accepted for publication in Discrete Mathematics

The intersection problem for $PBD(5^*, 3)$s

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Abstract

For every $v \equiv 5 \pmod 6$ there exists a *pairwise balanced design* (PBD) of order v with exactly one block of size 5 and the rest of size 3. We will refer to such a PBD as a $PBD(5^*, 3)$. A flower in a $PBD(5^*, 3)$ is the set of all blocks containing a given point. If (S, B) is a $PBD(5^*, 3)$ and F is a flower, we will write F^* to indicate that F contains the block of size 5.

The *intersection problem* for $PBD(5^*, 3)$s is the determination of all pairs (v, k) such that there exists a pair of $PBD(5^*, 3)$s (S, B_1) and (S, B_2) of order v containing the same block b of size 5 such that $|(B_1 \setminus b) \cap (B_2 \setminus b)| = k$.

The *flower intersection problem* for $PBD(5^*, 3)$s is the determination of all pairs (v, k) such that there exists a pair of $PBD(5^*, 3)$s (S, B_1) and (S, B_2) of order v having a common flower F^* such that $|(B_1 \setminus F^*) \cap (B_2 \setminus F^*)| = k$.

In this paper we give a complete solution of both problems.

1 Introduction

A *Steiner triple system* (or more simply triple system) of order v is a pair (S, T), where T is a collection of edge disjoint triangles (or triples) which partitions the edge set of K_v with vertex set S. It is well-known that the spectrum for triple systems (the set of all v such that a triple system of order v exists) is precisely the set of all $v \equiv 1$ or 3 (mod 6) [4].

The *intersection problem* for triple systems is the determination of all pairs (v, k) such that there exists a pair of triple systems (S, T_1) and (S, T_2) of order v such that $|T_1 \cap T_2| = k$. In [6] C. C. Lindner and A. Rosa gave a complete solution of the intersection problem for triple systems.

A variant of the intersection problem for triple systems is the flower intersection problem. A *flower* in a triple system is the set of all triples containing a given point. The *flower intersection problem* for triple systems is the determination of all pairs (v, k) such that there exists a pair of triple systems (S, T_1) and (S, T_2) of order v having a common flower F such that $|(T_1 \setminus F) \cap (T_2 \setminus F)| = k$. In [3] D. G. Hoffman and C. C. Lindner gave a complete solution of the flower intersection problem for triple systems.

A *packing* of K_v with triples is a triple (S, B, L), where B is a collection of edge disjoint triples of the edge set of K_v and L is the set of unused edges, called the *leave*. If $|B|$ is as large as possible, alternatively if $|L|$ is as small as possible, (S, B, L) is called a *maximum packing* of K_v with triples. It is well-known (see [5] for example) that a maximum packing of K_v with triples has leave $\emptyset$ if $v \equiv 1$ or 3 (mod 6) (a Steiner triple system), a 1-factor if $v \equiv 0$ or 2 (mod 6), a tripole (a spanning subgraph of odd degree containing $(v + 2)/2$ edges) if $v \equiv 4$ (mod 6), and a 4-cycle if $v \equiv 5$ (mod 6). The *intersection problem for maximum packings* asks for the set of all pairs (v, k) such that there exists a pair of maximum packings (S, B_1, L) and (S, B_2, L) of order v, with the same leave L, such that $|B_1 \cap B_2| = k$. The solution of the intersection problem for Steiner triple systems gives a solution of the intersection problem for maximum packings (the leave is $\emptyset$) for $v \equiv 1$ or 3 (mod 6). A bit of reflection reveals that the flower intersection problem for Steiner triple systems is equivalent to the intersection problem for maximum packings of K_v with triples when $v \equiv 0$ or 2 (mod 6).

Hence the solution of the flower intersection problem for triple systems gives a solution of the intersection problem for maximum packings of K_v with triples when $v \equiv 0$ or 2 (mod 6). In [8] Gaetano Quattrocchi gave a complete solution of the intersection problem for maximum packings for $v \equiv 4$ or 5 (mod 6).

When $v \equiv 5$ (mod 6) there does not exist a Steiner triple system of order v. However we can get "very close". In particular, for every $v \equiv 5$ (mod 6) there exists a *pairwise balanced design* (PBD) of order v with *exactly one* block of size 5 and the rest of size 3. We will refer to such a PBD as a $PBD(5^*, 3)$.

The *intersection problem* for $PBD(5^*, 3)$s is the determination of all pairs (v, k) such that there exists a pair of $PBD(5^*, 3)$s (S, B_1) and (S, B_2) of order v containing the same block b of size 5 such that $|(B_1 \setminus b) \cap (B_2 \setminus b)| = k$. We will denote the set of all such k by $I[v]$.

As with triple systems a flower in a $PBD(5^*, 3)$ is the set of all blocks containing a given point. If (S, B) is a $PBD(5^*, 3)$ and F is a flower, we will write F^* to indicate that F contains the block of size 5. The flower intersection problem for $PBD(5^*, 3)$s is the determination of all pairs (v, k) such that there exists a pair of $PBD(5^*, 3)$s (S, B_1) and (S, B_2) of order v having a common flower F^* such that $|(B_1 \setminus F^*) \cap (B_2 \setminus F^*)| = k$. We will denote the set of all such k by $\overline{I}[v]$. The object of this paper is to give a complete solution of both problems.

Apart from being of interest in their own right, these solutions can be used to obtain Quattrocchi's results on maximum packings of K_n with triples as a corollary (modulo a few exceptions). (See Section 4.)

2 The intersection problem for PBD$(5^*, 3)$s

The number of triples in any $PBD(5^*, 3)$ of order v will be denoted by $t_v = (v^2 - v - 20)/6$. Let $J[v] = \{0, 1, 2, ..., t_v - 6, t_v - 4, t_v\}$, i.e. the set $J[v]$ contains all nonnegative integers not exceeding t_v with the exception of $t_v - 5, t_v - 3, t_v - 2$, and $t_v - 1$.

It is straightforward to see that $I[v] \subseteq J[v]$. In what follows we will prove that $J[v] \subseteq I[v]$

for all v with few exceptions. Trivially, $I[5] = \{0\}$.

Lemma 2.1 $I[11] = \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$.

Proof Let (P, B) be a $PBD(5^*, 3)$ of order 11 containing the block b of size 5. Then b induces a 1-factorization of K_6 with vertex set $P \setminus b$. Hence the intersection problem for $PBD(5^*, 3)$s of order 11 is equivalent to the intersection problem for 1-factorizations of K_6 (the determination of the set $I'[6]$ of all k such that there exist two 1-factorizations (X, G) and (X, H), where $G = \{G_i | i = 1, 2, ..., 5\}, H = \{H_i | i = 1, 2, ..., 5\}$, and $\sum_{i=1}^{5} |G_i \cap H_i| = k$). Since $I'[6] = \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$ (see [7]), $I[11] = \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$. □

The 17 Construction: Let $X = \{1, 2, 3, 4, 5\}$ and set $S = \{\infty_1, \infty_2\} \cup (X \times \{1, 2, 3\})$. Let $(X, \circ)$ be any quasigroup of order 5 such that $1 \circ 1 = 1$, and define a $PBD(5^*, 3)$ (S, B) as follows:

(1) The block of size 5 is $\{\infty_1, \infty_2, (1, 1), (1, 2), (1, 3)\}$.

(2) Let $(\{\infty_1, \infty_2\} \cup X, T)$ be any Steiner triple systems such that $\{\infty_1, \infty_2, 1\} \in T$. Put a copy of $T \setminus \{\infty_1, \infty_2, (1, i)\}$ on $\{\infty_1, \infty_2\} \cup (X \times \{i\})$, $i = 1, 2, 3$, and put these triples in B.

(3) For each $i, j \in X$, except $i = j = 1$, put the triple $\{(i, 1), (j, 2), (i \circ j, 3)\}$ in B. (It is important to note that the triple $\{(1, 1), (1, 2), (1, 3)\}$ is not used. So there are 24 type (3) triples.)

Then (S, B) is a $PBD(5^*, 3)$.

Lemma 2.2 $I[17] = J[17]$.

Proof Let (S, B_1) and (S, B_2) be $PBD(5^*, 3)$s constructed using the 17 Construction.

(2) For each $i = 1, 2, 3$, use a pair of STSs intersecting in 1 or 7 triples. (This will contribute 0 or 6 triples to $B_1 \cap B_2$ for each $i = 1, 2, 3$.)

(3) Let $(X, \circ_1)$ and $(X, \circ_2)$ be a pair of quasigroups intersecting in $z \in \{1, 2, 3, ..., 25\} \setminus \{20, 22, 23, 24\}$ products [2]. (Remember that any quasigroup used for the type (3) triples

must have the property that $1 \circ 1 = 1$.) Since the triple $\{(1,1),(1,2),(1,3)\}$ is not used, these quasigroups contribute $z-1$ triples to $B_1 \cap B_2$. Trivially, $I[17] = \{0,6\} + \{0,6\} + \{0,6\} + (\{0,1,2,...,24\} \setminus \{19,21,22,23\}) = \{0,1,2,...,42\} \setminus \{37,39,40,41\}$. □

Now we will give a construction to handle the cases $n \equiv 5 \pmod 6 \geq 23$.

The $6n+5$ Construction: Let $6n+5 \geq 23$, (X,G,T) a $(6n,6,3)-$GDD, and b a set of size 5. Define a collection of blocks B on $b \cup X$ as follows:

(1) $b \in B$,

(2) For each group $g \in G$, define a $PBD(5^*,3)$ of order 11 on $b \cup g$, where b is the block of size 5, and put the triples in B.

(3) $T \subseteq B$.

Then $(b \cup X, B)$ is a $PBD(5^*,3)$ of order $6n+5$. □

Lemma 2.3 *$J[v] \subseteq I[v]$ for all $v \geq 23$.*

Proof Let $v = 6n+5$, $(b \cup X, B_1)$ and $(b \cup X, B_2)$ be $PBD(5^*,3)$s constructed using the $6n+5$ Construction from the $(6n,6,3)-$GDDs (X,G,T_1) and (X,G,T_2), where $|T_1 \cap T_2| \in \{0,1,2,...,$
$6n(n-1)\} \setminus \{6n(n-1)-5, 6n(n-1)-3, 6n(n-1)-2, 6n(n-1)-1\}$ [1].

For each group $g \in G$ use a pair of $PBD(5^*,3)$s of order 11 intersecting in 0 or 15 triples. Then $I[6n+5] = n\{0,15\} + |T_1 \cap T_2| = \{0,1,2,...,6n^2+9n\} \setminus \{6n^2+9n-5, 6n^2+9n-3,$ $6n^2+9n-2, 6n^2+9n-1\}$. □

Combining Lemmas 2.1, 2.2, and 2.3 gives the following theorem.

Theorem 2.4 *$I[11] = \{0,1,2,3,5,6,7,9,15\}$ and $I[v] = J[v]$ for all $v \geq 17$.* □

3 The flower intersection problem for PBD$(5^*,3)$s

Since the number of triples in any $PBD(5^*,3)$ of order v is $(v^2-v-20)/6$ and the number of triples in a common flower centered at a point x, where x is an element of the block of size 5 is

$(v-5)/2$, the number of common triples outside of the flower can be at most $(v^2-4v-5)/6$. Let $\bar{t}_v = (v^2-4v-5)/6$ and let $\overline{J}[v] = \{0, 1, 2, ..., \bar{t}_v - 6, \bar{t}_v - 4, \bar{t}_v\}$, i.e. the set $\overline{J}[v]$ contains all nonnegative integers not exceeding $\bar{t}_v$ with the exception of $\bar{t}_v - 5, \bar{t}_v - 3, \bar{t}_v - 2$, and $\bar{t}_v - 1$.

For every v, $\overline{I}[v] \subseteq \overline{J}[v]$. In what follows we will prove that $\overline{J}[v] \subseteq \overline{I}[v]$ for all v with a few exceptions. Trivially, $\overline{I}[5] = \{0\}$.

Lemma 3.1 $\overline{I}[11] = \{0, 2, 3, 4, 6, 12\}$.

Proof As with the proof of Lemma 2.1 the flower intersection problem is equivalent to the intersection problem of 1-factorizations of K_6 with a common 1-factor (the determination of the set $I''[6]$ of all k such that there exist two 1-factorizations (S, G) and (S, H), where $G = \{G_i | i = 1, 2, ..., 5\}$ and $H = \{H_i | i = 1, 2, ..., 5\}$ with $G_1 = H_1$ and $\sum_{i=2}^{5} |G_i \cap H_i| = k$). In [7] it is shown that $I''[6] = \{0, 2, 3, 4, 6, 12\}$, so $\overline{I}[11] = \{0, 2, 3, 4, 6, 12\}$. □

Lemma 3.2 $\overline{I}[17] = \overline{J}[17]$.

Proof Let (S, B_1) and (S, B_2) be $PBD(5^*, 3)$s constructed using the 17 Construction.

(2) For each $i = 1, 2, 3$, use a pair of STSs having the common flower $\{\infty_1, \infty_2, (1, i)\}$, $\{\infty_1, (2, i), (3, i)\}, \{\infty_1, (4, i), (5, i)\}$, and intersecting in 3 or 7 triples. (This will contribute 0 or 4 triples to $B_1 \cap B_2$ for each $i = 1, 2, 3$.)

(3) Let $(X, \circ_1)$ and $(X, \circ_2)$ be a pair of quasigroups intersecting in $z \in \{1, 2, 3, ..., 25\} \setminus \{20, 22, 23, 24\}$ products as in (3) of Lemma 2.2. (Remember that any quasigroup used for the type (3) triples must have the property that $1 \circ 1 = 1$.) Since the triple $\{(1, 1), (1, 2), (1, 3)\}$ is not used, these quasigroups contribute $z - 1$ triples to $B_1 \cap B_2$. Then $\overline{I}[17] = \{0, 4\} + \{0, 4\} + \{0, 4\} + (\{0, 1, 2, ..., 24\} \setminus \{19, 21, 22, 23\}) = \{0, 1, 2, ..., 36\} \setminus \{31, 33, 34, 35\}$. □

Lemma 3.3 $\overline{J}[v] = \overline{I}[v]$ *for all* $v \geq 23$.

Proof Let $v = 6n + 5$, $(b \cup X, B_1)$ and $(b \cup X, B_2)$ be $PBD(5^*, 3)$s constructed using the $6n + 5$ Construction from the $(6n, n, 3)-$GDDs (X, G, T_1) and (X, G, T_2), where $|T_1 \cap T_2| \in \{0, 1, 2, ..., 6n(n-1)\} \setminus \{6n(n-1) - 5, 6n(n-1) - 3, 6n(n-1) - 2, 6n(n-1) - 1\}$.

For each group $g \in G$ use a pair of $PBD(5^*, 3)$s of order 11 in Lemma 3.1 intersecting in 0 or 12 triples. Then $\overline{I}[6n+5] = n\{0, 12\} + |T_1 \cap T_2| = \{0, 1, 2, ..., 6n^2+6n\} \setminus \{6n^2+6n-5, 6n^2+6n-3, 6n^2+6n-2, 6n^2+6n-1\}$. □

Now combining Lemmas 3.1, 3.2, and 3.3, gives the following theorem.

Theorem 3.4 *$\overline{I}[11] = \{0, 2, 3, 4, 6, 12\}$ and $\overline{I}[v] = \overline{J}[v]$ for all $v \geq 17$.* □

4 Concluding remarks

The intersection problem for maximum packings of K_v with triples has been solved in a series of papers [3, 6, 8]. In [8] Quattrocchi gave a complete solution of the intersection problem for maximum packings of K_v with triples for $v \equiv 4$ and 5 (mod 6). The solution of the flower intersection problem and the intersection problem for $PBD(5^*, 3)$s obtained in this paper gives a complete solution for the intersection problem for maximum packings of K_v with triples for $v \equiv 4$ and 5 (mod 6), respectively, as a corollary (modulo a few exceptions).

If $v \equiv 5$ (mod 6), replace the block of size 5 with 2 triples and a 4-cycle. This gives a solution to the intersection problem with the 2 exceptions of 0 and 1 in each case. If $v \equiv 4$ (mod 6), delete the vertex x of the common flower and replace $b \setminus \{x\}$ with a triple and a $K_{1,3}$ star. This gives a solution to intersection problem with the single exception of 0 in each case.

5 Acknowledgment

The author wishes to thank Professors D. G. Hoffman and C. C. Lindner for helpful comments during the preparation of this paper.

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