Maximum uniformly resolvable decompositions of K_v and $K_v - I$ into 3-stars and 3-cycles

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Abstract

Let K_v denote the complete graph of order v and $K_v - I$ denote K_v minus a 1-factor. In this article we investigate uniformly resolvable decompositions of K_v and $K_v - I$ into r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles. We completely determine the spectrum in the case where the number of resolution classes of 3-stars is maximum.

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1 Introduction

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph G is a decomposition of the edges of G into isomorphic copies of graphs from $\mathcal{H}$, the copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of G appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of G is sometimes also referred to as an *$\mathcal{H}$-factorization of G*, a class can be called a *$\mathcal{H}$-factor of G*. The case where $\mathcal{H}$ is a single edge (K_2) is known as a *1-factorization of G* and it is well known to exist for $G = K_v$ if and only if v is even. A single class of a 1-factorization, a pairing of all points, is also known as a *1-factor* or a *perfect matching*.

In many cases we wish to impose further constraints on the classes of an $\mathcal{H}$-decomposition. For example, a class is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees [11] which finds necessary and sufficient conditions for the existence of uniform $\{K_2, K_3\}$-decompositions of K_v. Uniformly resolvable decompositions of K_v have also been studied in [3], [6], [7], [9], [10], [13] and [14]. Moreover, recently Dinitz, Ling and Danziger [4] have solved the question of the existence of a uniformly resolvable decomposition of K_v into r classes of K_2 and s classes of K_4 *in the case in which the number s of K_4-factors is maximum*.

1.1 Definitions and notation

For any four vertices a_1, a_2, a_3, a_4, let the 3-*star*, $K_{1,3}$, be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$. In what follows, we will denote it by $(a_1; a_2, a_3, a_4)$.

Let $K_{m(n)}$ denote the complete multipartite graph with m parts each of size n, that is, $K_{m(n)}$ has the vertex set $\{\bigcup_{i=1}^{m} X^i\}$ with $|X^i| = n$ for $i = 1, 2, \ldots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and the edge set $\{\{u, v\} : u \in X^i,\ v \in X^j,\ 1 \leq i < j \leq m\}$.

Let $C_{m(n)}$ denote the graph with the vertex set $\{\bigcup_{i=1}^{m} X^i\}$ with $|X^i| = n$ for $i = 1, 2, \ldots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and the edge set $\{\{u, v\} : u \in X^i,\ v \in X^j,\ i - j \equiv 1 \pmod m$ or $j - i \equiv 1 \pmod m\}$. For constructions below we shall also need the particular case $|X^i| = 12$. Then let $X^i = \{x^i_h : h = 0, 1, \ldots, 11\}$ and for each $j \in \{0, 1, 2, 3\}$ let $X^i_j = \{x^i_{3j}, x^i_{3j+1}, x^i_{3j+2}\}$, so that $X^i = \bigcup_{j=0}^{3} X^i_j$. Define, for each $i = 1, 2, \ldots, m$ and $r, s \in \{0, 1, 2, 3\}$, the following sets of 3-stars:

$$\begin{aligned} R^i_{r,s} &= \{X^i_r; X^{i+1}_s, X^{i+1}_{s+1}, X^{i+1}_{s+2}\} \\ &= \{\{x^i_{3r}; x^{i+1}_{3s}, x^{i+1}_{3s+1}, x^{i+1}_{3s+2}\}, \{x^i_{3r+1}; x^{i+1}_{3s+3}, x^{i+1}_{3s+4}, x^{i+1}_{3s+5}\}, \\ &\quad \{x^i_{3r+2}; x^{i+1}_{3s+6}, x^{i+1}_{3s+7}, x^{i+1}_{3s+8}\}\} \end{aligned}$$

where superscript addition is meant modulo 12.

A resolvable $\mathcal{H}$-decomposition of $K_{m(n)}$ is known as a *resolvable group divisible design* $\mathcal{H}$-RGDD *of type* n^m, where the parts of size n are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-RGDD. We shall use the terms "point" and "vertex" as synonyms.

1.2 Our results

In this paper we study the existence of a uniformly resolvable decomposition of K_v and of $K_v - I$, having the following type:

> r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles.

We will use the notation $(K_{1,3}, K_3)$-URD$(v; r, s)$ for such a uniformly resolvable decomposition of K_v when v is odd, and for that of $K_v - I$ when v is even. We will specify whether the system is a decomposition of K_v or of $K_v - I$ only when it is not clear in the context whether v is odd or even. Further, we will use the notation $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{m(n)}(C_{m(n)})$ to denote a uniformly resolvable decomposition of $K_{m(n)}(C_{m(n)})$ into r classes containing only copies of 3-stars and s classes containing only copies of 3-cycles. As r determines s if v is fixed, we will also use the simplified notation $K_{1,3}$-RGDD(r) for $(K_{1,3}, K_3)$-URGDD(r, s) when v is understood.

Determining the spectrum of triples (v, r, s) which admit a $(K_{1,3}, K_3)$-URD$(v; r, s)$ appears to be a rather hard problem in general. Similarly to the work [4], here we concentrate on the extremal case in which the number of resolution classes of 3-stars is maximum. In particular, we will prove the following result in this paper:

Main Theorem. *For each* $v \equiv 0 \pmod{12}$*, there exists a* $(K_{1,3}, K_3)$*-URD* $(v; \frac{2(v-6)}{3}, 2)$ *of* $K_v - I$*.*

2 Necessary conditions

In this section we will give necessary conditions for the existence of a uniformly resolvable decomposition of K_v and $K_v - I$ into r classes of 3-stars and s classes of 3-cycles.

Lemma 2.1. *A* $(K_{1,3}, K_3)$*-URD*$(v; r, s)$*, with* $r > 0$ *and* $s > 0$*, does not exist for any* $v \geq 4$ *of* K_v*.*

Proof. Assume that there exists a $(K_{1,3}, K_3)$-URD$(v; r, s)$ D of K_v with $r > 0$ and $s > 0$. By resolvability it follows that $v \equiv 0 \pmod{12}$, say $v = 12u$. Counting the edges of K_v that appear in D we obtain

$$\frac{3rv}{4} + \frac{3sv}{3} = \frac{v(v-1)}{2}$$

and hence

$$3r + 4s = 2(v-1). \tag{1}$$

The equality (1) implies that $r \equiv 2 \pmod 4$ and $s \equiv 1 \pmod 3$. Let $r = 2 + 4t$ and $s = 1 + 3h$ with $t, h \geq 0$. Denote by B the set of the r parallel classes of 3-stars and by R the set of the s parallel classes of 3-cycles. Since the classes of R are regular of degree 2, we have that every vertex x of K_v is incident with $2s$ edges in R and $(12u-1)-(2+6h) = 12u-6h-3$ edges in B. Assume that the vertex x appears in a classes with degree 3 and in b classes with degree 1 in B. Since

$$a + b = 2 + 4t \quad \text{and} \quad 3a + b = 12u - 6h - 3,$$

it follows that

$$2a = 12u - 6h - 3 - 2 - 4t = 2(6u - 3h - 1 - 2t) - 3,$$

which is a contradiction, since $2a$ cannot be odd. □

Given $v \equiv 0 \pmod{12}$, define

$$J(v) = \{(4x, \frac{v-2}{2} - 3x) : x = 1, \ldots, \frac{v-6}{6}\}.$$

Lemma 2.2. *If there exists a $(K_{1,3}, K_3)$-URD$(v; r, s)$ of $K_v - I$ with $r > 0$ and $s > 0$ then $v \equiv 0 \pmod{12}$ and $(r, s) \in J(v)$.*

Proof. The condition $v \equiv 0 \pmod{12}$ is trivial by the assumption that both r and s are positive. Let D be a $(K_{1,3}, K_3)$-URD$(v; r, s)$ of $K_v - I$. Counting the edges of $K_v - I$ that appear in D we obtain

$$\frac{3rv}{4} + \frac{3sv}{3} = \frac{v(v-2)}{2},$$

and hence that

$$3r + 4s = 2(v-2). \tag{2}$$

This equality implies that $r \equiv 0 \pmod 4$ and $s \equiv 2 \pmod 3$. Letting now $r = 4x$, the value of s is determined by (2) as $s = \frac{v-2}{2} - 3x$, where $3x \leq \frac{v-2}{2}$ must hold and x must be an integer. Thus, $x \leq \frac{v-6}{6}$ since v is a multiple of 6. This completes the proof. □

3 Constructions and related structures

In this section we will introduce some useful results and discuss constructions we will use in proving the main result. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [2] and its online updates. For some results below, we also cite this handbook instead of the original papers.

A resolvable K_3-decomposition of K_v is called a *Kirkman Triple System* (KTS(v)) and it is well known to exist if and only if $v \equiv 3 \pmod 6$ [2]. Let $u > 1$ be an integer. A 2-RGDD of type g^u exists if and only if gu is even. Moreover, a 3-RGDD of type g^u exists if and only if $g(u-1)$ is even and $gu \equiv 0 \pmod 3$, except when $(g,u) \in \{(2,6),(2,3),(6,3)\}$ [12]. In particular, a 3-RGDD of type 2^u is called a *Nearly Kirkman Triple System* (NKTS($2u$)) and is known to exist whenever $u \equiv 0 \pmod 3$, $u > 6$ [12].

We now recall the existence of some 4-RGDDs we will need in the proof.

Lemma 3.1. *[2] There exists a 4-RGDD of type*

- 6^t *for each* $t \equiv 0 \pmod 2$, $t > 4$, *except when* $t \in \{6, 54, 68\}$;
- 12^t *for each* $t \geq 4$, *except when* $t = 27$.

We also need the following definitions. Let (s_1,t_1) and (s_2,t_2) be two pairs of non-negative integers. Define $(s_1,t_1)+(s_2,t_2) = (s_1+s_2, t_1+t_2)$. If X and Y are two sets of pairs of non-negative integers, then $X+Y$ denotes the set $\{(s_1,t_1)+(s_2,t_2) : (s_1,t_1) \in X, (s_2,t_2) \in Y\}$. If X is a set of pairs of non-negative integers and h is a positive integer, then $h*X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Theorem 3.2. *Let v, g, t, u and x be non-negative integers such that $v = gtu$ and $x \in \{3,4\}$. If there exists*

(1) *an x-RGDD of type g^u;*

(2) *a $(K_{1,3}, K_3)$-URGDD(r_1,s_1) of $K_{x(t)}$ with $(r_1,s_1) \in J_1$;*

(3) *a $(K_{1,3}, K_3)$-URD$(gt; r_2, s_2)$ of $K_{gt} - I_i$, $i = 1,2,\ldots,u$, with $(r_2,s_2) \in J_2$;*

then there exists a $(K_{1,3}, K_3)$-URD$(v; r, s)$ of $K_v - I$ for each $(r,s) \in J_2 + h*J_1$, where $h = \frac{g(u-1)}{x-1}$ is the number of parallel classes of the x-RGDD of type g^u and $I = \bigcup_{i=1}^{u} I_i$.

Proof. Let $(X, \{G_1,\ldots,G_u\}, \mathcal{B})$ be an x-RGDD of type g^u, where the G_i, $i = 1,2,\ldots,u$, are the groups of size g, and $x \in \{3,4\}$. Let $R_1,\ldots,R_{\frac{g(u-1)}{x-1}}$ be the parallel classes of this x-RGDD. Give weight t to all points of this x-RGDD and place on each block of a given resolution class of $\mathcal{B}$ the same

$(K_{1,3}, K_3)$-URGDD(r_1, s_1) with $(r_1, s_1) \in J_1$. For each $i = 1, \ldots, u$, place on $G_i \times \{1, \ldots, t\}$ the same $(K_{1,3}, K_3)$-URD$(gt; r_2, s_2)$ of $K_{gt} - I_i$, $i = 1, 2, \ldots, u$, with $(r_2, s_2) \in J_2$. The result is a $(K_{1,3}, K_3)$-URD$(v; r, s)$ of $K_v - I$ for each $(r, s) \in J_2 + h * J_1$, where $h = \frac{g(u-1)}{x-1}$ is the number of parallel classes of the x-RGDD of type g^u and $I = \cup_{i=1}^{u} I_i$. □

Theorem 3.3. *If there exists $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{x(3)}$, then for each $t \geq 3$ there exists a $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{(xt)(3)}$ into rt parallel classes of 3-stars and rt parallel classes of 3-cycles.*

Proof. Let $K_1, \ldots, K_r$ be the parallel classes of $K_{x(3)}$ containing only copies of 3-stars and $C_1, \ldots, C_s$ be the parallel classes of $K_{x(3)}$ containing only copies of 3-cycles. Give weight t to all points of this $K_{x(3)}$ and for each block $(a; b, c, d)$ of a given resolution class of K_i, $i = 1, \ldots, r$, construct t parallel classes of 3-stars on $\{a, b, c, d\} \times \{1, \ldots, t\}$:

$$\{(a_i; b_{i+j-1}, c_{i+j-1}, d_{i+j-1})\}, \quad j = 1, \ldots, t.$$

For each 3-cycle (a, b, c) of a given resolution class of C_i, $i = 1, \ldots, s$, construct a 3-RGDD of type t^3 on $\{\{a\} \times \{1, \ldots, t\}\} \cup \{\{b\} \times \{1, \ldots, t\}\} \cup \{\{c\} \times \{1, \ldots, t\}\}$ having t parallel classes of 3-cycles, which comes from [12]. The result is a $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{(xt)(3)}$ into rt parallel classes of 3-stars and st parallel classes of 3-cycles. □

Theorem 3.4. *If there exists a $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{x(3)}$, then for each $t \geq 2$ there exists a $(K_{1,3}, K_3)$-URGDD(r, s) of $K_{(xt)(3)}$ into st parallel classes of 3-stars.*

Proof. The proof is similar to Theorem 3.3. □

4 Small cases

Lemma 4.1. *There exists a $(K_{1,3}, K_3)$-URGDD$(4, 0)$ of $K_{4(2)}$.*

Proof. Take the groups to be $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$ and the classes as listed below:
$\{(0; 2, 4, 6), (1; 3, 5, 7)\}$, $\{(2; 4, 1, 6), (3; 5, 0, 7)\}$, $\{(5; 2, 0, 7), (4; 1, 3, 6)\}$,
$\{(6; 1, 3, 5), (7; 0, 4, 2)\}$. □

Lemma 4.2. *There exists a $(K_{1,3}, K_3)$-URD$(12; 4, 2)$ of $K_{12} - I$.*

Proof. Let $V(K_{12})$=$\mathbb{Z}_{12}$, I= $\{(1, 10), (2, 8), (3, 4), (5, 11), (6, 9), (7, 0)\}$ and the classes as listed below:
$\{(1, 2, 3), (4, 5, 6), (7, 8, 9), (0, 10, 11)\}$, $\{(1, 5, 9), (4, 8, 11), (2, 6, 0), (3, 7, 10)\}$;
$\{(1; 4, 7, 11), (2; 5, 9, 10), (3; 6, 8, 0)\}$,$\{$ $(4; 2, 9, 10), (5; 3, 8, 0), (6; 1, 7, 11)\}$,
$\{(7; 2, 4, 5), (8; 1, 6, 10), (9; 3, 0, 11)\}$,$\{(0; 1, 4, 8), (10; 5, 6, 9), (11; 2, 3, 7)\}$. □

Lemma 4.3. *There exists a* $(K_{1,3}, K_3)$*-URGDD*$(8, 2)$ *of* $K_{3(8)}$

Proof. Let $\{a_1, \ldots, a_8\}$, $\{b_1, \ldots, b_8\}$ and $\{c_1, \ldots, c_8\}$ be the groups and the classes as listed below:

$\{(a_0, b_7, c_2), (a_1, b_4, c_3), (a_2, b_5, c_0), (a_3, b_6, c_1), (a_4, b_3, c_6), (a_5, b_0, c_7), (a_6, b_1, c_4), (a_7, b_2, c_5)\}$,

$\{(a_0, b_0, c_1), (a_1, b_1, c_2), (a_2, b_2, c_3), (a_3, b_3, c_0), (a_4, b_4, c_5), (a_5, b_5, c_6), (a_6, b_6, c_7), (a_7, b_7, c_4)\}$,

$\{(a_0; b_1, b_2, b_3), (b_0; c_0, c_2, c_6), (c_4; a_1, a_2, a_3), (b_7; c_1, c_5, c_7), (c_3; a_4, a_5, a_6), (a_7; b_4, b_5, b_6)\}$,

$\{(a_1; b_0, b_2, b_3), (b_1; c_1, c_3, c_7), (c_5; a_0, a_2, a_3), (b_4; c_2, c_4, c_6), (c_0; a_7, a_5, a_6), (a_4; b_7, b_5, b_6)\}$,

$\{(a_2; b_1, b_0, b_3), (b_2; c_0, c_2, c_4), (c_6; a_1, a_0, a_3), (b_5; c_3, c_5, c_7), (c_1; a_4, a_7, a_6), (a_5; b_4, b_7, b_6)\}$,

$\{(a_3; b_1, b_2, b_0), (b_3; c_1, c_3, c_5), (c_7; a_1, a_2, a_0), (b_6; c_0, c_4, c_6), (c_2; a_4, a_5, a_7), (a_6; b_4, b_5, b_7)\}$,

$\{(a_0; b_4, b_5, c_4), (b_7; c_0, c_7, a_1), (c_2; b_6, a_2, a_3), (b_0; c_3, c_5, a_6), (c_7; a_4, a_7, b_3), (a_5; b_1, b_2, c_1)\}$,

$\{(a_1; b_5, b_6, c_5), (b_4; c_1, a_2, c_7), (c_3; a_3, a_0, b_7), (b_1; a_7, c_0, c_6), (c_4; a_4, a_5, b_0), (a_6; b_2, b_3, c_2)\}$,

$\{(a_2; b_6, b_7, c_6), (b_5; a_3, c_2, c_4), (c_0; a_1, a_0, b_4), (b_2; a_4, c_1, c_7), (c_5; b_1, a_5, a_6), (a_7; b_0, b_3, c_3)\}$,

$\{(a_3; b_4, b_7, c_7), (b_6; a_0, c_3, c_5), (c_1; a_1, a_2, b_5), (b_3; c_2, c_4, a_5), (c_6; b_2, a_6, a_7), (a_4; c_0, b_0, b_1)\}$.

□

Lemma 4.4. *There exists a* $(K_{1,3}, K_3)$*-URD*$(24; 12, 2)$.

Proof. Take a $(K_{1,3}, K_3)$-URGDD$(8, 2)$ of type 8^3, which comes from Lemma 4.3. Place in each of the groups the same $(K_{1,3}, K_3)$-URGDD$(4, 0)$ of type 2^4, which comes from Lemma 4.1. This completes the proof. □

Lemma 4.5. *There exists a* $(K_{1,3}, K_3)$*-URD*$(48; 28, 2)$.

Proof. Take a $K_{1,3}$-URGDD$(16, 0)$ of type 24^2 which is known to exist [5]. Place in each of the groups of size 24 the same $(K_{1,3}, K_3)$-URD$(24; 12, 2)$, which comes from Lemma 4.4. This completes the proof. □

Lemma 4.6. *There exists a* $(K_{1,3}, K_3)$*-URGDD*$(16, 0)$ *of* $C_{3(12)}$.

Proof. Take the classes of 3-stars listed below:
$\{R^1_{0,0} \cup R^2_{3,0} \cup R^3_{3,1}\}$, $\{R^1_{0,1} \cup R^2_{0,1} \cup R^3_{0,1}\}$, $\{R^1_{0,2} \cup R^2_{1,2} \cup R^3_{1,1}\}$,
$\{R^1_{0,3} \cup R^2_{2,3} \cup R^3_{2,1}\}$, $\{R^1_{1,0} \cup R^2_{3,1} \cup R^3_{0,2}\}$, $\{R^1_{1,1} \cup R^2_{0,2} \cup R^3_{1,2}\}$,
$\{R^1_{1,2} \cup R^2_{1,3} \cup R^3_{2,2}\}$, $\{R^1_{1,3} \cup R^2_{2,0} \cup R^3_{3,2}\}$, $\{R^1_{2,0} \cup R^2_{3,2} \cup R^3_{1,3}\}$,
$\{R^1_{2,1} \cup R^2_{0,3} \cup R^3_{2,3}\}$, $\{R^1_{2,2} \cup R^2_{1,0} \cup R^3_{3,3}\}$, $\{R^1_{2,3} \cup R^2_{2,1} \cup R^3_{0,3}\}$,

$\{R^1_{3,0} \cup R^2_{3,3} \cup R^3_{2,0}\}$, $\{R^1_{3,1} \cup R^2_{0,0} \cup R^3_{3,0}\}$, $\{R^1_{3,2} \cup R^2_{1,1} \cup R^3_{0,0}\}$,
$\{R^1_{3,3} \cup R^2_{2,2} \cup R^3_{1,0}\}$.

□

Lemma 4.7. *There exists a* $(K_{1,3}, K_3)$*-URD*$(72; 44, 2)$.

Proof. Take a $K_{1,3}$-RGDD(16) of type 12^3 which comes from Lemma 4.6. Give weight 2 to every point of this $K_{1,3}$-RGDD and apply Theorem 3.4 with $t = 2$. Place in each of the groups of size 24 the same $(K_{1,3}, K_3)$-URD$(24; 12, 2)$, which comes from Lemma 4.4. Applying Theorem 3.2 with $g = 12$, $t = 2$ and $u = 3$ we obtain the result. □

Lemma 4.8. *There exists a* $(K_{1,3}, K_3)$*-URD*$(648; 428, 2)$.

Proof. Start with a 3-RGDD of type 2^{27} [12]. Give weight 12 to every point of this 3-RGDD and place in each block of a given resolution class of the 3-RGDD the same $(K_{1,3})$-RGDD of type 12^3 with 16 classes of 3-stars, which comes from Lemma 4.6. Fill in each of the groups of sizes 24 with the same $(K_{1,3}, K_3)$-URD$(24; 12, 2)$. Applying Theorem 3.2 with $g = 2$, $t = 12$ and $u = 27$ we obtain a $(K_{1,3}, K_3)$-URD$(648; 428, 2)$. □

Lemma 4.9. *There exists a* $(K_{1,3}, K_3)$*-URD*$(816; 540, 2)$.

Proof. Start with a 4-RGDD of type 12^{34} [2]. Give weight 2 to every point of this 4-RGDD and place in each block of a given resolution class of the 4-RGDD the same $K_{1,3}$-RGDD of type 2^4 with 4 classes of 3-stars, which comes from Lemma 4.1. Fill in each of the groups of sizes 24 with the same $(K_{1,3}, K_3)$-URD$(24; 12, 2)$. Applying Theorem 3.2 with $g = 12$, $t = 2$ and $u = 34$ we obtain a $(K_{1,3}, K_3)$-URD$(816; 540, 2)$. □

Lemma 4.10. *There exists a* $(K_{1,3}, K_3)$*-URGDD*$(16, 0)$ *of* $C_{5(12)}$.

Proof. Take the classes of 3-stars listed below:
$\{R^1_{0,0} \cup R^2_{3,0} \cup R^3_{3,0} \cup R^4_{3,0} \cup R^5_{3,1}\}$, $\{R^1_{0,1} \cup R^2_{0,1} \cup R^3_{0,1} \cup R^4_{0,1} \cup R^5_{0,1}\}$,
$\{R^1_{0,2} \cup R^2_{1,2} \cup R^3_{1,2} \cup R^4_{1,2} \cup R^5_{1,1}\}$, $\{R^1_{0,3} \cup R^2_{2,3} \cup R^3_{2,3} \cup R^4_{2,3} \cup R^5_{2,1}\}$,
$\{R^1_{1,0} \cup R^2_{3,1} \cup R^3_{0,2} \cup R^4_{1,3} \cup R^5_{2,2}\}$, $\{R^1_{1,1} \cup R^2_{0,2} \cup R^3_{1,3} \cup R^4_{2,0} \cup R^5_{3,2}\}$,
$\{R^1_{1,2} \cup R^2_{1,3} \cup R^3_{2,0} \cup R^4_{3,1} \cup R^5_{0,2}\}$, $\{R^1_{1,3} \cup R^2_{2,0} \cup R^3_{3,1} \cup R^4_{0,2} \cup R^5_{1,2}\}$,
$\{R^1_{2,0} \cup R^2_{3,2} \cup R^3_{1,0} \cup R^4_{3,2} \cup R^5_{1,3}\}$, $\{R^1_{2,1} \cup R^2_{0,3} \cup R^3_{2,1} \cup R^4_{0,3} \cup R^5_{2,3}\}$,
$\{R^1_{2,2} \cup R^2_{1,0} \cup R^3_{3,2} \cup R^4_{1,0} \cup R^5_{3,3}\}$, $\{R^1_{2,3} \cup R^2_{2,1} \cup R^3_{0,3} \cup R^4_{2,1} \cup R^5_{0,3}\}$,
$\{R^1_{3,0} \cup R^2_{3,3} \cup R^3_{2,2} \cup R^4_{1,1} \cup R^5_{0,0}\}$, $\{R^1_{3,1} \cup R^2_{0,0} \cup R^3_{3,3} \cup R^4_{2,2} \cup R^5_{1,0}\}$,
$\{R^1_{3,2} \cup R^2_{1,1} \cup R^3_{0,0} \cup R^4_{3,3} \cup R^5_{2,0}\}$, $\{R^1_{3,3} \cup R^2_{2,2} \cup R^3_{1,1} \cup R^4_{0,0} \cup R^5_{3,0}\}$.

□

5 Main results

Lemma 5.1. *For every* $v \equiv 0 \pmod{24}$ *there exists a* $(K_{1,3}, K_3)$*-URD* $(v; \frac{2(v-6)}{3}, 2)$.

Proof. Let $v = 24s$. The cases $s = 1, 2, 3, 27, 34$ are covered by Lemmas 4.4, 4.5, 4.7, 4.8 and 4.9. For $s > 4$, $s \neq 27$, 34, start with a 4-RGDD G of type 6^{2s} [2]. Give weight 2 to each point of this 4-RGDD and place in each block of a given resolution class of the 4-RGDD the same $(K_{1,3}, K_3)$-URGDD$(4, 0)$ of type 2^4, which comes from Lemma 4.1. Fill in each of the groups of sizes 12 with the same $(K_{1,3}, K_3)$-URD$(12; 4, 2)$, which comes from Lemma 4.2. Applying Theorem 3.2 with $g = 6$, $t = 2$ and $u = 2s$ we obtain a uniformly resolvable decomposition of $K_v - I$, $I = \bigcup_{i=1}^{2s} I_i$, into $8(2s-1) + 4 = \frac{2(v-6)}{3}$ classes of 3-stars and 2 classes of 3-cycles. □

Lemma 5.2. *There exists a* $(K_{1,3}, K_3)$*-URGDD*$(16, 0)$ *of* $C_{(3+4p)(12)}$, $p \geq 1$.

Proof. Take the classes of 3-stars listed below:
$\{R^1_{0,0} \cup \{\bigcup_{j=2}^{2+4p} R^j_{3,0}\} \cup R^{3+4p}_{3,1}\}$, $\{\bigcup_{j=1}^{3+4p} R^j_{0,1}\}$,
$\{R^1_{0,2} \cup \{\bigcup_{j=2}^{2+4p} R^j_{1,2}\} \cup R^{3+4p}_{1,1}\}$, $\{R^1_{0,3} \cup \{\bigcup_{j=2}^{2+4p} R^j_{2,3}\} \cup R^{3+4p}_{2,1}\}$,
$\{R^1_{1,0} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{1,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{2,0}\} \cup R^{3+4p}_{0,2}\}$,
$\{R^1_{1,1} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{1,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{3,1}\} \cup R^{3+4p}_{1,2}\}$,
$\{R^1_{1,2} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{1,3}\} \cup (\bigcup_{j=0}^{p-1} R^{3+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{0,2}\} \cup R^{3+4p}_{2,2}\}$,
$\{R^1_{1,3} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{1,3}\} \cup R^{3+4p}_{3,2}\}$,
$\{R^1_{2,0} \cup \{\bigcup_{j=0}^{2p} R^{2+2j}_{3,2}\} \cup \{\bigcup_{j=0}^{2p-1} R^{3+2j}_{1,0}\} \cup R^{3+4p}_{1,3}\}$,
$\{R^1_{2,1} \cup \{\bigcup_{j=0}^{2p} R^{2+2j}_{0,3}\} \cup \{\bigcup_{j=0}^{2p-1} R^{3+2j}_{2,1}\} \cup R^{3+4p}_{2,3}\}$,
$\{R^1_{2,2} \cup \{\bigcup_{j=0}^{2p} R^{2+2j}_{1,0}\} \cup \{\bigcup_{j=0}^{2p-1} R^{3+2j}_{3,2}\} \cup R^{3+4p}_{3,3}\}$,
$\{R^1_{2,3} \cup \{\bigcup_{j=0}^{2p} R^{2+2j}_{2,1}\} \cup \{\bigcup_{j=0}^{2p-1} R^{3+2j}_{0,3}\} \cup R^{3+4p}_{0,3}\}$,
$\{R^1_{3,0} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{0,0}\} \cup R^{3+4p}_{2,0}\}$,
$\{R^1_{3,1} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{1,1}\} \cup R^{3+4p}_{3,0}\}$,
$\{R^1_{3,2} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{2,2}\} \cup R^{3+4p}_{0,0}\}$,
$\{R^1_{3,3} \cup \{\bigcup_{j=0}^{p} R^{2+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{5+4j}_{3,3}\} \cup R^{3+4p}_{1,0}\}$.
□

Lemma 5.3. *There exists a* $(K_{1,3}, K_3)$*-URGDD*$(16, 0)$ *of* $C_{(1+4p)(12)}$, $p \geq 2$.

Proof. Take the classes of 3-stars listed below:
$\{R^1_{0,0} \cup \{\bigcup_{j=2}^{4p} R^j_{3,0}\} \cup R^{1+4p}_{3,1}\}$, $\{\bigcup_{j=1}^{1+4p} R^j_{0,1}\}$,
$\{R^1_{0,2} \cup \{\bigcup_{j=2}^{4p} R^j_{1,2}\} \cup R^{1+4p}_{1,1}\}$, $\{R^1_{0,3} \cup \{\bigcup_{j=2}^{4p} R^j_{2,3}\} \cup R^{1+4p}_{2,1}\}$,
$\{R^1_{1,0} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{1,3}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{2,0}\} \cup R^{1+4p}_{2,2}\}$,
$\{R^1_{1,1} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{1,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{3,1}\} \cup R^{1+4p}_{3,2}\}$,
$\{R^1_{1,2} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{1,3}\} \cup (\bigcup_{j=0}^{p-1} R^{3+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{0,2}\} \cup R^{1+4p}_{0,2}\}$,

$\{R^1_{1,3} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{2,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{3,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{0,2}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{1,3}\} \cup R^{1+4p}_{1,2}\}$,
$\{R^1_{2,0} \cup \{\bigcup_{j=0}^{2p-1} R^{2j+2}_{3,2}\} \cup \{\bigcup_{j=0}^{2p-2} R^{3+2j}_{1,0}\} \cup R^{1+4p}_{1,3}\}$,
$\{R^1_{2,1} \cup \{\bigcup_{j=0}^{2p-1} R^{2+2j}_{0,3}\} \cup \{\bigcup_{j=0}^{2p-2} R^{3+2j}_{2,1}\} \cup R^{1+4p}_{2,3}\}$,
$\{R^1_{2,2} \cup \{\bigcup_{j=0}^{2p-1} R^{2+2j}_{1,0}\} \cup \{\bigcup_{j=0}^{2p-2} R^{3+2j}_{3,2}\} \cup R^{1+4p}_{3,3}\}$,
$\{R^1_{2,3} \cup \{\bigcup_{j=0}^{2p-1} R^{2+2j}_{2,1}\} \cup \{\bigcup_{j=0}^{2p-2} R^{3+2j}_{0,3}\} \cup R^{1+4p}_{0,3}\}$,
$\{R^1_{3,0} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{0,0}\} \cup R^{1+4p}_{0,0}\}$,
$\{R^1_{3,1} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{1,1}\} \cup R^{1+4p}_{1,0}\}$,
$\{R^1_{3,2} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{3,3}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{2,2}\} \cup R^{1+4p}_{2,0}\}$,
$\{R^1_{3,3} \cup \{\bigcup_{j=0}^{p-1} R^{2+4j}_{2,2}\} \cup \{\bigcup_{j=0}^{p-1} R^{3+4j}_{1,1}\} \cup \{\bigcup_{j=0}^{p-1} R^{4+4j}_{0,0}\} \cup \{\bigcup_{j=0}^{p-2} R^{5+4j}_{3,3}\} \cup R^{1+4p}_{3,0}\}$. □

Lemma 5.4. *For every $v \equiv 12 \pmod{24}$ there exists a $(K_{1,3}, K_3)$-URD $(v; \frac{2(v-6)}{3}, 2)$.*

Proof. Let $v = 12(2t+1)$, $t \geq 0$. The case $v = 12$ is covered by Lemma 4.2. For $t \geq 1$ start with a $(2t+1)$-cycle system (X, C) [1, 8]. Give weight 12 to each point of X and replace each $(2t+1)$-cycle of C with a $(K_{1,3}, K_3)$-URGDD$(16, 0)$ of $C_{(1+2t)12}$, which comes from Lemmas 4.6, 4.10, 5.2 and 5.3. For each $a_i \in X$, $i = 0, \ldots, 2t$, place in $a_i \times \mathbb{Z}_{12}$ the same URD$(12; 4, 2)$, which comes from Lemma 4.2. Since $|C| = t$, the result is a uniformly resolvable decomposition of $K_{12+24t} - I$, $I = \bigcup_{i=0}^{2t} I_i$, into $16t + 4 = \frac{2(v-6)}{3}$ classes of 3-stars and 2 classes of 3-cycles. □

Combining Lemmas 5.1 and 5.4 we obtain the main theorem of this article.

Theorem 5.5. *For every $v \equiv 0 \pmod{12}$, there exists a $(K_{1,3}, K_3)$-URD $(v; \frac{2(v-6)}{3}, 2)$.*

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