]} \mu \mathbf{E}\left[\int_{\beta_1^n}^{\eta_1^n} f'\left(\hat{Q}^n(s-)\right) \sqrt{n} \mathrm{d}s \right]. \tag{A.141}$$

By (9), (A.63) (and recall Figure A.3.1b), there exists constants $C_9, C_{10} \in \mathbb{R}_{++}$ such that

$$|f'(x)| \le C_9 + C_{10}|x|, \qquad \forall x \in \mathbb{R}. \tag{A.142}$$

Recall that D^n is the random variable denoting the cumulative number of demands arriving in the set-up stage. By (A.3), the first and the second moments of D^n are as follows.

$$\mathbf{E}\left[D^n\right] = \frac{\lambda^n}{\tau}, \qquad \mathbf{E}\left[\left(D^n\right)^2\right] = \frac{\lambda^n(2\lambda^n + \tau)}{\tau^2}, \qquad \forall n \in \mathbb{N}_+. \tag{A.143}$$

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Recall that $n\eta_k^n - n\beta_k^n \sim \exp(\tau)$, and thus $\eta_k^n - \beta_k^n \sim \exp(n\tau)$ for all $k \in \mathbb{N}_+$. Recall also that κ_1 denotes the length of the set-up process in the first renewal cycle. Let $\{\tilde{\kappa}_1(j), j \in \mathbb{N}\}$ be a sequence of i.i.d. random variables such that $\tilde{\kappa}_1(0) \sim \exp(\tau + \lambda^n)$. By (A.16),

$$\left(\kappa_1 \middle| D^n = i\right) = \sum_{j=0}^{i} \tilde{\kappa}_1(j), \qquad \forall i \in \mathbb{N},\, n \in \mathbb{N}_+. \tag{A.144}$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left[\int_{\beta_1^n}^{\eta_1^n} f'\left(\hat{Q}^n(s-)\right)\sqrt{n}\mathrm{d}s\right] &= \sum_{i=0}^{\infty} \mathbf{E}\left[\int_{\beta_1^n}^{\eta_1^n} f'\left(\hat{Q}^n(s-)\right)\sqrt{n}\mathrm{d}s \,\middle|\, D^n = i\right] \mathbf{P}\left(D^n = i\right) \\
&= \sum_{i=0}^{\infty} \mathbf{E}\left[\sum_{j=0}^{i} f'\left(\hat{q}_1^n - \frac{j}{\sqrt{n}}\right)\frac{\tilde{\kappa}_1(j)}{\sqrt{n}}\right] \mathbf{P}\left(D^n = i\right) \\
&= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} f'\left(\hat{q}_1^n - \frac{j}{\sqrt{n}}\right)\frac{1}{(\tau+\lambda^n)\sqrt{n}}\right) \mathbf{P}\left(D^n = i\right),
\end{aligned} \tag{A.145}$$

where the second equality follows from (A.144). Then,

$$\begin{aligned}
&\left|\mathbf{E}\left[\int_{\beta_1^n}^{\eta_1^n} f'\left(\hat{Q}^n(s-)\right)\sqrt{n}\mathrm{d}s\right]\right| \\
&\quad\le \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \left| f'\left(\hat{q}_1^n - \frac{j}{\sqrt{n}}\right)\right| \frac{1}{(\tau+\lambda^n)\sqrt{n}}\right) \mathbf{P}\left(D^n = i\right) \\
&\quad\le \frac{1}{(\tau+\lambda^n)\sqrt{n}} \sum_{i=0}^{\infty} \left[\sum_{j=0}^{i} \left(C_9 + C_{10}\left(|\hat{q}_1^n| + \frac{j}{\sqrt{n}}\right)\right)\right] \mathbf{P}\left(D^n = i\right) \\
&\quad= \frac{1}{(\tau+\lambda^n)\sqrt{n}} \sum_{i=0}^{\infty} \left(\left(C_9 + C_{10}\left|\hat{q}_1^n\right|\right)(i+1) + C_{10}\frac{i(i+1)}{2\sqrt{n}}\right) \mathbf{P}\left(D^n = i\right) \\
&\quad= \frac{1}{(\tau+\lambda^n)\sqrt{n}} \left[\left(C_9 + C_{10}\left|\hat{q}_1^n\right|\right)\left(1 + \frac{\lambda^n}{\tau}\right) + \frac{C_{10}}{2\sqrt{n}}\frac{2\lambda^n(\tau+\lambda^n)}{\tau^2}\right] \\
&\quad= \frac{1}{\sqrt{n}} \left[\left(C_9 + C_{10}\left|\hat{q}_1^n\right|\right)\frac{1}{\tau} + \frac{C_{10}}{2\sqrt{n}}\frac{2\lambda^n}{\tau^2}\right] \\
&\quad\to 0 \qquad \text{as } n \to \infty,
\end{aligned} \tag{A.146}$$

where the first inequality follows from (A.145), the second inequality follows from (A.142), the second equality follows from (A.143), and the convergence result follows from (A.101a).

Consequently, by (A.139), (A.141), and (A.146), we have

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_5^n(t)\right] = 0. \tag{A.147}$$

Finally, Lemma A.4.4 follows from (A.125), (A.130), (A.140), and (A.147).

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A.5. Proof of Theorem 3

Under the sequence of policies π and $\tilde{\pi}$, the equality in (A.106) holds for all $t \in \mathbb{R}_{++}$ and $n \geq n_0$, where n_0 is defined in Lemma A.2.3. For notational convenience, we will omit the notation π and $\tilde{\pi}$ henceforth. Let us fix an arbitrary $n \geq n_0$.

Case 1: $B(\gamma_p) > c$. By (18), $\gamma^* \in (\gamma_r, \gamma_p)$. Consider the turning-off policy with the parameters $(\lfloor \sqrt{n} q_1(\gamma^*) \rfloor, \lfloor \sqrt{n} q_2(\gamma^*) \rfloor)$. Observe that $\hat{Q}^n(t) \leq q_2(\gamma^*)$ for all $t \in \mathbb{R}_+$. By (20a) and (A.106), for all $t \in \mathbb{R}_{++}$,

$$\begin{aligned} 0 = \gamma^* &- \frac{1}{t}\mathbf{E}\left[\int_{0+}^{t} \left(h\left(\hat{Q}^n(s-)\right)^+ + b\left(\hat{Q}^n(s-)\right)^-\right) \mathrm{d}s\right] \\ &- \frac{\mu}{t}\mathbf{E}\left[\int_{0+}^{t} f'\left(\hat{Q}^n(s-)\right)\left(\mathrm{d}\hat{I}^n(s) + \mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right)\right] \\ &+ \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\left(\hat{Q}^n(s-)\right)\mathrm{d}s\right] \\ &+ \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} R^n(s)\right]. \end{aligned} \tag{A.148}$$

Because $\hat{I}^n(t) = 0$ for all $t \in \mathbb{R}_+$ and by (5) and (A.148),

$$\begin{aligned} \hat{J}^n(t) &= \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) + \int_{0+}^{t}\left(h\left(\hat{Q}^n(s-)\right)^+ + b\left(\hat{Q}^n(s-)\right)^-\right)\mathrm{d}s\right] \\ &= \gamma^* + \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - \mu\int_{0+}^{t} f'\left(\hat{Q}^n(s-)\right)\left(\mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right)\right] \\ &\quad + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} \frac{1}{2} f''\left(\hat{Q}^n(s-)\right)\left(\Delta\hat{Q}^n(s)\right)^2 - \mu\int_{0+}^{t} f''\left(\hat{Q}^n(s-)\right)\mathrm{d}s\right] \\ &\quad + \frac{1}{t}\mathbf{E}\left[\sum_{0<s\leq t:\, |\Delta\hat{Q}^n(s)|>0} R^n(s)\right]. \end{aligned} \tag{A.149}$$

The following result is about the second term in the RHS of the last equality above.

LEMMA A.5.1. *Suppose that $B(\gamma_p) > c$. Under the sequence of turning-off policies with the parameters $(\lfloor \sqrt{n} q_1(\gamma^*) \rfloor, \lfloor \sqrt{n} q_2(\gamma^*) \rfloor)$ for all $n \in \mathbb{N}_+$, we have*

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - \mu\int_{0+}^{t} f'\left(\hat{Q}^n(s-)\right)\left(\mathrm{d}\hat{F}^n(s) + \mathrm{d}\hat{U}^n(s)\right)\right] = 0.$$

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The proof of Lemma A.5.1 is presented in Section A.5.1.

Next, by (A.149), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\limsup_{t\to\infty}\hat{J}^n(t)\\
&\quad\le\gamma^*+\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t)-r\hat{F}^n(t)-\mu\int_{0+}^{t}f'\big(\hat{Q}^n(s-)\big)\left(\mathrm{d}\hat{F}^n(s)+\mathrm{d}\hat{U}^n(s)\right)\right]\\
&\qquad+\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}\frac{1}{2}f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2-\mu\int_{0+}^{t}f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\\
&\qquad+\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}R^n(s)\right]\\
&\quad=\gamma^*,
\end{aligned}\tag{A.150}$$

where the inequality and the equality follow from Lemmas A.5.1, A.4.3, and A.4.2, respectively, and we replace the limit supremums with the limit signs when the associated limit exists.

Finally,

$$\gamma^*\le\liminf_{n\to\infty}\liminf_{t\to\infty}\hat{J}^n(t)=\liminf_{n\to\infty}\lim_{t\to\infty}\hat{J}^n(t)\le\limsup_{n\to\infty}\lim_{t\to\infty}\hat{J}^n(t)=\limsup_{n\to\infty}\limsup_{t\to\infty}\hat{J}^n(t)\le\gamma^*,$$

where the first inequality follows from Theorem 2, both equalities follow from Lemma A.2.5 and the fact that $\lim_{t\to\infty}\hat{J}^n(t)=\lim_{t\to\infty}J^n(t)/\sqrt{n}$ for all $n\ge n_0$, and the last inequality follows from (A.150). Consequently, we obtain the desired result by the sandwich theorem.

Case 2: $B(\gamma_p)\le c$. By (18), $\gamma^*=\gamma_p$. Fix an arbitrary $K\in\mathbb{N}_+$. Consider the idling policy with the parameters $(\lfloor\sqrt{n}q^*\rfloor-K,\lfloor\sqrt{n}q^*\rfloor)$. Because $\hat{Q}^n(t)\le q^*<q_2(\gamma^*)$ for all $t\in\mathbb{R}_+$ by (15) and by (20a) and (A.106), the equality (A.148) holds under the considered sequence of idling policies. Furthermore, because $\hat{F}^n(t)=\hat{U}^n(t)=0$ for all $t\in\mathbb{R}_+$ and by (5) and (A.148),

$$\begin{aligned}
\hat{J}^n(t)&=\frac{1}{t}\mathbf{E}\left[-p\hat{I}^n(t)+\int_{0+}^{t}\left(h\left(\hat{Q}^n(s-)\right)^+ +b\left(\hat{Q}^n(s-)\right)^-\right)\mathrm{d}s\right]\\
&=\gamma^*+\frac{1}{t}\mathbf{E}\left[-p\hat{I}^n(t)-\mu\int_{0+}^{t}f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{I}^n(s)\right]\\
&\qquad+\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}\frac{1}{2}f''\big(\hat{Q}^n(s-)\big)\left(\Delta\hat{Q}^n(s)\right)^2-\mu\int_{0+}^{t}f''\big(\hat{Q}^n(s-)\big)\mathrm{d}s\right]\\
&\qquad+\frac{1}{t}\mathbf{E}\left[\sum_{0<s\le t:\,|\Delta\hat{Q}^n(s)|>0}R^n(s)\right].
\end{aligned}\tag{A.151}$$

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The following result is about the second term in the RHS of the last equality above.

LEMMA A.5.2. *Suppose that $B(\gamma_p) \leq c$. Fix an arbitrary $K \in \mathbb{N}_+$. Under the sequence of idling policies with the parameters $(\lfloor\sqrt{n}q^*\rfloor - K, \lfloor\sqrt{n}q^*\rfloor)$ for all $n \in \mathbb{N}_+$, we have*

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left[-p\hat{I}^n(t) - \mu\int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{I}^n(s)\right] = 0.$$

The proof of Lemma A.5.2 is presented in Section A.5.2.

The rest of the proof is very similar to the one under the case $B(\gamma_p) > c$ and thus we skip it.

A.5.1. Proof of Lemma A.5.1

Because $B(\gamma_p) > c$, we have $\gamma^* \in (\gamma_r, \gamma_p)$ by (18). Under the turning-off policy with the parameters $(\lfloor\sqrt{n}q_1(\gamma^*)\rfloor, \lfloor\sqrt{n}q_2(\gamma^*)\rfloor)$, the equality (A.124) holds and is equal to

$$\frac{1}{t}\mathbf{E}\left[\hat{R}_1^n(t) - \hat{R}_3^n(t)\right] + \frac{1}{t}\mathbf{E}\left[\hat{R}_3^n(t) - \hat{R}_2^n(t)\right] + \frac{1}{t}\mathbf{E}\left[\hat{R}_2^n(t) - \hat{R}_4^n(t)\right] - \frac{1}{t}\mathbf{E}\left[\hat{R}_5^n(t)\right]. \tag{A.152}$$

The iterated limits of the first, the third, and the last terms in (A.152) converge to 0 by (A.130), (A.140), and (A.147), respectively. Therefore, if we can prove that

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_3^n(t) - \hat{R}_2^n(t)\right] = 0, \tag{A.153}$$

then the desired result follows. Hence, we will prove (A.153) henceforth.

By (A.126) and (A.131), for all $n \in \mathbb{N}_+$,

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\hat{R}_3^n(t) - \hat{R}_2^n(t)\right] \\ &\quad = \frac{n}{L_1^n}\left[f\left(\frac{\lfloor\sqrt{n}q_1(\gamma^*)\rfloor}{\sqrt{n}}\right) - f\left(\frac{\lfloor\sqrt{n}q_2(\gamma^*)\rfloor}{\sqrt{n}}\right) - \frac{r}{\mu}\left(\frac{\lfloor\sqrt{n}q_2(\gamma^*)\rfloor - \lfloor\sqrt{n}q_1(\gamma^*)\rfloor}{\sqrt{n}}\right) + c\right].\end{aligned} \tag{A.154}$$

Because f is continuous and by (20b), we have

$$\lim_{n\to\infty}\left[f\left(\frac{\lfloor\sqrt{n}q_1(\gamma^*)\rfloor}{\sqrt{n}}\right) - f\left(\frac{\lfloor\sqrt{n}q_2(\gamma^*)\rfloor}{\sqrt{n}}\right) - \frac{r}{\mu}\left(\frac{\lfloor\sqrt{n}q_2(\gamma^*)\rfloor - \lfloor\sqrt{n}q_1(\gamma^*)\rfloor}{\sqrt{n}}\right) + c\right] = 0. \tag{A.155}$$

Finally, (A.153) follows from (A.139), (A.154), and (A.155).

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A.5.2. Proof of Lemma A.5.2

Because $B(\gamma_p) \leq c$, we have $\gamma^* = \gamma_p$ by (18). Under the idling policy with the parameters $(\lfloor\sqrt{n}q^*\rfloor - K, \lfloor\sqrt{n}q^*\rfloor)$, the server becomes idle when the inventory level reaches $\lfloor\sqrt{n}q^*\rfloor$ and becomes busy when the inventory level decreases to $\lfloor\sqrt{n}q^*\rfloor - K$. Therefore, the idle time process I^n starts increasing when the inventory level reaches $\lfloor\sqrt{n}q^*\rfloor$ and stops increasing when the inventory level decreases to $\lfloor\sqrt{n}q^*\rfloor - K$.

Observe that $f'(x)$ is strictly increasing in x for all $x \leq q^*$ and f' reaches its unique maximum at the point q^*, where $f'(q^*) = -p/\mu$, by (12), (13b), and (A.63). Furthermore, $\lfloor\sqrt{n}q^*\rfloor/\sqrt{n} \leq q^*$ for all $n \in \mathbb{N}_+$. Because the idle time process I^n increases at the inventory levels $\{\lfloor\sqrt{n}q^*\rfloor - K + 1, \lfloor\sqrt{n}q^*\rfloor - K + 2, \ldots, \lfloor\sqrt{n}q^*\rfloor\}$, for all $t \in \mathbb{R}_{++}$ and $n \in \mathbb{N}_+$,

$$p\hat{I}^n(t) \leq -\mu \int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{I}^n(s) \leq -\mu f'\left(\frac{\lfloor\sqrt{n}q^*\rfloor - K + 1}{\sqrt{n}}\right)\hat{I}^n(t),$$

which implies, for all $t \in \mathbb{R}_{++}$ and $n \in \mathbb{N}_+$,

$$\frac{p}{t}\mathbf{E}\left[\hat{I}^n(t)\right] \leq \frac{1}{t}\mathbf{E}\left[-\mu \int_{0+}^{t} f'\big(\hat{Q}^n(s-)\big)\mathrm{d}\hat{I}^n(s)\right] \leq -\mu f'\left(\frac{\lfloor\sqrt{n}q^*\rfloor - K + 1}{\sqrt{n}}\right)\frac{1}{t}\mathbf{E}\left[\hat{I}^n(t)\right]. \tag{A.156}$$

Next, we will use the renewal-reward theorem to prove the existence of the limit of the term in the LHS of (A.156) as $t \to \infty$. Recall Section A.2.1 for the definition of the renewal cycles. Expected time the server is idle in the first renewal cycle is equal to K/λ^n and the expected length of the first renewal cycle is given in (A.2) and is equal to

$$K\left(\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}\right).$$

Furthermore, expected time the server is idle in the zeroth renewal cycle is less than or equal to K/λ^n and the expected length of the zeroth renewal cycle is finite by Lemma A.2.1. Therefore, by the renewal-reward theorem, for all $n \in \mathbb{N}_+$,

$$\lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\hat{I}^n(t)\right] = \lim_{t\to\infty}\frac{1}{\sqrt{n}t}\mathbf{E}\left[I^n(nt)\right] = \lim_{t\to\infty}\frac{\sqrt{n}}{t}\mathbf{E}\left[I^n(t)\right] = \sqrt{n}\frac{\frac{K}{\lambda^n}}{K\left(\frac{1}{\lambda^n} + \frac{\sqrt{n}}{\theta}\right)}. \tag{A.157}$$

Because f' is continuous,

$$f'\left(\frac{\lfloor\sqrt{n}q^*\rfloor - K + 1}{\sqrt{n}}\right) \to f'(q^*) = -\frac{p}{\mu} \quad \text{as } n \to \infty. \tag{A.158}$$

Finally, the desired result follows from (A.156), (A.157), and (A.158).

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A.6. Challenges of analyzing a multi-class manufacturing system

In this section, we explain why extending our results to a system manufacturing multiple product types is not trivial. There are two main reasons which make that extension challenging. First, the resulting BCP is multi-dimensional and we cannot obtain a single-dimensional EWF straightforwardly unlike the cases we see in the literature (see for example Wein (1992), Ata and Barjesteh (2022), and Gao and Huang (2022)). Consequently, it is not clear how to solve the multi-dimensional BCP and derive a heuristic control policy for the pre-limit system.

Second, even if we achieve the aforementioned tasks, it is not clear how to prove the asymptotic optimality of the proposed heuristic control policy. Under the single-item case, we only focus on the idling, turning-off, and hybrid policies by Lemma 1. In the proofs, we extensively utilize the simple regenerative structures of those policies (recall Definition 2 and Figure 1). However, because the structures of the admissible policies under the multi-item case are expected to be more complicated than the ones under the single-item case, so do the proofs.

In the remaining part of this section, we will show why we cannot obtain a single-dimensional EWF straightforwardly. Suppose that there are $I \in \mathbb{N}_+$ different product types that are manufactured in the system and $\mathcal{I} := \{1, 2, \ldots, I\}$ denotes the set of product types. The subscript i will denote the product type henceforth. The constraints (1a) and (1b) are modified such that for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$Q_i^n(t) = S_i \circ T_i^n(t) - A_i^n(t), \qquad \forall i \in \mathcal{I},$$

$$\sum_{i \in \mathcal{I}} T_i^n(t) + I^n(t) + F^n(t) + U^n(t) = t,$$

where A_i^n and S_i are mutually independent Poisson processes with rates λ_i^n and μ_i, respectively, for all $i \in \mathcal{I}$ and $n \in \mathbb{N}_+$. The system load is updated as $\rho^n := \sum_{i \in \mathcal{I}} \lambda_i^n / \mu_i$ for all $n \in \mathbb{N}_+$. The modified Assumption 1 is as follows.

Assumption A.6.1. *For all $i \in \mathcal{I}$, $\lambda_i^n \to \lambda_i \in \mathbb{R}_{++}$ as $n \to \infty$. Furthermore, there exists a constant $\tilde{\theta} \in \mathbb{R}_{++}$ such that $\rho^n = 1 - \tilde{\theta}/\sqrt{n}$ for all $n \in \mathbb{N}_+$. Therefore, $\sqrt{n}\,(1 - \rho^n) = \tilde{\theta}$ for all $n \in \mathbb{N}_+$.*

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For all $i \in \mathcal{I}$, $t \in \mathbb{R}_+$, and $n \in \mathbb{N}_+$, let

$$\hat{X}_i^n(t) := \hat{S}_i^n \circ \bar{T}_i^n(t) - \hat{A}_i^n(t),$$
$$\hat{Y}_i^n(t) := \sqrt{n}\left(\bar{T}_i^n(t) - \frac{\lambda_i^n}{\mu_i} t\right).$$

The process $\hat{Y}_i^n$ denotes the deviation of the actual cumulative amount of time spent for the manufacturing of type i products from the average cumulative time that should be spent for the manufacturing of type i products in diffusion scale.

Equations (4) and (5) are modified such that for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$\hat{Q}_i^n(t) = \hat{Q}_i^n(0) + \hat{X}_i^n(t) + \mu_i \hat{Y}_i^n(t), \qquad \forall i \in \mathcal{I},$$
$$\hat{J}^n(t) := \frac{1}{t}\mathbf{E}\left[c\tau\sqrt{n}\hat{U}^n(t) - r\hat{F}^n(t) - p\hat{I}^n(t) + \sum_{i\in\mathcal{I}} \int_0^t \left(h_i\left(\hat{Q}_i^n(s)\right)^+ + b_i\left(\hat{Q}_i^n(s)\right)^-\right) \mathrm{d}s\right].$$

By Assumption A.6.1, for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$,

$$\sum_{i\in\mathcal{I}} \hat{Y}_i^n(t) + \hat{I}^n(t) + \hat{F}^n(t) + \hat{U}^n(t) = \tilde{\theta} t. \tag{A.159}$$

Because the system is in the conventional heavy-traffic regime (recall Assumption A.6.1), we expect

$$\left(\bar{T}_i^n, i \in \mathcal{I}\right) \Rightarrow \left(\frac{\lambda_i}{\mu_i} e, i \in \mathcal{I}\right), \qquad \text{as } n \to \infty, \tag{A.160}$$

where $\Rightarrow$ denotes weak convergence in space $\mathbb{D}^I$ endowed with the Skorokhod J_1 topology (see chapter 3 of Billingsley (1999)) and $e(t) = t$ for all $t \in \mathbb{R}_+$. The convergence result in (A.160) implies that the cumulative amount of time spent for the manufacturing of each product type weakly converges to its nominal rate in fluid scale (cf. Lemma A.2.2).

By (A.160), the functional central limit theorem (see theorems 4.3.5 and 7.3.2 of Whitt (2002)), joint convergence when one limit is deterministic (see theorem 11.4.5 of Whitt (2002)), the random time-change theorem (see theorem 13.2.2 of Whitt (2002)), and the continuous mapping theorem (see theorem 3.4.4 of Whitt (2002)), we have

$$\left(\hat{X}_i^n, i \in \mathcal{I}\right) \Rightarrow \left(X_i, i \in \mathcal{I}\right), \qquad \text{as } n \to \infty,$$

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where X_i is a driftless Brownian motion with variance $2\lambda_i$, $X_i(0)=0$ for all $i\in\mathcal{I}$, and X_i is independent of X_j for all $j\neq i$.

We formulate the following BCP:

$$\min \limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\left(\sum_{0<s\leq t:\,\Delta F(s)>0}(c-r\Delta F(s))\right)-pI(t)+\sum_{i\in\mathcal{I}}\int_0^t\left(h_i\left(Q_i(s)\right)^+ + b_i\left(Q_i(s)\right)^-\right)\mathrm{d}s\right] \tag{A.161a}$$

$$\text{s.t. } Q_i(t)=Q_i(0)+X_i(t)+\mu_i Y_i(t), \qquad \forall i\in\mathcal{I},\, t\in\mathbb{R}_+, \tag{A.161b}$$

$$\sum_{i\in\mathcal{I}} Y_i(t)+I(t)+F(t)=\tilde{\theta}t, \qquad \forall t\in\mathbb{R}_+, \tag{A.161c}$$

$$I(0)=F(0)=0 \text{ and } I \text{ and } F \text{ are nondecreasing, } \mathbb{F}\text{-adapted, and in } \mathbb{D}, \tag{A.161d}$$

$$Y_i(0)=0, \text{ and } Y_i \text{ is } \mathbb{F}\text{-adapted and in } \mathbb{D} \text{ for all } i\in\mathcal{I}, \tag{A.161e}$$

$$F \text{ is piecewise constant and } \Delta I(t)\Delta F(t)=0 \text{ for all } t\in\mathbb{R}_+, \tag{A.161f}$$

$$\text{If } \Delta F(t)>0 \text{ for some } t\in\mathbb{R}_{++}, \text{ then } \mu_i\Delta Y_i(t)=-\lambda_i\Delta F(t) \text{ for all } i\in\mathcal{I}, \tag{A.161g}$$

$$\mathbf{E}\left[\left(Q_i(0)\right)^2\right]<\infty \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\mathbf{E}\left[\left(Q_i(t)\right)^2\right]=0, \qquad \forall i\in\mathcal{I}, \tag{A.161h}$$

where the decision variable is the process $(I,F,Q_i,Y_i,i\in\mathcal{I})$. Constraint (A.161c) follows from (A.159). Constraints (A.161b) and (A.161g) imply that if the process F makes a jump, then the inventory process Q_i makes a downward jump for all $i\in\mathcal{I}$. In other words, if the server is turned-off, inventory levels of all product types decrease. Observe that if there is a single product type, that is, if $I=1$, then the BCP (A.161) is equivalent to the BCP (6).

Let $W:=\sum_{i\in\mathcal{I}}Q_i/\mu_i$ denote the workload process. By (A.161b) and (A.161c),

$$W(t)=W(0)+X(t)+\tilde{\theta}t-I(t)-F(t), \qquad \forall t\in\mathbb{R}_+,$$

where $X:=\sum_{i\in\mathcal{I}}X_i/\mu_i$ is a driftless Brownian motion with variance $\sum_{i\in\mathcal{I}}2\lambda_i/\mu_i^2$. There are studies in the literature (see for example Wein (1992) and Ata and Barjesteh (2022)) in which it is proven

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that the multi-dimensional BCP has a single-dimensional EWF, where the single-dimensional workload process, W, is controlled instead of the multi-dimensional process $(Q_i, i \in \mathcal{I})$. That result follows from the following trivial and path-wise optimal workload allocation rule: Let

$$j \in \arg\min_{i \in \mathcal{I}} h_i \mu_i \qquad \text{and} \qquad k \in \arg\min_{i \in \mathcal{I}} b_i \mu_i.$$

- If the workload is non-negative at time $t \in \mathbb{R}_+$, that is, if $W(t) \geq 0$, it is better to keep all the workload as the product type j, that is, the workload allocation $Q_j(t) = \mu_j W(t)$ and $Q_i(t) = 0$ for all $i \in \mathcal{I} \setminus \{j\}$ minimizes the instantaneous total holding and backorder cost.
- If the workload is negative at time $t \in \mathbb{R}_+$, that is, if $W(t) < 0$, it is better to keep all the workload as the product type k, that is, the workload allocation $Q_k(t) = \mu_k W(t)$ and $Q_i(t) = 0$ for all $i \in \mathcal{I} \setminus \{k\}$ minimizes the instantaneous total holding and backorder cost.

Therefore, controlling the single-dimensional workload process under the aforementioned simple workload allocation rule is equivalent to controlling the multi-dimensional inventory level process. Consequently, one can formulate a single-dimensional EWF of the multi-dimensional BCP and solve it without loss of optimality. Observe that in the EWF, holding (backorder) cost is incurred only due to the product type j (k).

The aforementioned EWF derivation technique does not work in our case. The reason is that the aforementioned simple workload allocation rule cannot be implemented due to the constraint (A.161g). For example, suppose that $W(t-) > 0$ for some $t \in \mathbb{R}_{++}$, $Q_j(t-) = \mu_j W(t-)$ and $Q_i(t-) = 0$ for all $i \in \mathcal{I} \setminus \{j\}$ and thus the instantaneous total holding and backorder cost is minimized under the workload allocation at time $t-$, and the process F makes a jump at time t. Then, $Q_j(t) = \mu_j W(t-) - \lambda_j \Delta F(t)$ and $Q_i(t) = -\lambda_i \Delta F(t)$ for all $i \in \mathcal{I} \setminus \{j\}$ by (A.161b) and (A.161g). Therefore, it is not possible to always keep all the workload as the product type j when the inventory level is positive. Similarly, it is not possible to always keep all the workload as the product type k when the inventory level is negative. Furthermore, on the time intervals at which the process F makes a jump, holding or backorder cost is incurred due to all product types rather than a single product type. Consequently, the classical EWF derivation technique does not work in our case. Nevertheless, solving the multi-dimensional BCP (A.161) is an interesting future research topic.

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A.7. Detailed Numerical Results

We present the detailed numerical results associated with the experiments with exponentially and Erlang-2 distributed inter-arrival times in Sections A.7.1 and A.7.2, respectively. First, we introduce some new notation.

- "PP type" and "OP type" denote the type of the proposed and the optimal policies, respectively.
  - "0" denotes the idling policy.
  - "1" denotes the turning-off policy.
  - "2" denotes the hybrid policy.
- J_i^* denotes the optimal long-run average cost at instance i for all $i \in \{1, 2, \ldots, 80\}$.
- $J_i(PP)$ denotes the long-run average cost under the proposed policy at instance i.
- $J_i(BCP) = \sqrt{n}\gamma^*$ at instance i, that is, it is the BCP approximation for the optimal long-run average cost.

A.7.1. Experiments with exponentially distributed inter-arrival times

Tables A.7.1 and A.7.2 present the detailed results of the numerical experiments discussed in Section 7.1.

A.7.2. Experiments with Erlang-2 distributed inter-arrival times

Tables A.7.3 and A.7.4 present the detailed results of the numerical experiments discussed in Section 7.2. Because the threshold values (q_0^n, q_1^n, q_2^n) depend on the phase of the next arrival, we use the notation $q_i(k)$ for all $i \in \{0, 1, 2\}$ and $k \in \{1, 2\}$ such that k denotes the phase of the next arrival. An arrival happens after it completes the second phase.

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Table A.7.1 Results of the numerical experiments under exponentially distributed inter-arrival times where $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, and λ^n and c^n are given below.

Ins. #	c^n	λ^n	PP type	q_1^n	q_2^n	$J_i(PP)$	$J_i(BCP)$	OP type	q_1^n	q_2^n	J_i^*
1	50	0.5	1	-3	16	-32.41	-35.88	1	-3	18	-32.50
2		0.525	1	-3	15	-29.69	-33.66	1	-3	18	-29.94
3		0.55	1	-3	15	-27.13	-31.45	1	-2	18	-27.41
4		0.575	1	-3	15	-24.59	-29.24	1	-2	18	-24.92
5		0.6	1	-2	15	-22.15	-27.01	1	-2	19	-22.46
6		0.625	1	-2	15	-19.68	-24.78	1	-2	19	-20.01
7		0.65	1	-2	15	-17.22	-22.53	1	-1	19	-17.61
8		0.675	1	-2	15	-14.76	-20.26	1	-1	19	-15.22
9		0.7	1	-1	15	-12.41	-17.95	1	0	19	-12.84
10		0.725	1	-1	15	-9.95	-15.60	1	0	19	-10.47
11		0.75	1	0	15	-7.56	-13.17	1	1	19	-8.08
12		0.775	1	0	15	-4.98	-10.65	0	4	5	-5.84
13		0.8	1	1	16	-2.68	-7.98	0	5	6	-3.81
14		0.825	1	1	16	0.25	-5.11	0	6	7	-1.56
15		0.85	1	3	17	2.88	-1.91	0	7	8	1.01
16		0.875	1	4	19	6.06	1.81	0	9	10	4.12
17		0.9	1	7	22	9.82	6.45	0	12	13	8.15
18		0.925	1	11	26	15.50	12.93	0	16	17	14.02
19		0.95	1	20	35	25.51	24.02	0	26	27	24.47
20		0.975	1	48	62	53.97	53.60	0	53	54	53.50
21	100	0.5	1	-5	21	-29.90	-30.33	1	-4	21	-29.99
22		0.525	1	-5	21	-27.33	-28.28	1	-3	21	-27.44
23		0.55	1	-5	20	-24.77	-26.23	1	-3	21	-24.95
24		0.575	1	-4	20	-22.41	-24.19	1	-3	21	-22.49
25		0.6	1	-4	20	-19.94	-22.15	1	-3	21	-20.06
26		0.625	1	-4	20	-17.50	-20.11	1	-2	21	-17.67
27		0.65	1	-3	19	-15.17	-18.06	1	-2	21	-15.31
28		0.675	1	-3	19	-12.78	-16.01	1	-2	21	-12.98
29		0.7	1	-3	19	-10.39	-13.93	1	2	3	-11.13
30		0.725	1	-2	19	-8.18	-11.83	0	3	4	-9.47
31		0.75	1	-2	19	-5.78	-9.68	0	3	4	-7.70
32		0.775	1	-1	19	-3.52	-7.45	0	4	5	-5.84
33		0.8	1	0	19	-1.17	-5.10	0	5	6	-3.81
34		0.825	1	0	19	1.56	-2.55	0	6	7	-1.56
35		0.85	1	1	20	4.23	0.31	0	7	8	1.01
36		0.875	1	3	22	7.08	3.68	0	9	10	4.12
37		0.9	1	5	24	10.97	7.95	0	12	13	8.15
38		0.925	1	10	28	16.23	14.06	0	16	17	14.02
39		0.95	1	19	37	26.00	24.78	0	26	27	24.47
40		0.975	1	46	64	54.23	53.98	0	53	54	53.50

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Table A.7.2 Results of the numerical experiments under exponentially distributed inter-arrival times where $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, and λ^n and c^n are given below.

Ins. #	c^n	λ^n	PP type	q_1^n	q_2^n	$J_i(PP)$	$J_i(BCP)$	OP type	q_1^n	q_2^n	J_i^*
41	150	0.5	1	-6	25	-27.59	-26.03	1	-4	23	-27.77
42		0.525	1	-6	25	-25.03	-24.10	1	-4	23	-25.24
43		0.55	1	-6	25	-22.51	-22.17	1	-4	23	-22.76
44		0.575	1	-6	24	-20.05	-20.26	1	-4	23	-20.31
45		0.6	1	-5	24	-17.78	-18.35	1	-3	23	-17.91
46		0.625	1	-5	23	-15.38	-16.45	0	1	2	-15.81
47		0.65	1	-4	23	-13.16	-14.55	0	2	3	-14.32
48		0.675	1	-4	22	-10.80	-12.65	0	2	3	-12.77
49		0.7	1	-4	22	-8.47	-10.74	0	2	3	-11.13
50		0.725	1	-3	22	-6.35	-8.82	0	3	4	-9.47
51		0.75	0	4	5	-7.65	-6.95	0	3	4	-7.70
52		0.775	0	5	6	-5.71	-5.09	0	4	5	-5.84
53		0.8	0	5	6	-3.81	-3.07	0	5	6	-3.81
54		0.825	0	6	7	-1.56	-0.83	0	6	7	-1.56
55		0.85	0	8	9	1.08	1.74	0	7	8	1.01
56		0.875	0	10	11	4.20	4.84	0	9	10	4.12
57		0.9	0	12	13	8.15	8.86	0	12	13	8.15
58		0.925	0	17	18	14.04	14.73	0	16	17	14.03
59		0.95	0	26	27	24.53	25.23	0	26	27	24.53
60		0.975	0	54	55	53.51	54.20	0	53	54	53.50
61	200	0.5	1	-8	29	-25.21	-22.40	1	-5	25	-25.76
62		0.525	0	1	2	-21.64	-20.83	1	-5	25	-23.23
63		0.55	0	2	3	-19.91	-19.42	1	-5	25	-20.75
64		0.575	0	2	3	-18.57	-17.99	0	1	2	-18.81
65		0.6	0	2	3	-17.20	-16.53	0	1	2	-17.34
66		0.625	0	2	3	-15.79	-15.05	0	1	2	-15.81
67		0.65	0	2	3	-14.32	-13.54	0	2	3	-14.32
68		0.675	0	3	4	-12.60	-11.98	0	2	3	-12.77
69		0.7	0	3	4	-11.09	-10.38	0	2	3	-11.13
70		0.725	0	4	5	-9.27	-8.71	0	3	4	-9.47
71		0.75	0	4	5	-7.65	-6.95	0	3	4	-7.70
72		0.775	0	5	6	-5.71	-5.09	0	4	5	-5.84
73		0.8	0	5	6	-3.81	-3.07	0	5	6	-3.81
74		0.825	0	6	7	-1.56	-0.83	0	6	7	-1.56
75		0.85	0	8	9	1.08	1.74	0	7	8	1.01
76		0.875	0	10	11	4.20	4.84	0	9	10	4.12
77		0.9	0	12	13	8.15	8.86	0	12	13	8.15
78		0.925	0	17	18	14.04	14.73	0	16	17	14.03
79		0.95	0	26	27	24.53	25.23	0	26	27	24.53
80		0.975	0	54	55	53.51	54.20	0	53	54	53.50

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Table A.7.3 Results of the numerical experiments under Erlang-2 distributed inter-arrival times where $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, and λ^n and c^n are given below.

Ins.	c^n	λ^n	PP type	q_1^n	q_2^n	$J_i(PP)$	$J_i(BCP)$	OP type	$q_0^n(1)$	$q_0^n(2)$	$q_1^n(1)$	$q_1^n(2)$	$q_2^n(1)$	$q_2^n(2)$	J_i^*
1	50	0.5	1	-4	15	-32.39	-36.06	2	-4	-3	-3		18	18	-32.65
2		0.525	1	-3	15	-29.88	-33.88	2	-4	-3	-3		18	18	-30.11
3		0.55	1	-3	14	-27.16	-31.69	2	-3	-3		-2	18	18	-27.60
4		0.575	1	-3	14	-24.66	-29.51	2	-3	-3		-2	18	18	-25.13
5		0.6	1	-3	14	-22.19	-27.33	2	-3	-2	-2		18	18	-22.70
6		0.625	1	-3	14	-19.74	-25.15	2	-3	-2	-2		18	18	-20.29
7		0.65	1	-2	14	-17.42	-22.96	2	-2	-2		-1	18	18	-17.92
8		0.675	1	-2	14	-15.05	-20.76	2	-2	-1	-1		18	18	-15.58
9		0.7	1	-2	13	-12.38	-18.55	2	-2	-1	-1		18	18	-13.26
10		0.725	1	-2	13	-9.99	-16.31	2	-1	-1		0	18	18	-10.97
11		0.75	1	-1	13	-7.76	-14.04	0				3	3	4	-8.95
12		0.775	1	-1	13	-5.34	-11.70	0			3		4	4	-7.23
13		0.8	1	0	14	-3.36	-9.26	0				4	4	5	-5.41
14		0.825	1	0	14	-0.73	-6.67	0			4		5	5	-3.41
15		0.85	1	1	15	1.61	-3.86	0			5		6	6	-1.17
16		0.875	1	2	16	4.39	-0.65	0				7	7	8	1.47
17		0.9	1	4	18	7.49	3.24	0				9	9	10	4.81
18		0.925	1	7	21	11.88	8.50	0			12		13	13	9.52
19		0.95	1	14	27	19.39	17.22	0			19		20	20	17.71
20		0.975	1	35	48	40.48	39.80	0			40		41	41	39.76
21	100	0.5	1	-6	21	-29.89	-30.46	2	-5	-4	-4		20	20	-30.13
22		0.525	1	-5	20	-27.50	-28.43	2	-4	-4		-3	20	20	-27.60
23		0.55	1	-5	20	-24.98	-26.40	2	-4	-4		-3	20	20	-25.12
24		0.575	1	-5	19	-22.44	-24.39	2	-4	-3	-3		20	20	-22.67
25		0.6	1	-4	19	-20.15	-22.38	2	-4	-3	-3		20	20	-20.27
26		0.625	1	-4	19	-17.75	-20.38	2	-3	-3		-2	20	20	-17.90
27		0.65	1	-4	18	-15.28	-18.38	2	-3	-3		-2	20	20	-15.58
28		0.675	1	-4	18	-12.92	-16.38	0				2	2	3	-13.67
29		0.7	1	-3	18	-10.80	-14.39	0				2	2	3	-12.16
30		0.725	1	-3	17	-8.36	-12.38	0			2		3	3	-10.59
31		0.75	1	-2	17	-6.27	-10.36	0				3	3	4	-8.95
32		0.775	1	-2	17	-3.97	-8.30	0			3		4	4	-7.23
33		0.8	1	-1	17	-1.85	-6.17	0				4	4	5	-5.41
34		0.825	1	-1	17	0.61	-3.91	0			4		5	5	-3.41
35		0.85	1	0	18	2.83	-1.45	0			5		6	6	-1.17
36		0.875	1	1	18	5.61	1.38	0				7	7	8	1.47
37		0.9	1	3	20	8.51	4.88	0				9	9	10	4.81
38		0.925	1	6	23	12.66	9.74	0			12		13	13	9.52
39		0.95	1	13	29	19.89	18.06	0			19		20	20	17.71
40		0.975	1	33	50	40.79	40.22	0			40		41	41	39.76

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Table A.7.4 Results of the numerical experiments under Erlang-2 distributed inter-arrival times where $\mu = 1$, $\tau = 0.2$, $h = 1$, $b = 3$, $p^n = 50$, $r^n = 100$, and λ^n and c^n are given below.

Ins.	c^n	λ^n	PP type	q_1^n	q_2^n	$J_i(PP)$	$J_i(BCP)$	OP type	$q_0^n(1)$	$q_0^n(2)$	$q_1^n(1)$	$q_1^n(2)$	$q_2^n(1)$	$q_2^n(2)$	J_i^*
41	150	0.5	1	-7	25	-27.57	-26.14	2	-5	-5		-4	22	22	-27.90
42		0.525	1	-7	24	-25.05	-24.22	2	-5	-5		-4	23	23	-25.38
43		0.55	1	-6	24	-22.73	-22.32	2	-5	-4	-4		23	23	-22.90
44		0.575	1	-6	23	-20.28	-20.42	2	-5	-4	-4		23	23	-20.48
45		0.6	1	-6	23	-17.85	-18.54	1			-4		23	23	-18.10
46		0.625	1	-5	22	-15.63	-16.67	0			1		2	2	-16.61
47		0.65	1	-5	22	-13.30	-14.81	0			1		2	2	-15.16
48		0.675	0	2	3	-13.57	-13.05	0				2	2	3	-13.67
49		0.7	0	2	3	-12.12	-11.53	0				2	2	3	-12.16
50		0.725	0	2	3	-10.59	-9.97	0			2		3	3	-10.59
51		0.75	0	3	4	-8.87	-8.34	0				3	3	4	-8.95
52		0.775	0	3	4	-7.23	-6.63	0			3		4	4	-7.23
53		0.8	0	4	5	-5.35	-4.80	0				4	4	5	-5.41
54		0.825	0	4	5	-3.41	-2.81	0			4		5	5	-3.41
55		0.85	0	5	6	-1.17	-0.57	0			5		6	6	-1.17
56		0.875	0	7	8	1.50	2.07	0				7	7	8	1.47
57		0.9	0	9	10	4.82	5.40	0				9	9	10	4.81
58		0.925	0	12	13	9.53	10.11	0			12		13	13	9.52
59		0.95	0	19	20	17.71	18.29	0			19		20	20	17.71
60		0.975	0	40	41	39.76	40.34	0			40		41	41	39.76
61	200	0.5	0	1	2	-23.34	-22.92	2	-7	-6	-6		24	24	-25.87
62		0.525	0	1	2	-22.04	-21.56	2	-6	-5	-5		25	25	-23.35
63		0.55	0	1	2	-20.73	-20.19	2	-6	-5	-5		25	25	-20.89
64		0.575	0	1	2	-19.39	-18.80	0				1	1	2	-19.45
65		0.6	0	1	2	-18.02	-17.40	0				1	1	2	-18.03
66		0.625	0	1	2	-16.61	-15.98	0			1		2	2	-16.61
67		0.65	0	1	2	-15.16	-14.53	0			1		2	2	-15.16
68		0.675	0	2	3	-13.57	-13.05	0				2	2	3	-13.67
69		0.7	0	2	3	-12.12	-11.53	0				2	2	3	-12.16
70		0.725	0	2	3	-10.59	-9.97	0			2		3	3	-10.59
71		0.75	0	3	4	-8.87	-8.34	0				3	3	4	-8.95
72		0.775	0	3	4	-7.23	-6.63	0			3		4	4	-7.23
73		0.8	0	4	5	-5.35	-4.80	0				4	4	5	-5.41
74		0.825	0	4	5	-3.41	-2.81	0			4		5	5	-3.41
75		0.85	0	5	6	-1.17	-0.57	0			5		6	6	-1.17
76		0.875	0	7	8	1.50	2.07	0				7	7	8	1.47
77		0.9	0	9	10	4.82	5.40	0				9	9	10	4.81
78		0.925	0	12	13	9.53	10.11	0			12		13	13	9.53
79		0.95	0	19	20	17.71	18.29	0			19		20	20	17.71
80		0.975	0	40	41	39.76	40.34	0			40		41	41	39.76

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