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AVERAGES OF VALUES OF L-SERIES

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Dedicated to the memory of Professor Cemal Koç

Abstract. We obtain an exact formula for the average of values of L-series over two independent odd characters. The average of any positive moment of values at $s = 1$ is then expressed in terms of finite cotangent sums subject to congruence conditions. As consequences, bounds on such cotangent sums, limit points for the average of first moment of L-series at $s = 1$ and the average size of positive moments of character sums related to the class number are deduced.

1. Introduction

Let χ be a Dirichlet character modulo k and

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

be the corresponding L-series for $\mathrm{Re}(s) > 1$. Asymptotic formulas for the power mean averages of special values of L-series were studied from various perspectives in recent literature. In particular, Zhang [10], [12] obtained the asymptotic formulas

$$\sum_{\chi\neq\chi_0}|L(1,\chi)|^2=\frac{\pi^2}{6}\varphi(k)\prod_{p|k}\left(1-\frac{1}{p^2}\right)-\frac{\varphi(k)^2}{k^2}\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2+o(\log\log k),$$

$$\sum_{\chi\neq\chi_0}|L(1,\chi)|^4=\frac{5\pi^4}{72}\varphi(k)\prod_{p|k}\frac{(p^2-1)^3}{p^4(p^2+1)}+O\left(e^{\frac{4\log k}{\log\log k}}\right)$$

and

$$\sum_{\chi\neq\chi_0}|L(1,\chi)|^{2r}=\varphi(k)\sum_{\substack{n\geq 1\\(n,k)=1}}\frac{d_r(n)^2}{n^2}+O\left(e^{\frac{2r\log k}{\log\log k}}\right)$$

for $r \geq 3$, where the summations are over all nonprincipal characters modulo $k \geq 3$, $d_r(n)$ is the generalized number of divisors function that counts distinct ordered

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ways of writing n as a product of r divisors and φ is Euler's totient function. Zhang [11] further gave an asymptotic formula for

$$\sum_{\substack{\chi \\ \chi(-1)=-1}} |L(1,\chi)|^{2r}$$

for $r \geq 2$ as q tends to infinity, where the summation extends over all odd characters of conductor q. When k is a square-full integer, Zhang [13] obtained the exact evaluation

$$\sum_{\substack{\chi \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2 \varphi(k)^3}{12k^2} \prod_{p|k} \left(1+\frac{1}{p}\right),$$

where the summation extends over all primitive odd characters modulo k. Applying the theory of Bernoulli polynomials and generalized Dedekind sums, Liu and Zhang [7] gave exact evaluations of averages of the form

$$\frac{2}{\varphi(k)} \sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi})$$

for any positive integers m, n of the same parity, where the summation is over all odd (or all even) characters modulo k. For additional results involving averages of products of two L-series with integer arguments, we refer the reader to the work of Xu and Zhang [9]. For any complex number z, let

$$G(z,\chi) := \sum_{m=0}^{k-1} \chi(m) e^{\frac{2\pi i m z}{k}}$$

be the Gauss sum. The author [3] (see also [1]) recently obtained evaluations of special values of L-series in terms of weighted averages of Gauss sums of the form

$$\frac{1}{k^r} \sum_{j=1}^{k-1} j^r G(j,\chi)$$

for $r \geq 0$. Making use of this connection, here we prove the following average result in the spirit of [9] over two independent odd characters modulo k. For any $n \geq 1$, we let

$$J_n(k) = k^n \prod_{p|k} \left(1-\frac{1}{p^n}\right)$$

be Jordan's totient function of order $n \geq 1$.

Theorem 1. *For $k \geq 3$, the following formulas hold:*

$$\frac{4}{\varphi(k)^2} \sum_{\substack{\chi_1,\chi_2 (\mathrm{mod}\ k) \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) = \pi^4 \left(\frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4}\right)$$

$$= \pi^4 \sum_{\substack{1\leq m\leq k-1 \\ (m,k)=1}} \frac{-k^{-3}}{\left(e^{\frac{2\pi i m}{k}}-1\right)} + \frac{(4k^{-4}-3k^{-3})}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2} + \frac{(12k^{-4}-2k^{-3})}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} + \frac{8k^{-4}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4}.$$

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We remark that Theorem 1 is only a representative of many similar averaging results of values of L-series. In general, it turns out that if there are h independent odd characters (or h independent even characters) $\chi_1, ..., \chi_h$ modulo k and $h+1$ values of the L-series satisfying suitable parity conditions, then it is possible to give an exact formula for the average of products of these values in terms of Jordan's totient function and in terms of finite sums involving negative powers of $e^{\frac{2\pi i m}{k}} - 1$. As an alternative, we show that any moment of values at $s = 1$ can be exactly determined in terms of finite cotangent sums subject to congruence conditions modulo k, which is reminiscent of Kloosterman sums.

Theorem 2. *For any integers $k \geq 3$ and $r \geq 1$, the formula*

$$\frac{2}{\varphi(k)} \sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}} L(1,\chi)^r = \frac{\pi^r}{2^{r-1}k^r} \sum_{\substack{m_1,...,m_r \pmod k \\ m_1...m_r \equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right) ... \cot\left(\frac{\pi m_r}{k}\right)$$

holds, where each m_j runs over a reduced residue system modulo k. Consequently, one has the estimate

$$\frac{2}{\varphi(k)} \left| \sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}} L(1,\chi)^r \right| \leq \frac{3^{r-1}\pi^r}{2^{2r-1}}.$$

For any given $\epsilon > 0$ and for all k large enough only in terms of ϵ, the above bound becomes

$$\frac{(2+\epsilon)^{r-1}\pi^r}{2^{2r-1}}.$$

Moreover, for any $r \geq 1$, we have

$$\lim_{k\to\infty} \frac{2}{\varphi(k)} \left| \sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}} L(1,\chi)^r \right| = 1.$$

Let us mention that the distribution of normalized r-tuples $\left(\frac{m_1}{k}, ..., \frac{m_r}{k}\right)$ subject to congruence conditions such as $m_1...m_r \equiv 1 \pmod k$ was recently studied by the author, Stan and Zaharescu [4] in connection with Lehmer's problem on the parity of inverses modulo k. Consequences of Theorem 2 are now in order.

Corollary 1. *For any integer $k \geq 3$, the inequality*

$$\left| \sum_{\substack{m \pmod k \\ (m,k)=1}} \cot\left(\frac{\pi m}{k}\right) \cot\left(\frac{\pi \overline{m}}{k}\right) \right| \leq \frac{J_2(k)}{3} - \varphi(k)$$

holds, where $m\overline{m} \equiv 1 \pmod k$ and $1 \leq \overline{m} \leq k-1$. Moreover, for any $r \geq 1$, we have the asymptotic relation

$$\left| \sum_{\substack{m_1,...,m_r \pmod k \\ m_1...m_r \equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right) ... \cot\left(\frac{\pi m_r}{k}\right) \right| = (1+o(1))\frac{2^{r-1}k^r}{\pi^r}.$$

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Note that when $k = 3, 4, 6$, the inequality in Corollary 1 becomes an equality. Although the asymptotic behavior of odd moments of $|L(1,\chi)|$ over odd characters is not yet fully understood, we still have good control on the limit points of the average of first moment.

Corollary 2. *The inequalities*

$$1 \leq \liminf_{k\to\infty} \frac{2}{\varphi(k)} \sum_{\substack{\chi \pmod{k} \\ \chi(-1)=-1}} |L(1,\chi)| \leq \limsup_{k\to\infty} \frac{2}{\varphi(k)} \sum_{\substack{\chi \pmod{k} \\ \chi(-1)=-1}} |L(1,\chi)| \leq \frac{\pi}{\sqrt{6}} = 1.282...$$

hold.

A natural question is whether the set of values

$$\frac{2}{\varphi(k)} \sum_{\substack{\chi \pmod{k} \\ \chi(-1)=-1}} |L(1,\chi)|$$

is dense or not in the interval formed by its smallest and largest limit points. If $q \geq 7$ is a prime with $q \equiv 3 \pmod 4$ and $\chi(j) = \left(\frac{j}{q}\right)$ is the Legendre symbol modulo q, then the class number is given by the remarkable formulas (see [6])

$$\frac{\sqrt{q}}{\pi} L(1,\chi) = h(-q) = -\frac{1}{q} \sum_{j=1}^{q-1} j\chi(j).$$

Therefore, using Siegel's estimate $|L(1,\chi)| \gg_\epsilon \frac{1}{q^\epsilon}$ for any $\epsilon > 0$, we see that

$$q^{\frac{3}{2}-\epsilon} \ll_\epsilon \left| \sum_{j=1}^{q-1} j\chi(j) \right|$$

with an ineffective constant. In contrast with this situation, our final result shows that all positive moments of such character sums behave regularly on average. For more connections between the class number and specific trigonometric sums, one may consult the work of Berndt and Zaharescu [5].

Corollary 3. *Let q be an odd prime. Then for any $r \geq 1$, the formula*

$$\sum_{\substack{m_1,\dots,m_r \pmod{q} \\ m_1 \dots m_r \equiv 1 \pmod{q}}} \cot\left(\frac{\pi m_1}{q}\right) \dots \cot\left(\frac{\pi m_r}{q}\right) = \frac{(2i)^r}{q^r(q-1)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \left(G(1,\chi) \sum_{j=1}^{q-1} j\overline{\chi}(j) \right)^r$$

holds. Moreover, we have

$$q^{\frac{1}{2}} \cot\left(\frac{\pi}{q}\right) \leq \frac{2}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \left| \sum_{j=1}^{q-1} j\chi(j) \right| \leq \frac{q^{\frac{3}{2}}}{\sqrt{6}}$$

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and for any $r \geq 1$,

$$q^{\frac{3r}{2}} \ll_r \frac{2}{q-1} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \left| \sum_{j=1}^{q-1} j\chi(j) \right|^r \ll_r q^{\frac{3r}{2}}$$

with effective constants depending only on r.

2. Proof of Theorem 1

First we observe that if χ_1 and χ_2 are odd characters modulo k, then $\overline{\chi_1\chi_2}$ is an even character modulo k. Therefore, as a consequence of the evaluation of L-series values given in [3], we obtain

$$L(1,\chi_1) = \frac{\pi i}{k^2} \sum_{j_1=1}^{k-1} j_1 \; G(j_1,\chi_1), \tag{2.1}$$

$$L(1,\chi_2) = \frac{\pi i}{k^2} \sum_{j_2=1}^{k-1} j_2 \; G(j_2,\chi_2), \tag{2.2}$$

$$L(2,\overline{\chi_1\chi_2}) = \begin{cases} \frac{\pi^2}{k^3} \sum_{j_3=1}^{k-1} j_3^2 \; G(j_3,\overline{\chi_1\chi_2}) & \text{if } \chi_1\chi_2 \neq \chi_0 \\ \\ \frac{\pi^2}{k^3} \left(\frac{k^2\varphi(k)}{2} + \sum_{j_3=1}^{k-1} j_3^2 \; G(j_3,\overline{\chi_1\chi_2}) \right) & \text{if } \chi_1\chi_2 = \chi_0, \end{cases} \tag{2.3}$$

where χ_0 is the principal character modulo k. Using (2.3), note the decomposition

$$\begin{aligned} &\sum_{\substack{\chi_1,\chi_2 \pmod k \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) \\ &\quad = \sum_{\substack{\chi_1,\chi_2 \pmod k \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1 \\ \chi_1\chi_2\neq\chi_0}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) + \sum_{\substack{\chi_1,\chi_2 \pmod k \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1 \\ \chi_1\chi_2=\chi_0}} L(1,\chi_1)L(1,\chi_2)L(2,\chi_0) \\ &\quad = \frac{\pi^2}{k^3} \sum_{\substack{\chi_1,\chi_2 \pmod k \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2) \sum_{j_3=1}^{k-1} j_3^2 \; G(j_3,\overline{\chi_1\chi_2}) \\ &\qquad + \frac{\pi^2\varphi(k)}{2k} \sum_{\substack{\chi_1,\chi_2 \pmod k \\ \chi_1(-1)=-1 \\ \chi_2(-1)=-1 \\ \chi_1\chi_2=\chi_0}} L(1,\chi_1)L(1,\chi_2). \end{aligned}$$

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Using (2.1), (2.2) and the definition of the Gauss sum, we see that

(2.4)

$$\begin{aligned}
&\frac{\pi^2}{k^3}\sum_{\substack{\chi_1,\chi_2 \pmod k\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)\sum_{j_3=1}^{k-1} j_3^2\ G(j_3,\overline{\chi_1\chi_2})\\
&=-\frac{\pi^4}{k^7}\sum_{\substack{\chi_1,\chi_2 \pmod k\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}}\left(\sum_{j_1=1}^{k-1} j_1\ G(j_1,\chi_1)\right)\left(\sum_{j_2=1}^{k-1} j_2\ G(j_2,\chi_2)\right)\left(\sum_{j_3=1}^{k-1} j_3^2\ G(j_3,\overline{\chi_1\chi_2})\right)\\
&=-\frac{\pi^4}{k^7}\sum_{1\le j_1,j_2,j_3\le k-1} j_1j_2j_3^2\sum_{\substack{\chi_1,\chi_2 \pmod k\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}}\left(\sum_{m_1=1}^{k-1}\chi_1(m_1)e^{\frac{2\pi i m_1j_1}{k}}\right)\\
&\quad\times\left(\sum_{m_2=1}^{k-1}\chi_2(m_2)e^{\frac{2\pi i m_2j_2}{k}}\right)\left(\sum_{m_3=1}^{k-1}\overline{\chi_1(m_3)}\,\overline{\chi_2(m_3)}e^{\frac{2\pi i m_3j_3}{k}}\right)\\
&=-\frac{\pi^4}{k^7}\sum_{1\le j_1,j_2,j_3\le k-1} j_1j_2j_3^2\sum_{1\le m_1,m_2,m_3\le k-1} e^{\frac{2\pi i m_1j_1}{k}}e^{\frac{2\pi i m_2j_2}{k}}e^{\frac{2\pi i m_3j_3}{k}}\\
&\quad\times\left(\sum_{\chi_1}\chi_1(m_1)\overline{\chi_1(m_3)}\right)\left(\sum_{\chi_2}\chi_2(m_2)\overline{\chi_2(m_3)}\right).
\end{aligned}$$

By the orthogonality of characters modulo k, one gets

$$\sum_{\chi_1}\chi_1(m_1)\overline{\chi_1(m_3)}=\begin{cases}\frac{\varphi(k)}{2} & \text{if } m_1=m_3\\ -\frac{\varphi(k)}{2} & \text{if } m_1=-m_3\\ 0 & \text{otherwise,}\end{cases}\tag{2.5}$$

$$\sum_{\chi_2}\chi_2(m_2)\overline{\chi_2(m_3)}=\begin{cases}\frac{\varphi(k)}{2} & \text{if } m_2=m_3\\ -\frac{\varphi(k)}{2} & \text{if } m_2=-m_3\\ 0 & \text{otherwise}\end{cases}\tag{2.6}$$

for $k\ge 3$. Clearly, (2.5) and (2.6) introduce four cases for m_1,m_2,m_3, namely, $m_1=m_2=m_3$, $m_1=-m_2=m_3$, $m_1=-m_2=-m_3$ and $m_1=m_2=-m_3$. Taking these cases into account and rewriting (2.4), we deduce that

(2.7)

$$\frac{4\pi^2}{\varphi(k)^2k^3}\sum_{\substack{\chi_1,\chi_2 \pmod k\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)\sum_{j_3=1}^{k-1} j_3^2\ G(j_3,\overline{\chi_1\chi_2})=-\frac{\pi^4}{k^7}(A-2B+C),$$

where A,B,C are defined as

$$A:=\sum_{\substack{1\le m\le k-1\\ (m,k)=1}}\left(\sum_{1\le j_1\le k-1} j_1e^{\frac{2\pi i m j_1}{k}}\right)^2\left(\sum_{1\le j_3\le k-1} j_3^2e^{\frac{2\pi i m j_3}{k}}\right),\tag{2.8}$$

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(2.9)
$$B := \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \left(\sum_{1\le j_1\le k-1} j_1 e^{\frac{2\pi i m j_1}{k}}\right)\left(\sum_{1\le j_2\le k-1} j_2 e^{-\frac{2\pi i m j_2}{k}}\right)\left(\sum_{1\le j_3\le k-1} j_3^2 e^{\frac{2\pi i m j_3}{k}}\right),$$

(2.10)
$$C := \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \left(\sum_{1\le j_1\le k-1} j_1 e^{\frac{2\pi i m j_1}{k}}\right)^2 \left(\sum_{1\le j_3\le k-1} j_3^2 e^{-\frac{2\pi i m j_3}{k}}\right).$$

To simplify A, B, C, we may use the following formulas, which are elementary to obtain:

$$\sum_{j=1}^{k-1} j e^{\frac{2\pi i m j}{k}} = \frac{k}{\left(e^{\frac{2\pi i m}{k}}-1\right)}, \quad \sum_{j=1}^{k-1} j e^{-\frac{2\pi i m j}{k}} = -k - \frac{k}{\left(e^{\frac{2\pi i m}{k}}-1\right)},$$
$$\sum_{j=1}^{k-1} j^2 e^{\frac{2\pi i m j}{k}} = \frac{k^2}{\left(e^{\frac{2\pi i m}{k}}-1\right)} - \frac{2k e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2},$$
$$\sum_{j=1}^{k-1} j^2 e^{-\frac{2\pi i m j}{k}} = -k^2 - \frac{k^2}{\left(e^{\frac{2\pi i m}{k}}-1\right)} - \frac{2k e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2}.$$

From (2.8)-(2.10), we may rewrite A, B, C in the form

(2.11)
$$A = \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{2k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4},$$

(2.12)
$$B = \sum_{\substack{1\le m\le k-1\\(m,k)=1}} -\frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2} + \frac{2k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} + \frac{2k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4},$$

(2.13)
$$C = \sum_{\substack{1\le m\le k-1\\(m,k)=1}} -\frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2} - \frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{2k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4}.$$

Combining (2.7), (2.11)-(2.13), we see that

(2.14)
$$\frac{4\pi^2}{\varphi(k)^2 k^3} \sum_{\substack{\chi_1,\chi_2 \pmod k\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2) \sum_{j_3=1}^{k-1} j_3^2\ G(j_3, \overline{\chi_1\chi_2})$$
$$= -\frac{\pi^4}{k^7} \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2} + \frac{2k^4}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{4k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{8k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4}$$
$$= -\frac{\pi^4}{k^7} \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{(k^4-4k^3)}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2} + \frac{(2k^4-12k^3)}{\left(e^{\frac{2\pi i m}{k}}-1\right)^3} - \frac{8k^3}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4}.$$

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Next we simplify the sum in the second line of (2.14). Using the fact that $\sin\left(\frac{2\pi m}{k}\right)$ is an odd function in the range of summation, one has

$$\begin{aligned}(2.15)\qquad &\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{k^4}{\left(e^{\frac{2\pi im}{k}}-1\right)^2}=k^4\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{e^{-\frac{2\pi im}{k}}}{(e^{\frac{\pi im}{k}}-e^{-\frac{\pi im}{k}})^2}\\&=-\frac{k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{\cos\left(\frac{2\pi m}{k}\right)-i\sin\left(\frac{2\pi m}{k}\right)}{\sin^2\left(\frac{\pi m}{k}\right)}=-\frac{k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{\cos\left(\frac{2\pi m}{k}\right)}{\sin^2\left(\frac{\pi m}{k}\right)}\\&=-\frac{k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{1-2\sin^2\left(\frac{\pi m}{k}\right)}{\sin^2\left(\frac{\pi m}{k}\right)}=-\frac{k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{1}{\sin^2\left(\frac{\pi m}{k}\right)}+\frac{k^4\varphi(k)}{2}.\end{aligned}$$

By the evaluation procedure of L-series values given in [3] (see also [2]), it is possible to derive the formula

$$(2.16)\qquad \sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{1}{\sin^2\left(\frac{\pi m}{k}\right)}=\frac{J_2(k)}{3}.$$

Combining (2.15) and (2.16), we obtain

$$(2.17)\qquad \sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{k^4}{\left(e^{\frac{2\pi im}{k}}-1\right)^2}=-\frac{k^4J_2(k)}{12}+\frac{k^4\varphi(k)}{2}.$$

Observing that $\cos\left(\frac{3\pi m}{k}\right)$ is an odd function in the range of summation, we have

$$\begin{aligned}(2.18)\qquad &\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{2k^4}{\left(e^{\frac{2\pi im}{k}}-1\right)^3}=2k^4\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{\cos\left(\frac{3\pi m}{k}\right)-i\sin\left(\frac{3\pi m}{k}\right)}{(2i)^3\sin^3\left(\frac{3\pi m}{k}\right)}\\&=\frac{k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{3\cos^2\left(\frac{\pi m}{k}\right)\sin\left(\frac{\pi m}{k}\right)-\sin^3\left(\frac{\pi m}{k}\right)}{\sin^3\left(\frac{\pi m}{k}\right)}\\&=\frac{3k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{\cos^2\left(\frac{\pi m}{k}\right)}{\sin^2\left(\frac{\pi m}{k}\right)}-\frac{k^4\varphi(k)}{4}\\&=\frac{3k^4}{4}\sum_{\substack{1\le m\le k-1\\(m,k)=1}}\frac{1-\sin^2\left(\frac{\pi m}{k}\right)}{\sin^2\left(\frac{\pi m}{k}\right)}-\frac{k^4\varphi(k)}{4}=\frac{k^4J_2(k)}{4}-k^4\varphi(k),\end{aligned}$$

where we used (2.16). Similarly, one obtains

$$(2.19)\qquad \sum_{\substack{1\le m\le k-1\\(m,k)=1}}-\frac{4k^3e^{\frac{2\pi im}{k}}}{\left(e^{\frac{2\pi im}{k}}-1\right)^3}=-\frac{k^3J_2(k)}{6}.$$

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Next we have

$$\sum_{\substack{1\le m\le k-1\\(m,k)=1}} -\frac{8k^3 e^{\frac{2\pi i m}{k}}}{\left(e^{\frac{2\pi i m}{k}}-1\right)^4} = -\frac{k^3}{2}\sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{\cos\left(\frac{2\pi m}{k}\right)-i\sin\left(\frac{2\pi m}{k}\right)}{\sin^4\left(\frac{\pi m}{k}\right)} \tag{2.20}$$

$$= -\frac{k^3}{2}\sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{1}{\sin^4\left(\frac{\pi m}{k}\right)} + k^3\sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{1}{\sin^2\left(\frac{\pi m}{k}\right)} = -\frac{k^3 J_4(k)}{90} + \frac{2k^3 J_2(k)}{9},$$

where we used (2.16) again and the formula (see [3], [2])

$$\sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{1}{\sin^4\left(\frac{\pi m}{k}\right)} = \frac{J_4(k)}{45} + \frac{2J_2(k)}{9}.$$

Combining (2.14), (2.17)-(2.20), we see that

$$\frac{4\pi^2}{\varphi(k)^2 k^3}\sum_{\substack{\chi_1,\chi_2 (\mathrm{mod}\ k)\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)\sum_{j_3=1}^{k-1} j_3^2\, G(j_3,\overline{\chi_1\chi_2}) \tag{2.21}$$

$$= \pi^4\left(\frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4} - \frac{J_2(k)}{6k^3} + \frac{\varphi(k)}{2k^3}\right).$$

Using (2.1), (2.2), together with orthogonality of characters and arguing similarly as above, we also have

$$\frac{\pi^2\varphi(k)}{2k}\sum_{\substack{\chi_1,\chi_2 (\mathrm{mod}\ k)\\ \chi_1(-1)=-1\\ \chi_2(-1)=-1\\ \chi_1\chi_2=\chi_0}} L(1,\chi_1)L(1,\chi_2) = \frac{\pi^2\varphi(k)}{2k}\sum_{\substack{\chi (\mathrm{mod}\ k)\\ \chi(-1)=-1}} L(1,\chi)L(1,\overline{\chi})$$

$$= -\frac{\pi^4\varphi(k)^2}{4k^5}\left(\sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{k^2}{\left(e^{\frac{2\pi i m}{k}}-1\right)} + \sum_{\substack{1\le m\le k-1\\(m,k)=1}} \frac{2k^2}{\left(e^{\frac{2\pi i m}{k}}-1\right)^2}\right) \tag{2.22}$$

$$= \frac{\pi^4\varphi(k)^2}{4}\left(\frac{J_2(k)}{6k^3} - \frac{\varphi(k)}{2k^3}\right).$$

Gathering (2.14), (2.21) and (2.22), one can complete the proof of Theorem 1.

3. Proof of Theorem 2

Using (2.1) and orthogonality of characters, we have

$$\frac{2}{\varphi(k)}\sum_{\substack{\chi (\mathrm{mod}\ k)\\ \chi(-1)=-1}} L(1,\chi)^r = \frac{2}{\varphi(k)}\left(\frac{\pi i}{k^2}\right)^r \sum_{\substack{\chi (\mathrm{mod}\ k)\\ \chi(-1)=-1}} \left(\sum_{j=1}^{k-1} jG(j,\chi)\right)^r \tag{3.1}$$

$$= \frac{2\pi^r i^r}{\varphi(k)k^{2r}}\sum_{1\le j_1,\dots,j_r\le k-1} j_1\dots j_r \sum_{1\le m_1,\dots,m_r\le k-1} e^{\frac{2\pi i m_1 j_1}{k}}\dots e^{\frac{2\pi i m_r j_r}{k}}\sum_{\chi}\chi(m_1)\dots\chi(m_r)$$

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$$= \frac{\pi^r i^r}{k^{2r}} \left(\sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \prod_{s=1}^{r} \left(\sum_{j_s=1}^{k-1} j_s e^{\frac{2\pi i m_s j_s}{k}} \right) \right. \tag{3.2}$$

$$\left. - \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv -1 \pmod k}} \prod_{s=1}^{r} \left(\sum_{j_s=1}^{k-1} j_s e^{\frac{2\pi i m_s j_s}{k}} \right) \right)$$

$$= \frac{\pi^r i^r}{k^r} \left(\sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 (\mathrm{mod}\, k)}} \prod_{s=1}^{r} \frac{1}{(e^{\frac{2\pi i m_s}{k}}-1)} - \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv -1 (\mathrm{mod}\, k)}} \prod_{s=1}^{r} \frac{1}{(e^{\frac{2\pi i m_s}{k}}-1)} \right).$$

Note that by replacing m_1 with $-m_1$, the congruence $m_1...m_r \equiv -1 \pmod k$ transforms to $m_1...m_r \equiv 1 \pmod k$. Therefore, (3.1) further equals

$$= \frac{\pi^r i^r}{k^r} \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \prod_{s=1}^{r} \frac{1}{(e^{\frac{2\pi i m_s}{k}}-1)} - \frac{1}{(e^{-\frac{2\pi i m_1}{k}}-1)} \prod_{s=2}^{r} \frac{1}{(e^{\frac{2\pi i m_s}{k}}-1)} \tag{3.3}$$

$$= \frac{\pi^r i^r}{k^r} \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \frac{(e^{\frac{2\pi i m_1}{k}}+1)}{(e^{\frac{2\pi i m_1}{k}}-1)} \dots (e^{\frac{2\pi i m_r}{k}}-1)$$

$$\sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \frac{(e^{\frac{\pi i m_1}{k}}+e^{-\frac{\pi i m_1}{k}})e^{-\frac{\pi i m_2}{k}} \dots e^{-\frac{\pi i m_r}{k}}}{(e^{\frac{\pi i m_1}{k}}-e^{-\frac{\pi i m_1}{k}})(e^{\frac{\pi i m_2}{k}}-e^{-\frac{\pi i m_2}{k}}) \dots (e^{\frac{\pi i m_r}{k}}-e^{-\frac{\pi i m_r}{k}})}$$

$$= \frac{\pi^r}{2^{r-1}k^r} \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right) \left(\cot\left(\frac{\pi m_2}{k}\right) - i\right) \dots \left(\cot\left(\frac{\pi m_r}{k}\right) - i\right).$$

If i is chosen from one of the above parentheses in the summation, then the remaining m_j's are independent of each other, and using the fact that $\cot\left(\frac{\pi m}{k}\right)$ is an odd function in the range of summation, we deduce that there is no contribution from such terms. Consequently, (3.1) and (3.2) give

$$\frac{2}{\varphi(k)} \sum_{\substack{\chi (\mathrm{mod}\ k) \\ \chi(-1)=-1}} L(1,\chi)^r = \frac{\pi^r}{2^{r-1}k^r} \sum_{\substack{1\le m_1,\dots,m_r\le k-1 \\ m_1\dots m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right) \dots \cot\left(\frac{\pi m_r}{k}\right). \tag{3.4}$$

For $1 \le j \le r$, we have

$$\left|\cot\left(\frac{\pi m_j}{k}\right)\right| \le \frac{1}{\left|\sin\left(\frac{\pi m_j}{k}\right)\right|}. \tag{3.5}$$

We may assume without loss of generality that each m_j lies in the interval $\left(-\frac{k}{2}, \frac{k}{2}\right)$ so that the estimate

$$\left|\sin\left(\frac{\pi m_j}{k}\right)\right| \ge \frac{2|m_j|}{k} \tag{3.6}$$

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follows. Therefore, (3.4) and (3.5) give

$$\left|\cot\left(\frac{\pi m_j}{k}\right)\right| \le \frac{k}{2|m_j|} \tag{3.7}$$

for every $1 \le j \le r$. Using (3.6), we obtain

(3.8)

$$\left|\sum_{\substack{1\le m_1,\dots,m_r\le k-1\\ m_1\dots m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right)\dots\cot\left(\frac{\pi m_r}{k}\right)\right| \le \frac{k^r}{2^r} \sum_{\substack{m_1,\dots,m_r\in\left(-\frac{k}{2},\frac{k}{2}\right)\\ m_1\dots m_r\equiv 1 \pmod k}} \frac{1}{|m_1|\dots|m_r|}$$

$$= \frac{k^r}{2^r}\prod_{j=1}^{r-1}\left(\sum_{m_j\in\left(-\frac{k}{2},\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|}\right),$$

where $m_j\overline{m_j} \equiv 1 \pmod k$ and $\overline{m_j} \in \left(-\frac{k}{2},\frac{k}{2}\right)$. By symmetry, we have

$$\sum_{m_j\in\left(-\frac{k}{2},\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|} = 2\sum_{m_j\in\left(0,\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|}. \tag{3.9}$$

Moreover, note that

$$\sum_{m_j\in\left(0,\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|} = 1 + \sum_{m_j\in\left(1,\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|}. \tag{3.10}$$

If $m_j > 1$ and $m_j\overline{m_j} \equiv 1 \pmod k$, then $|m_j||\overline{m_j}| \ge k+1$ and the number of such $m_j \in \left(1,\frac{k}{2}\right)$ is at most $\frac{\varphi(k)}{2}$. Combining (3.8) and (3.9), we see that

$$\sum_{m_j\in\left(-\frac{k}{2},\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|} \le 2\left(1+\frac{\varphi(k)}{2(k+1)}\right) \le 3 \tag{3.11}$$

for every $1 \le j \le r-1$. Finally, (3.3), (3.7) and (3.10) give

$$\frac{2}{\varphi(k)}\left|\sum_{\substack{\chi \pmod k\\ \chi(-1)=-1}} L(1,\chi)^r\right| \le \frac{3^{r-1}\pi^r}{2^{2r-1}}.$$

To prove the remaining claims, we estimate the sum in (3.10) more carefully for large k. Note that if $m_j \in \left(1,\frac{k}{2}\right)$ and $m_j\overline{m_j} \equiv 1 \pmod k$ with $\overline{m_j} \in \left(-\frac{k}{2},\frac{k}{2}\right)$, then $|m_j||\overline{m_j}| \in \{nk+1 \ : \ 1 \le n \le k/4\}$. For each integer n with $1 \le n \le k/4$, the number of m_j with $|m_j||\overline{m_j}| = nk+1$ is at most $\tau(nk+1)$, where τ is the number of divisors. In this way, (3.10) becomes

$$\sum_{m_j\in\left(-\frac{k}{2},\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|} \le 2\left(1+\sum_{1\le n\le k/4}\frac{\tau(nk+1)}{nk+1}\right). \tag{3.12}$$

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Noting that $nk+1 \leq \frac{k^2}{4}+1 \leq k^2$, we have $\tau(nk+1) \ll e^{\frac{C\log k}{\log\log k}}$ for some constant $C>0$. It follows that

$$\sum_{1\leq n\leq k/4} \frac{\tau(nk+1)}{nk+1} \leq \frac{1}{k}\sum_{1\leq n\leq k/4} \frac{\tau(nk+1)}{n} \ll \frac{(\log k)e^{\frac{C\log k}{\log\log k}}}{k}.$$

Combining this with (3.11), we have for all k large enough in terms of ϵ that

$$\sum_{m_j\in\left(-\frac{k}{2},\frac{k}{2}\right)} \frac{1}{|m_j||\overline{m_j}|} \leq 2+\epsilon$$

and consequently the bound $\frac{(2+\epsilon)^{r-1}\pi^r}{2^{2r-1}}$ is obtained for all such k. Observing that the number of r-tuples $(m_1,...,m_r)\in\{1,k-1\}^r$ satisfying $m_1...m_r\equiv 1 \pmod k$ is 2^{r-1}, we may write

$$\begin{aligned}(3.13)\qquad & 2^{r-1}\left(\cot\left(\frac{\pi}{k}\right)\right)^r - \sum_{j=1}^{r}\left|\sum_{\substack{1\leq m_1,...,m_r\leq k-1\\ m_j\neq 1,k-1\\ m_1...m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right)...\cot\left(\frac{\pi m_r}{k}\right)\right| \\ &\leq \left|\sum_{\substack{1\leq m_1,...,m_r\leq k-1\\ m_1...m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right)...\cot\left(\frac{\pi m_r}{k}\right)\right| \\ &\leq 2^{r-1}\left(\cot\left(\frac{\pi}{k}\right)\right)^r + \sum_{j=1}^{r}\left|\sum_{\substack{1\leq m_1,...,m_r\leq k-1\\ m_j\neq 1,k-1\\ m_1...m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right)...\cot\left(\frac{\pi m_r}{k}\right)\right|.\end{aligned}$$

Estimating as above, we see that

$$(3.14)\qquad \left|\sum_{\substack{1\leq m_1,...,m_r\leq k-1\\ m_j\neq 1,k-1\\ m_1...m_r\equiv 1 \pmod k}} \cot\left(\frac{\pi m_1}{k}\right)...\cot\left(\frac{\pi m_r}{k}\right)\right| \leq \frac{3^{r-2}k^{r-1}e^{\frac{C\log k}{\log\log k}}}{2^r}$$

for some constant $C>0$. Using the fact that $\left(\cot\left(\frac{\pi}{k}\right)\right)^r \sim \left(\frac{k}{\pi}\right)^r$, we may obtain from (3.12) and (3.13) applied to all $1\leq j\leq r$ that

$$(3.15)\qquad \left|\sum_{1\leq m_1,...,m_r\leq k-1 m_1...m_r\equiv 1 \pmod k} \cot\left(\frac{\pi m_1}{k}\right)...\cot\left(\frac{\pi m_r}{k}\right)\right| = (1+o(1))\frac{2^{r-1}k^r}{\pi^r}$$

as k tends to infinity. Now using (3.3), we complete the proof of Theorem 2.

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4. Proof of Corollary 1

Taking $r = 2$ in Theorem 2, we have

$$\frac{\pi^2}{2k^2}\left|\sum_{\substack{m(\mathrm{mod}\ k)\\(m,k)=1}}\cot\left(\frac{\pi m}{k}\right)\cot\left(\frac{\pi\overline{m}}{k}\right)\right| = \frac{2}{\varphi(k)}\left|\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}L(1,\chi)^2\right| \leq \frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|^2 \tag{4.1}$$

for $k \geq 3$. It is known that (see [8], [14], [2])

$$\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|^2 = \frac{\pi^2}{2k^2}\left(\frac{J_2(k)}{3}-\varphi(k)\right) \tag{4.2}$$

for $k \geq 3$. The desired inequality and asymptotic relation now follow from (4.1), (4.2) and (3.14).

5. Proof of Corollary 2

Taking $r = 1$ in Theorem 2, we see that

$$\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}L(1,\chi) = \frac{\pi}{k}\cot\left(\frac{\pi}{k}\right) \tag{5.1}$$

for $k \geq 3$. Therefore, using (5.1), one obtains

$$\frac{\pi}{k}\cot\left(\frac{\pi}{k}\right) = \frac{2}{\varphi(k)}\left|\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}L(1,\chi)\right| \leq \frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|. \tag{5.2}$$

Since

$$\lim_{k\to\infty}\frac{\pi}{k}\cot\left(\frac{\pi}{k}\right) = 1,$$

we deduce from (5.2) that

$$1 \leq \liminf_{k\to\infty}\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|.$$

On the other hand, from (4.2) we obtain

$$\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|^2 \leq \frac{\pi^2}{6}. \tag{5.3}$$

Using (5.3) and the Cauchy-Schwarz inequality, we have

$$\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)| \leq \left(\frac{2}{\varphi(k)}\sum_{\substack{\chi(\mathrm{mod}\ k)\\\chi(-1)=-1}}|L(1,\chi)|^2\right)^{\frac{1}{2}} \leq \frac{\pi}{\sqrt{6}}, \tag{5.4}$$

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and

$$\limsup_{k\to\infty}\frac{2}{\varphi(k)}\sum_{\substack{\chi \pmod k\\ \chi(-1)=-1}}|L(1,\chi)|\leq\frac{\pi}{\sqrt{6}}$$

follows from (5.4).

6. Proof of Corollary 3

Since q is an odd prime, every odd character χ modulo q is primitive and

$$G(j,\chi)=\overline{\chi}(j)G(1,\chi)$$

for $1\leq j\leq q-1$. Therefore, from (2.1) one obtains

$$L(1,\chi)=\frac{\pi i}{q^2}\sum_{j=1}^{q-1}jG(j,\chi)=\frac{\pi i}{q^2}G(1,\chi)\sum_{j=1}^{q-1}j\overline{\chi}(j). \tag{6.1}$$

Taking $k=q$ in Theorem 2, we have

$$\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}L(1,\chi)^r=\frac{\pi^r}{2^{r-1}q^r}\sum_{\substack{1\leq m_1,\dots,m_r\leq q-1\\ m_1\dots m_r\equiv 1 \pmod q}}\cot\left(\frac{\pi m_1}{q}\right)\dots\cot\left(\frac{\pi m_r}{q}\right). \tag{6.2}$$

The desired formula

$$\sum_{\substack{m_1,\dots,m_r \pmod q\\ m_1\dots m_r\equiv 1 (\mathrm{mod} q)}}\cot\left(\frac{\pi m_1}{q}\right)\dots\cot\left(\frac{\pi m_r}{q}\right)=\frac{(2i)^r}{q^r(q-1)}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}\left(G(1,\chi)\sum_{j=1}^{q-1}j\overline{\chi}(j)\right)^r$$

follows from (6.1) and (6.2). Since $|G(1,\chi)|=\sqrt{q}$ and $L(1,\chi)\neq 0$, from (6.1) one obtains

$$\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}\left|\sum_{j=1}^{q-1}j\chi(j)\right|^r=\frac{2q^{\frac{3r}{2}}}{\pi^r(q-1)}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}|L(1,\chi)|^r>0 \tag{6.3}$$

for any $r\geq 1$. As a consequence of Theorem 2, we can find a constant $C_r>0$ depending only on r, such that

$$C_r\leq\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}|L(1,\chi)|^r \tag{6.4}$$

for all odd primes q. (6.3) and (6.4) now give that

$$q^{\frac{3r}{2}}\ll_r\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}\left|\sum_{j=1}^{q-1}j\chi(j)\right|^r.$$

Moreover, observe that

$$\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi(-1)=-1}}|L(1,\chi)|^r\leq\frac{2}{q-1}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}}|L(1,\chi)|^r\leq D_r \tag{6.5}$$

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for some constant $D_r > 0$ depending only on r by using the asymptotic formulas of Zhang [10], [12] when $r \geq 2$ is even (or one may argue directly by converting to cotangent sums) and by using the Cauchy-Schwarz inequality to get

$$\frac{2}{q-1} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |L(1,\chi)|^r \leq 2 \left(\frac{1}{q-1} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} |L(1,\chi)|^{2r} \right)^{\frac{1}{2}}$$

when $r \geq 1$ is odd. (6.3) and (6.5) give

$$\frac{2}{q-1} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \left| \sum_{j=1}^{q-1} j\chi(j) \right|^r \ll_r q^{\frac{3r}{2}}$$

as desired. Finally, taking $k = q$ in (5.2) and (5.4) and using (6.1), one gets

$$q^{\frac{1}{2}} \cot\left(\frac{\pi}{q}\right) \leq \frac{2}{q-1} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \left| \sum_{j=1}^{q-1} j\chi(j) \right| \leq \frac{q^{\frac{3}{2}}}{\sqrt{6}}.$$

This completes the proof of Corollary 3.

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