Stickelberger elements and Kolyvagin systems

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ABSTRACT. In this paper, we construct (many) Kolyvagin systems out of Stickelberger elements utilizing ideas borrowed from our previous work on Kolyvagin systems of Rubin-Stark elements. The applications of our approach are two-fold: First, *assuming Brumer's conjecture*, we prove results on the *odd* parts of the ideal class groups of CM fields which are abelian over a totally real field, and deduce Iwasawa's main conjecture for totally real fields (for totally odd characters). Although this portion of our results have already been established by Wiles unconditionally (and refined by Kurihara using an Euler system argument, when Wiles' work is assumed), the approach here fits well in the general framework the author has developed elsewhere to understand Euler / Kolyvagin system machinery when the *core Selmer rank* is $r > 1$ (in the sense of Mazur and Rubin). As our second application, we establish a rather curious link between the Stickelberger elements and Rubin-Stark elements by using the main constructions of this article hand in hand with the 'rigidity' of the collection of Kolyvagin systems proved by Mazur, Rubin and the author.

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1. INTRODUCTION

The Euler / Kolyvagin system machinery is designed to bound the size of a Selmer group. In all well-known cases the bounds obtained relate to L-values, and thus provide a link between arithmetic and analytic data. Well-known prototypes for such a relation between arithmetic and analytic data are the Birch and Swinnerton-Dyer conjecture (more generally, Bloch-Kato conjectures) and the main conjectures of Iwasawa theory. The Kolyvagin system machinery has been successfully applied by many to obtain deep results towards proving these conjectures.

B. Howard, B. Mazur and K. Rubin show in [MR04] that the existence of Kolyvagin systems relies on a cohomological invariant, what they call the *core Selmer rank* (c.f., [MR04, Definition 4.1.11]). When the core Selmer rank is one, they determine the structure of the Selmer group completely in terms of a Kolyvagin system. However, when the core Selmer rank is greater than one, not much could be said. One of the principal objectives of the current article is to explore this more mysterious case in depth for a particular Galois representation, for which the core Selmer rank is greater than 1.

Let k be a totally real field of degree $[k:\mathbb{Q}]=r$ and let $G_k:=\mathrm{Gal}(\overline{k}/k)$ be its absolute Galois group. Fix once and for all an odd rational prime p, and let $\psi: G_k \to \mathbb{Z}_p^{\times}$ be a totally even character. Consider the G_k-representation $T':=\mathbb{Z}_p(1)\otimes\psi^{-1}$. The core Selmer rank of the Galois representation T' is r, and T' leads us to one of the basic instances when the core Selmer rank is greater than one. In [Büy09a, Büy09b], the author has studied the Kolyvagin system machinery for T'. The main idea in these two papers is to modify the relevant Selmer group appropriately and construct Kolyvagin systems out of Rubin-Stark elements defined in [Rub96] so as to control this modified Selmer group.

In this paper, we consider the G_k-representation $T=\mathbb{Z}_p(\chi)$, where $\chi: G_k \to \mathbb{Z}_p^{\times}$ is a totally odd character. The G_k-representation T turns out to have core Selmer rank r. As in [Büy09a, Büy09b], we introduce certain modified Selmer groups associated with the representation T. The Euler system that gives rise to the Kolyvagin system which controls the modified Selmer group is obtained from the Stickelberger elements.

Before we state the main results of this article, we set some notation which will be in effect throughout the paper. Let p, k, G_k and r be as above, and let $\chi: G_k \to \mathbb{Z}_p^{\times}$ be a totally odd character, which is different from the Teichmüller character ω that gives the action of G_k on the p-th roots of unity. Let k_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of k. Consider the following properties:

(A1) For any prime $\wp$ of k above p, we have $\chi(\wp)\neq 1$.
(A2) Any prime of k above p totally ramifies in k_∞/k.

Note that (A2) is true, for example, if $k/\mathbb{Q}$ is unramified.

Let L be the fixed field of $\ker(\chi)$ inside a fixed algebraic closure $\overline{k}$ of k, write $\Delta=\mathrm{Gal}(L/k)$. For any number field K containing L, let A_K be the p-part of the ideal class group of K, and A_K^{χ} its χ-isotypic part. We fix S as the set of places of k which consists of all infinite places of k, all places λ which divide the conductor $\mathfrak{f}_\chi$ of χ, as well as all the places of k above p. Finally, let $\theta_{L,S}=\theta_L\in\mathbb{Z}_p[\Delta]$ be the *Stickelberger element* (defined precisely in [Kur03, §1.2], see also §3 below) relative to S. For the main results of this article, we will assume the χ-part of Brumer's conjecture:

Assumption 1.1. θ_K^{χ} annihilates A_K^{χ}.

We remark here that Wiles [Wil90a] proved that Brumer's conjecture as stated above follows from his proof [Wil90b] of the *main conjecture of Iwasawa theory* for totally real fields. In this paper, we prove the other way around, namely that, assuming Brumer's conjecture, one might also prove the main conjecture (see Theorem B below; see also Kurihara's work [Kur03]).

The first application of the treatment here is the following (Theorem 5.3 below):

Theorem A. *Suppose the hypothesis* (A1) *and Assumption 1.1 hold. Then,*

$$|A_L^\chi| = |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

With a bit more work, we can prove the Iwasawa theoretic version of Theorem A, which we state below. Set $\Gamma = \mathrm{Gal}(k_\infty/k)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$, as usual. Let $\mathcal{L}_{\omega\chi^{-1}} \in \Lambda$ denote the Deligne-Ribet p-adic L-function (see [DR80]). We recall in (5.2) the basic interpolation property which characterizes $\mathcal{L}_{\omega\chi^{-1}}$. Let $\mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}$ be a certain twisting operator on Λ (see §5.2 below for its definition). For any abelian group A, let $A^\vee := \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ denote its Pontryagin dual, and finally, let $\mathrm{char}(M)$ denote the characteristic ideal of a finitely generated Λ-module M (with the convention that $\mathrm{char}(M) = 0$ unless M is Λ-torsion). Then we are able to prove (see Theorem 5.8 and Corollary 5.10 for a slightly improved version so as to include the case $\boldsymbol{\mu}_p \subset L$):

Theorem B. *Suppose the hypotheses* (A1)-(A2) *as well as Assumption 1.1 hold. Assume also that L does not contain pth roots of unity. Then,*

$$\mathrm{char}\left((\varinjlim_n A_{L_n}^\chi)^\vee\right) = \mathrm{Tw}_{\langle\rho_{\mathrm{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}}).$$

These results have already been obtained by Wiles [Wil90b] without appealing to the Euler system machinery, even without the assumptions of Theorem B above. Kurihara [Kur03] was also able to give another proof of Theorem B (in fact he proved considerably more), also using an Euler system argument, though significantly different then ours. The novelty in this paper is a new treatment of the Euler / Kolyvagin system machinery *à la* what we call $\boldsymbol{\mathcal{L}}$*-restricted Euler systems* (see §4.1, more particularly, Definition 4.10 and Example 4.11), when the *core Selmer rank* (in the sense of Mazur and Rubin [MR04]) of the Galois representation[1] in question is greater than one. One benefit of this new approach presented in this paper is a rather surprising link between Stickelberger elements and the (conjectural) Rubin-Stark elements, which we prove below in Theorem 5.16; see also Theorem C in this section below and the paragraph that follows its statement.

As in our earlier papers [Büy09a, Büy09b], we improve the Euler system / Kolyvagin system machinery of [Rub00, PR98, Kat99, MR04] to prove Theorem A and B. The main obstruction to apply the machinery of [MR04] directly in our setting is the fact that, what Mazur and Rubin call the canonical Selmer structure in loc.cit. produces a Selmer group too big to control using the theory developed in [MR04]. In order to deal with this matter we proceed as follows:

(1) We first 'refine' the canonical Selmer structure by introducing more restrictive local conditions at p. We achieve this by choosing a *line* $\boldsymbol{\mathcal{L}}$ inside a certain local cohomology group at p (see §2.3). This step has to do with the issue discussed in Remark 3.5(i) below.

[1]Here, we study a particular Galois representation that admits an Euler system of Stickelberger element-type. One may hope to utilize the approach we develop in this paper for an Euler system obtained from Mazur-Tate elements attached to modular forms; however, we do not treat this example in the current paper.

(2) We introduce what we call the module of $\boldsymbol{\mathcal{L}}$*-restricted Euler systems* (c.f., Definition 4.10). The point in doing so is that, an ordinary Euler system a priori gives rise to a Kolyvagin system only for the canonical Selmer structure, and not necessarily for the refined Selmer structure defined in Step (1). On the other hand, an $\boldsymbol{\mathcal{L}}$-restricted Euler system does give rise to a Kolyvagin system for the refined Selmer structure.
(3) We then obtain $\boldsymbol{\mathcal{L}}$-restricted Euler systems starting from Stickelberger elements (c.f., Example 4.11), which we apply to deduce Theorem A and B above. For this part, one needs to determine the structure of semi-local cohomology groups and using this information, choose a useful collection of homomorphisms (see Proposition 4.9 below). The choice of such a collection also appears in [Rub00, Proposition D.1.3], where Rubin *explicitly* constructs one –in a different way from ours; see Remark 3.5 below for a comparison of our construction here to Rubin's work [Rub00, §III.4] in the case $k = \mathbb{Q}$. Rubin's construction is useful only when the base field k is $\mathbb{Q}$, whereas we abstractly show that a collection of homomorphisms with the necessary properties exists for an arbitrary totally real base field k.

The Galois representation (and the Euler system attached to it) which we treat in this paper needs to be handled in a slightly different manner than the case of Rubin-Stark elements (which was studied in [Büy09a, Büy09b]), as far as the Euler / Kolyvagin system machinery is concerned. In a forthcoming paper [Büy08], the set up from the current article and that from [Büy07] are combined together to treat the theory of Kolyvagin systems which descend from Euler systems[2] for an *arbitrary* self-dual Galois representation whose core Selmer rank is $r > 1$.

Our method to improve the results of [PR98, Rub00, MR04] relies on the choice of a rank-one direct summand (which we call $\boldsymbol{\mathcal{L}}$ above) inside the semi-local cohomology group at p. This makes our approach seem less natural. We address this issue in Remark 2.32 and show that the module generated by the 'leading terms' of the Kolyvagin systems constructed this way does *not* depend on the decomposition of the semi-local cohomology group at p into rank-one direct summands; see Theorem 2.33.

Besides the standard applications (i.e., Theorem A and Theorem B above) of our construction of what we call an $\boldsymbol{\mathcal{L}}$*-restricted Kolyvagin system* (c.f., Definition 2.36, Theorem 4.15 and Remark 4.8) out of Stickelberger elements, we also prove the following statement regarding the local Iwasawa theory of Rubin-Stark elements, which was proved in [Büy09b] assuming the truth of the main conjecture:

Theorem C. *Let $\psi : G_k \to \mathbb{Z}_p^\times$ be a totally even character. Suppose that both ψ and $\chi = \omega\psi^{-1}$ satisfy the hypothesis* (A1)*, and assume that* (A2) *holds as well as Assumption 1.1. Then,*

$$\mathrm{char}\left(\wedge_\Lambda^r H^1(k_p, T' \otimes \Lambda)/\Lambda \cdot c^\psi_{k_\infty}\right) = \mathcal{L}_\psi.$$

Here, $T' = \mathbb{Z}_p(1) \otimes \psi^{-1}$ and $c^\psi_{k_\infty} := \{c^\psi_{k_n}\}$ is the ψ-part of the collection of (conjectural) Rubin-Stark elements (which we assume to exist) along the cyclotomic tower, see [Büy09b, §3] for a precise definition of these elements.

In fact, we are able to prove considerably more than Theorem C in regard of the Rubin-Stark elements. In Theorem 5.16(i) below, we obtain a relation between the Stickelberger elements and Rubin-Stark elements (note that the existence of the latter is conjectural), making use of the formalism of $\boldsymbol{\mathcal{L}}$-restricted Euler systems we develop in this paper; as well as the rigidity of

[2]More precisely, *Euler systems of rank r* in the terminology of [PR98].

the module of Λ-adic Kolyvagin systems proved in [Büy07]. This, we believe, is interesting on its own right. We also note that, the "rigidity phenomenon" which plays an important role for the link we obtain here (between the Stickelberger elements and Rubin-Stark elements) has recently been utilized by Mazur and Rubin [MR09] (in a rather similar fashion) to prove an important portion of Darmon's refinement of Gross' conjecture.

We finally remark that, thanks to (an appropriate variant of) Proposition 4.5 below, we may by-pass the need of appealing to Krasner's Lemma in [Büy09a, Büy09b] and hence we may remove the hypothesis that χ is unramified at all primes $\wp \subset k$ above p on the main results of [Büy09a, Büy09b].

Notation: Besides what we have fixed above, the following notation will be in effect throughout.

For any field F, let G_F denote the Galois group of a fixed separable closure $\overline{F}$ of F. For any abelian group A, write

$$A^\wedge := \text{Hom}(\text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

for its p-adic completion. Suppose in addition that Δ acts on A, we then write A^χ for the χ-isotypic component of $A^\wedge$.

For k_∞/k as above, let k_n/k be the unique subfield of degree p^n. We set $\Gamma_n = \text{Gal}(k_n/k)$ and write $L_n = Lk_n$. For any prime $\mathfrak{q} \subset k$, let $k(\mathfrak{q})$ denote the p-part of the ray class field extension of k modulo $\mathfrak{q}$. For any square free integral ideal $\mathfrak{q}_1 \cdots \mathfrak{q}_n = \tau \subset k$, we set $k(\tau)$ as the composite

$$k(\tau) = k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_n).$$

Set $\Delta_\tau = \text{Gal}(k(\tau)/k)$. We let $L(\tau) = Lk(\tau)$, $k_n(\tau) = k_n k(\tau)$ and $L_n(\tau) = k_n L(\tau)$.

2. Local conditions and Selmer groups

Much of this section is a review of the Kolyvagin system machinery and the terminology of [MR04], which we will refer to until the end of this paper. The reader who is comfortable with the language of loc.cit. may safely jump to §3 after a glance at §§2.3.1-2.3.2, then at Proposition 2.21, Corollary 2.22 and Propositions 2.23, 2.29, Theorem 2.37 below. We also note that Remark 2.32 (particularly Theorem 2.33) should be of interest for the general understanding of the Kolyvagin system machinery when the core Selmer core rank is greater than one.

2.1. Selmer structures on $T = \mathbb{Z}_p(\chi)$.

Below we use the notation that was set in §1. Recall that $\Gamma := \text{Gal}(k_\infty/k)$ and $\Lambda := \mathbb{Z}_p[[\Gamma]]$ is the cyclotomic Iwasawa algebra.

We first recall Mazur and Rubin's definition of a *Selmer structure*, in particular the *canonical Selmer structure* on T and $T \otimes \Lambda$.

2.1.1. *Local conditions.* Let R be a complete local noetherian ring, and let M be a $R[[G_k]]$-module which is free of finite rank over R. In this paper, we will be interested in the case when $R = \Lambda$ or its certain quotients, and M is $T \otimes \Lambda$ or its corresponding quotients by ideals of Λ. For example, if we start with the Galois representation $T \otimes \Lambda$ with coefficients in Λ, one gets the representation T upon taking the quotient of $T \otimes \Lambda$ by the augmentation ideal of Λ.

For each place λ of k, a *local condition* $\mathcal{F}$ (at λ) on M is a choice of an R-submodule $H^1_{\mathcal{F}}(k_\lambda, M)$ of $H^1(k_\lambda, M)$. A local condition $\mathcal{F}$ at p is a choice of an R-submodule $H^1_{\mathcal{F}}(k_p, M)$ of the semi-local cohomology group $H^1(k_p, M) := \oplus_{\wp|p} H^1(k_\wp, M)$, where the direct sum is over all the primes $\wp$ of k which lie above p.

For examples of local conditions, see [MR04, Definition 1.1.6 and 3.2.1].

Suppose that $\mathcal{F}$ is a local condition (at λ) on M. If M' is a submodule of M (resp., M'' is a quotient module), then $\mathcal{F}$ induces local conditions (which we still denote by $\mathcal{F}$) on M' (resp., on M''), by taking $H^1_{\mathcal{F}}(k_\lambda, M')$ (resp., $H^1_{\mathcal{F}}(k_\lambda, M'')$) to be the inverse image (resp., the image) of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the natural maps induced by

$$M' \hookrightarrow M, \qquad M \twoheadrightarrow M''.$$

Definition 2.1. *Propagation* of a local condition $\mathcal{F}$ on M to a submodule M' (and a quotient M'' of M) is the local condition $\mathcal{F}$ on M' (and on M'') obtained following the procedure above.

For example, if I is an ideal of R, then a local condition on M induces local conditions on M/IM and $M[I]$, by *propagation*.

2.1.2. *Selmer structures and Selmer groups.* Notation from §2.1.1 is in effect throughout this section. We will denote G_{k_λ} by $\mathcal{D}_\lambda$, whenever we wish to identify this group with a closed subgroup of G_k; namely with a particular decomposition group at λ. We further define $\mathcal{I}_\lambda \subset \mathcal{D}_\lambda$ to be the inertia group and $\mathrm{Fr}_\lambda \in \mathcal{D}_\lambda/\mathcal{I}_\lambda$ to be the arithmetic Frobenius element at λ.

Definition 2.2. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

- A finite set $\Sigma(\mathcal{F})$ of places of k, including all infinite places and primes above p, and all primes where M is ramified.
- For every $\lambda \in \Sigma(\mathcal{F})$, a local condition (in the sense of §2.1.1) on M (which we view now as a $R[[\mathcal{D}_\lambda]]$-module), i.e., a choice of R-submodule

$$H^1_{\mathcal{F}}(k_\lambda, M) \subset H^1(k_\lambda, M).$$

If $\lambda \notin \Sigma(\mathcal{F})$, we will also write $H^1_{\mathcal{F}}(k_\lambda, M) = H^1_{\mathrm{f}}(k_\lambda, M)$, where the module $H^1_{\mathrm{f}}(k_\lambda, M)$ is the *finite* part of $H^1(k_\lambda, M)$, defined as in [MR04, Definition 1.1.6].

Definition 2.3. Define the *Cartier dual* of M to be the $R[[G_k]]$-module

$$M^* := \mathrm{Hom}(M, \mu_{p^\infty}),$$

where μ_{p^∞} stands for the p-power roots of unity.

Let λ be a prime of k. There is a perfect pairing

$$< , >_\lambda : H^1(k_\lambda, M) \times H^1(k_\lambda, M^*) \longrightarrow H^2(k_\lambda, \mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p,$$

called the local Tate pairing.

Definition 2.4. The *dual local condition* $\mathcal{F}^*$ on M^* of a local condition $\mathcal{F}$ on M is defined so that $H^1_{\mathcal{F}^*}(k_\lambda, M^*)$ is the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ with respect to the local Tate pairing $< , >_\lambda$.

The *dual Selmer structure* $\mathcal{F}^*$ is defined by setting $\Sigma(\mathcal{F}^*) = \Sigma(\mathcal{F})$ and choosing the local conditions on M^* as the dual local conditions $H^1_{\mathcal{F}^*}(k_\lambda, M^*) = H^1_{\mathcal{F}}(k_\lambda, M)^\perp$ at every prime $\lambda \in \Sigma(\mathcal{F}^*)$.

Definition 2.5. If $\mathcal{F}$ is a Selmer structure on M, we define the *Selmer module* $H^1_{\mathcal{F}}(k, M)$ to be the kernel of the sum of the restriction maps:

$$H^1_{\mathcal{F}}(k, M) := \ker \left(H^1(\mathrm{Gal}(k_{\Sigma(\mathcal{F})}/k), M) \to \bigoplus_{\lambda \in \Sigma(\mathcal{F})} \frac{H^1(k_\lambda, M)}{H^1_{\mathcal{F}}(k_\lambda, M)} \right). \tag{2.1}$$

Here, $k_{\Sigma(\mathcal{F})}$ is the maximal extension of k which is unramified outside $\Sigma(\mathcal{F})$. We also define the dual Selmer module $H^1_{\mathcal{F}^*}(k, M^*)$ in a similar fashion; just replace M by M^* and $\mathcal{F}$ by $\mathcal{F}^*$ in (2.1).

Example 2.6. In this example we recall [MR04, Definition 3.2.1 and 5.3.2].

(i) Let $R = \mathbb{Z}_p$ and let M be a free R-module endowed with a continuos action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_{\text{can}}$ on M by setting
- $\Sigma(\mathcal{F}_{\text{can}}) = \{\lambda : M \text{ is ramified at } \lambda\} \cup \{\wp | p\} \cup \{v|\infty\}$,
- $H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, M) = \begin{cases} H^1_{\text{f}}(k_\lambda, M) & , \quad \text{if } \lambda \in \Sigma(\mathcal{F}_{\text{can}}), \lambda \nmid p\infty, \\ H^1(k_\lambda, M) & , \quad \text{if } \lambda | p. \end{cases}$

Here, $H^1_{\text{f}}(k_\lambda, M) := \ker\{H^1(k_\lambda, M) \to H^1(k_\lambda, M \otimes \mathbb{Q}_p)\}$for every $\lambda \nmid p\mathfrak{f}_\chi$.

The Selmer structure $\mathcal{F}_{\text{can}}$ is called the *canonical Selmer structure* on M.

(ii) Let now $R = \Lambda$ be the cyclotomic Iwasawa algebra, and let $\mathbb{M}$ be a free R-module endowed with a continuos action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{M}$ by setting
- $\Sigma(\mathcal{F}_\Lambda) = \{\lambda : \mathbb{M} \text{ is ramified at } \lambda\} \cup \{\wp | p\} \cup \{v|\infty\}$,
- $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}) = H^1(k_\lambda, \mathbb{M})$ for $\lambda \in \Sigma(\mathcal{F}_\Lambda)$

The Selmer structure $\mathcal{F}_\Lambda$ is called the *canonical Selmer structure* on $\mathbb{M}$.

As in Definition 2.1, the induced Selmer structure on the quotients $\mathbb{M}/I\mathbb{M}$ is still denoted by $\mathcal{F}_\Lambda$. Note that $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}/I\mathbb{M})$ will not usually be the same as $H^1(k_\lambda, \mathbb{M}/I\mathbb{M})$. In particular, when I is the augmentation ideal inside Λ, $\mathcal{F}_\Lambda$ on $\mathbb{M}$ will not always propagate to $\mathcal{F}_{\text{can}}$ on $M = \mathbb{M} \otimes \Lambda/I$.

However, when $M = T$ and $\mathbb{M} = T \otimes \Lambda$ as in §1, it is not hard to see that $\mathcal{F}_\Lambda$ *does* propagate to $\mathcal{F}_{\text{can}}$.

Remark 2.7. When $R = \Lambda$ and $\mathbb{M} = T \otimes \Lambda$ with $T = \mathbb{Z}_p(\chi)$, we will see in §2.5.2 that the Selmer structure $\mathcal{F}_{\text{can}}$ of [Büy07, §2.1] on the quotients $T \otimes \Lambda/(f)$ may be identified, under the hypothesis (A1) on χ, with the propagation of $\mathcal{F}_\Lambda$ to the quotients $T \otimes \Lambda/(f)$, for every distinguished polynomial f inside Λ.

Definition 2.8. A *Selmer triple* is a triple $(M, \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ is a Selmer structure on M and $\mathcal{P}$ is a set of rational primes, disjoint from $\Sigma(\mathcal{F})$.

Remark 2.9. In our setting, i.e., when the Galois representation in question is $T \otimes \Lambda$ or its quotients by ideals of Λ, one may explicitly compute the cohomology groups in terms of certain groups of homomorphisms (c.f., [Rub00, §I.6.1-3]). Nevertheless, we will insist on using the cohomological language for the sake of notational consistency with [MR04] from which we borrow the main technical results. We also hope that the similarity of the ideas applied here and in [Büy09a, Büy09b] are more apparent this way.

2.2. **Computing Selmer groups explicitly.** In this section, we give an explicit description of the Selmer groups for the G_k-representations $T = \mathbb{Z}_p(\chi)$ (resp., for $T \otimes \Lambda$) and $T^* = \boldsymbol{\mu}_{p^\infty} \otimes \chi^{-1}$ (resp., for $(T \otimes \Lambda)^*$); following [Rub00, §I.6.2] and [MR04, §6.1].

2.2.1. *Selmer groups over k.* Recall that L is the CM field cut by χ. For any $m \in \mathbb{Z}^+$, it follows (as in [MR04, §6.1]) from the inflation-restriction sequence that

$$H^1(k, T/p^mT) = H^1(k, \mathbb{Z}/p^m\mathbb{Z}(\chi)) \cong \mathrm{Hom}(G_L, \mathbb{Z}/p^m\mathbb{Z})^{\chi^{-1}}, \tag{2.2}$$

and similarly for every prime λ of k,

$$H^1(k_\lambda, T/p^mT) \cong \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}/p^m\mathbb{Z}) \right)^{\chi^{-1}}. \tag{2.3}$$

Therefore, for the semi-local cohomology at a rational prime ℓ, we have

$$H^1(k_\ell, T/p^mT) \cong \bigoplus_{\lambda|\ell} \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}/p^m\mathbb{Z}) \right)^{\chi^{-1}}. \tag{2.4}$$

Passing to the inverse limit, we obtain

$$H^1(k, T) \cong \mathrm{Hom}(G_L, \mathbb{Z}_p)^{\chi^{-1}}, \tag{2.5}$$

and

$$H^1(k_\ell, T) \cong \bigoplus_{\lambda|\ell} \left(\bigoplus_{\mathfrak{q}|\lambda} \mathrm{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}_p) \right)^{\chi^{-1}}. \tag{2.6}$$

For the dual representation T^*, we have by the inflation-restriction sequence and by Kummer theory

$$H^1(k, T^*[p^m]) = H^1(k, \boldsymbol{\mu}_{p^m} \otimes \chi^{-1}) \cong \left(L^\times/(L^\times)^{p^m}\right)^\chi, \tag{2.7}$$

and similarly for every prime $\lambda \subset k$,

$$H^1(k_\lambda, T^*[p^m]) \cong \left(L_\lambda^\times/(L_\lambda^\times)^{p^m}\right)^\chi. \tag{2.8}$$

Also for the semi-local cohomology, we have

$$H^1(k_\ell, T^*[p^m]) \cong \left(L_\ell^\times/(L_\ell^\times)^{p^m}\right)^\chi, \tag{2.9}$$

where $L_\lambda := L \otimes_k k_\lambda$, the sum of the completions of L at the primes above λ, and $L_\ell := L \otimes_\mathbb{Q} \mathbb{Q}_\ell$. Taking direct limits, we see that

$$H^1(k, T^*) \cong \left(L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi, \tag{2.10}$$

and

$$H^1(k_\ell, T^*) \cong \left(L_\ell^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\chi. \tag{2.11}$$

Proposition 2.10. *The canonical Selmer structure $\mathcal{F}_{\mathrm{can}}$ on T (resp., $\mathcal{F}^*_{\mathrm{can}}$ on T^*) is given by*

- $\Sigma(\mathcal{F}_{\mathrm{can}}) = \Sigma(\mathcal{F}^*_{\mathrm{can}}) = \{\lambda : \lambda | p\mathfrak{f}_\chi\} \cup \{v|\infty\}$,

and by setting (using the identifications above):

- $H^1_{\mathcal{F}_{\mathrm{can}}}(k_\ell, T) = \left(\bigoplus_{\mathfrak{q}|\ell} \mathrm{Hom}(G_{L_\mathfrak{q}}/\mathcal{I}_\mathfrak{q}, \mathbb{Z}_p)\right)^{\chi^{-1}}$,
  $H^1_{\mathcal{F}^*_{\mathrm{can}}}(k_\ell, T^*) = (\mathcal{O}_{L,\ell}^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\chi$, *if* $\ell \neq p$,
- $H^1_{\mathcal{F}_{\mathrm{can}}}(k_p, T) = H^1(k_p, T)$,
  $H^1_{\mathcal{F}^*_{\mathrm{can}}}(k_p, T^*) = 0$.

Here, $\mathcal{I}_\mathfrak{q}$ stands for a fixed inertia group at $\mathfrak{q}$, and $\mathcal{O}_{L,\ell} := \mathcal{O}_L \otimes \mathbb{Z}_\ell$ is the sum of the local units inside $L_\ell = \oplus_{\mathfrak{q}|\ell} L_\mathfrak{q}$.

Proof. This is proved in [Rub00, §I.6.2-3]. □

Definition 2.11. We define the classical Selmer structure $\mathcal{F}_{\text{cl}}$ on T (and $\mathcal{F}^*_{\text{cl}}$ on T^*) by setting $\Sigma(\mathcal{F}_{\text{cl}}) = \Sigma(\mathcal{F}_{\text{can}})$, and by letting

- $H^1_{\mathcal{F}_{\text{cl}}}(k_\ell, T) = H^1_{\mathcal{F}_{\text{can}}}(k_\ell, T)$, and
  $H^1_{\mathcal{F}^*_{\text{cl}}}(k_\ell, T^*) = H^1_{\mathcal{F}^*_{\text{can}}}(k_\ell, T^*)$, if $\ell \neq p$,
- $H^1_{\mathcal{F}_{\text{cl}}}(k_p, T) = \Big(\bigoplus_{\mathfrak{q}|p} \text{Hom}(G_{L_\mathfrak{q}}/\mathcal{I}_\mathfrak{q}, \mathbb{Z}_p)\Big)^{\chi^{-1}}$, and
  $H^1_{\mathcal{F}^*_{\text{cl}}}(k_p, T^*) = (\mathcal{O}^\times_{L,p} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\chi$.

Remark 2.12. If we assume that (A1) holds, it follows from the proof of [Rub00, Proposition III.2.6] (see also [MR04, Lemma 6.1.2]) that $H^1_{\mathcal{F}_{\text{cl}}}(k_p, T) = 0$ and $H^1_{\mathcal{F}^*_{\text{cl}}}(k_p, T^*) = H^1(k_p, T^*)$. We therefore have the following exact sequences

$$0 \longrightarrow H^1_{\mathcal{F}_{\text{cl}}}(k, T) \longrightarrow H^1_{\mathcal{F}_{\text{can}}}(k, T) \xrightarrow{\text{loc}_p} H^1(k_p, T)$$

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(k, T^*) \longrightarrow H^1_{\mathcal{F}^*_{\text{cl}}}(k, T^*) \xrightarrow{\text{loc}^*_p} H^1(k_p, T^*)$$

Furthermore, the image of loc_p is the orthogonal complement of the image of loc^*_p, by the Poitou-Tate global duality theorem. We note that the classical Selmer group $H^1_{\mathcal{F}_{\text{cl}}}(k, T)$ (resp., $H^1_{\mathcal{F}^*_{\text{cl}}}(k, T^*)$) is denoted by $\mathcal{S}(k, T)$ (resp., by $\mathcal{S}(k, W^*)$) in [Rub00].

Proposition 2.13. *Let A_L denote the p-part of the ideal class group of L. Then, $H^1_{\mathcal{F}_{\text{cl}}}(k, T) = 0$ and $H^1_{\mathcal{F}^*_{\text{cl}}}(k, T^*) \cong A_L^\chi$.*

Proof. Proposition 6.1.3 of [MR04] gives

$$H^1_{\mathcal{F}_{\text{cl}}}(k, T) = \varprojlim_m \text{Hom}(A_L^\chi, \mathbb{Z}/p^m\mathbb{Z}) = \text{Hom}(A_L^\chi, \mathbb{Z}_p);$$

we note that the propagation of $\mathcal{F}_{\text{cl}}$ to $\mathbb{Z}/p^m\mathbb{Z}(\chi)$ coincides with the Selmer structure $\mathcal{F}^*$ of [MR04, §6.1]. Since A_L^χ is finite, it follows that $H^1_{\mathcal{F}_{\text{cl}}}(k, T) = 0$.

Similarly, the propagation of $\mathcal{F}^*_{\text{cl}}$ to $\boldsymbol{\mu}_{p^m} \otimes \chi^{-1}$ coincides with the Selmer structure $\mathcal{F}$ of [MR04, §6.1]. It therefore follows from [MR04, Proposition 6.1.3] that there is an exact sequence

$$0 \longrightarrow \big(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^m}\big)^\chi \longrightarrow H^1_{\mathcal{F}^*_{\text{cl}}}(k, T^*[p^m]) \longrightarrow A_L[p^m]^\chi \longrightarrow 0.$$

Taking the direct limit with respect to m, we obtain the following exact sequence:

$$0 \longrightarrow \big(\mathcal{O}_L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\big)^\chi \longrightarrow H^1_{\mathcal{F}^*_{\text{cl}}}(k, T^*) \longrightarrow A_L^\chi \longrightarrow 0.$$

Since χ is totally odd, it follows from [Tat84, Proposition I.3.4] that $(\mathcal{O}_L^\times)^\chi$ is finite, hence $\big(\mathcal{O}_L^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p\big)^\chi = 0$. This completes the proof of the Proposition. □

2.2.2. *Selmer groups over k_∞.* Let k_n denote the unique subfield k_∞, which is of degree p^n over k. We also set $L_n = L \cdot k_n$. Repeating the arguments of the previous section (replacing the totally real field k with the totally real field k_n), we prove the following:

Lemma 2.14. *There is a canonical identification*

$$\varinjlim_n H^1_{\mathcal{F}_{\text{cl}}}(k_n, T^*) = \varinjlim_n A^\chi_{L_n}.$$

2.3. **Modifying the local conditions at p.** When the core Selmer rank of a Selmer structure (in the sense of [MR04], see also §2.5 below) is greater than one, it produces a Selmer group which is difficult to control using the Kolyvagin system machinery of [MR04]. As we will see in §2.5, the Selmer structure $\mathcal{F}_{\text{can}}$ on T (resp., $\mathcal{F}_\Lambda$ on $T \otimes \Lambda$) has core Selmer rank $r = [k : \mathbb{Q}]$. Hence, to be able to utilize the Kolyvagin system machinery, we will need to modify $\mathcal{F}_{\text{can}}$ and $\mathcal{F}_\Lambda$ appropriately. This is what we do in this section.

Throughout this section we assume (A1) and (A2).

2.3.1. *Local conditions at p over k.*

Lemma 2.15. *Under our running hypotheses,*

$$H^1(k_p, T) := \bigoplus_{\wp|p} H^1(k_\wp, T)$$

is a free $\mathbb{Z}_p$-module of rank $r = [k : \mathbb{Q}]$.

Proof. We first prove this using the general structure theory of semi-local cohomology groups at p. All the references in this proof are to [Büy09b, Appendix A]. We note that the results quoted in loc.cit. are originally due to Benois, Colmez, Herr and Perrin-Riou.

By Theorem A.8(i), the Λ-torsion submodule $H^1(k_p, T \otimes \Lambda)_{\text{tors}}$ is isomorphic to $\oplus_{\wp|p} T^{H_{k_\wp}}$, where $H_{k_\wp} = \text{Gal}(\overline{k_\wp}/k_{\wp,\infty})$. Since we assume (A1), it follows that $H^1(k_p, T \otimes \Lambda)_{\text{tors}} = 0$. Theorem A.8(ii) now concludes that the Λ-module $H^1(k_p, T \otimes \Lambda)$ is free rank r. Furthermore,

$$\text{coker}[H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T)] = H^2(k_p, T \otimes \Lambda)[\gamma - 1],$$

where γ is any topological generator of Γ. However, it follows from [Büy07, Lemma 2.11] that $H^2(k_p, T \otimes \Lambda) = 0$, hence the map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1(k_p, T)$$

is surjective. Lemma now follows. □

Remark 2.16. There is a more direct proof of Lemma 2.15. In this remark, we include this alternative proof of this lemma.

By the explicit description of the semi-local cohomology groups in (2.6)

$$H^1(k_p, T) \cong \bigoplus_{\wp|p} \left(\bigoplus_{\mathfrak{q}|\wp} \text{Hom}(G_{L_\mathfrak{q}}, \mathbb{Z}_p) \right)^{\chi^{-1}}.$$

It follows at once from this description that $H^1(k_p, T)$ is $\mathbb{Z}_p$-torsion free, hence free. Further, since $\mathbb{Z}_p$ is an abelian group we may rewrite the equality above as

$$H^1(k_p, T) \cong \bigoplus_{\wp|p} \left(\bigoplus_{\mathfrak{q}|\wp} \text{Hom}(G^{\text{ab}}_{L_\mathfrak{q}}, \mathbb{Z}_p) \right)^{\chi^{-1}},$$

where $G_{L_\mathfrak{q}}^{\mathrm{ab}}$ stands for the abelianization of $G_{L_\mathfrak{q}}$. By local class field theory $G_{L_\mathfrak{q}}^{\mathrm{ab}} \cong L_\mathfrak{q}^\wedge$, the p-adic completion of the multiplicative group of $L_\mathfrak{q}$. Further, the valuation map $\mathrm{val}_\mathfrak{q}$ gives an isomorphism

$$L_\mathfrak{q}^\times \xrightarrow{\mathrm{val}_\mathfrak{q}} \mathbb{Z}_p \oplus \mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge}.$$

We therefore have

$$\begin{aligned} H^1(k_p,T) &\cong \mathrm{Hom}\left(\bigoplus_{\mathfrak{q}|p}(\mathbb{Z}_p \oplus \mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge}),\mathbb{Z}_p\right)^{\chi^{-1}} \\ &\cong \mathrm{Hom}\left(\left(\oplus_{\mathfrak{q}|p}\mathbb{Z}_p\right)^\chi \oplus \left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge}\right)^\chi, \mathbb{Z}_p\right). \end{aligned}$$

It follows from (A1) that $\left(\oplus_{\mathfrak{q}|p}\mathbb{Z}_p\right)^\chi = 0$, hence

$$H^1(k_p,T) \cong \mathrm{Hom}\left(\left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge}\right)^\chi, \mathbb{Z}_p\right).$$

To prove the Lemma, it suffices to check that the $\mathbb{Q}_p$-dimension of $V := \left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge} \otimes \mathbb{Q}_p\right)^\chi$ is equal to r. The p-adic logarithm gives a homomorphism $\mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge} \to \mathcal{O}_{L_\mathfrak{q}}$ with finite kernel and cokernel. Hence

$$V = \left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}}^{\times,\wedge} \otimes \mathbb{Q}_p\right)^\chi = \left(\oplus_{\mathfrak{q}|p}\mathcal{O}_{L_\mathfrak{q}} \otimes \mathbb{Q}_p\right)^\chi = (L \otimes \mathbb{Q}_p)^\chi$$

and therefore the $\mathbb{Q}_p$-dimension of V equals r by the normal basis theorem.

Definition 2.17. Fix a $\mathbb{Z}_p$-direct summand $\mathcal{L} \subset H^1(k_p,T)$ such that $\mathcal{L}$ is free of rank one. Fix also a generator $\varphi = \varphi_\mathcal{L}$ of $\mathcal{L}$. Define the *$\mathcal{L}$-modified Selmer structure* $\mathcal{F}_\mathcal{L}$ on T as follows:

- $\Sigma(\mathcal{F}_\mathcal{L}) = \Sigma(\mathcal{F}_{\mathrm{can}})$,
- if $\lambda \nmid p$, $H^1_{\mathcal{F}_\mathcal{L}}(k_\lambda,T) = H^1_{\mathcal{F}_{\mathrm{can}}}(k_\lambda,T)$,
- $H^1_{\mathcal{F}_\mathcal{L}}(k_p,T) := \mathcal{L} \subset H^1(k_p,T) = H^1_{\mathcal{F}_{\mathrm{can}}}(k_p,T)$.

2.3.2. *Local conditions at p over k_∞.* Set $\Gamma = \mathrm{Gal}(k_\infty/k)$ as before. Let $k_\wp$ denote the completion of k at $\wp$, and let $k_{\wp,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $k_\wp$. Since we assume (A2), we may identify $\mathrm{Gal}(k_{\wp,\infty}/k_\wp)$ by Γ for each $\wp|p$ and henceforth Γ will stand for any of these Galois groups. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra, as usual. We also fix a topological generator γ of Γ, and we set $\mathbf{X} = \gamma - 1$. We will occasionally identify Λ by the power series ring $\mathbb{Z}_p[[\mathbf{X}]]$.

Lemma 2.18. *Under the assumptions* (A1) *and* (A2)*,*

$$H^1(k_p, T \otimes \Lambda) := \oplus_{\wp|p} H^1(k_\wp, T \otimes \Lambda)$$

is a free Λ-module of rank r.

Proof. This is checked in the first part of the proof of Lemma 2.15. □

Definition 2.19. Fix a Λ-rank one direct summand $\mathbb{L} \subset H^1(k_p, T \otimes \Lambda)$ such that $\mathbb{L}$ maps onto $\mathcal{L}$ under the projection

$$H^1(k_p, T \otimes \Lambda) \twoheadrightarrow H^1(k_p,T). \tag{2.12}$$

Fix also a generator $\Phi = \Phi_\mathbb{L}$ of $\mathbb{L}$, which maps to $\varphi = \varphi_\mathcal{L}$ under the projection (2.12). Define the *$\mathbb{L}$-modified Selmer structure* $\mathcal{F}_\mathbb{L}$ on $T \otimes \Lambda$ as follows:

- $\Sigma(\mathcal{F}_{\mathbb{L}}) = \Sigma(\mathcal{F}_{\Lambda})$,
- if $\lambda \nmid p$, $H^1_{\mathcal{F}_{\mathbb{L}}}(k_\lambda, T\otimes\Lambda) = H^1_{\mathcal{F}_{\Lambda}}(k_\lambda, T\otimes\Lambda)$,
- $H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, T\otimes\Lambda) := \mathbb{L} \subset H^1(k_p, T\otimes\Lambda) = H^1_{\mathcal{F}_{\Lambda}}(k_p, T\otimes\Lambda)$.

Remark 2.20. By definition, the image of $H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, T\otimes\Lambda)$ is $H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T)$ under the map $H^1(k_p, T\otimes\Lambda) \to H^1(k_p, T)$. Further, it follows from [MR04, Lemma 5.3.1(ii)] for $\ell \neq p$ that $H^1_{\mathcal{F}_{\mathbb{L}}}(k_\ell, T\otimes\Lambda)$ also maps to $H^1_{\mathcal{F}_{\mathcal{L}}}(k_\ell, T)$ under the natural map $H^1(k_\ell, T\otimes\Lambda) \to H^1(k_\ell, T)$. In other words, $\mathcal{F}_{\mathbb{L}}$ propagates to $\mathcal{F}_{\mathcal{L}}$, and there is an induced map

$$H^1_{\mathcal{F}_{\mathbb{L}}}(k, T\otimes\Lambda) \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}}}(k, T).$$

2.4. **Global duality and a comparison of Selmer groups.** In this section, we compare the classical Selmer group (which we wish to relate to L-values) to the modified Selmer groups (for which we will apply the Kolyvagin system machinery and which we will compute in terms of L-values). The necessary tool to accomplish this comparison is Poitou-Tate global duality.

2.4.1. *Comparison over k.* The definition of the modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ and Remark 2.12 gives us the following exact sequences:

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathrm{cl}}}(k, T) \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}}}(k, T) \xrightarrow{\mathrm{loc}_p} \mathcal{L}$$

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*) \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \xrightarrow{\mathrm{loc}_p^*} \frac{H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_p, T^*)}{H^1_{\mathcal{F}^*_{\mathcal{L}}}(k_p, T^*)}$$

Poitou-Tate global duality (c.f., [Rub00, Theorem I.7.3], [Mil86, Theorem I.4.10]) states that the image of loc_p is the orthogonal compliment of the image of loc_p^* with respect to the local Tate pairing. Using this fact, together with Proposition 2.13, one may prove the following Proposition. See [Rub00, Theorem I.7.3] for details:

Proposition 2.21. *We have an exact sequence*

$$0 \to H^1_{\mathcal{F}_{\mathcal{L}}}(k, T) \xrightarrow{\mathrm{loc}_p} \mathcal{L} \xrightarrow{(\mathrm{loc}_p^*)^\vee} \left(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)\right)^\vee \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)\right)^\vee \to 0,$$

where the map $(\mathrm{loc}_p^*)^\vee$ is induced from localization at p and the local Tate pairing between $H^1(k_p, T)$ and $H^1(k_p, T^*)$.

Suppose $c \in H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)$ is any class. We still write c for the image of the class c inside $\mathcal{L} = H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T)$ under the (injective) map loc_p.

Corollary 2.22. *The following sequence is exact:*

$$0 \to \frac{H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)}{\mathbb{Z}_p \cdot c} \xrightarrow{\mathrm{loc}_p} \frac{\mathcal{L}}{\mathbb{Z}_p \cdot c} \xrightarrow{(\mathrm{loc}_p^*)^\vee} \left(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)\right)^\vee \to \left(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)\right)^\vee \to 0.$$

2.4.2. *Comparison over k_∞.* Repeating the argument of Proposition 2.21 for each field k_n (instead of k) and passing to inverse limit we obtain the following:

Proposition 2.23. *Both of the following sequences of Λ-modules are exact:*

(i) $0 \to {\scriptstyle H^1_{\mathcal{F}_{\mathbb{L}}}(k, T\otimes\Lambda)} \xrightarrow{\mathrm{loc}_p} \mathbb{L} \longrightarrow \left(\varinjlim_n A_{L_n}^{\chi}\right)^\vee \longrightarrow \left({\scriptstyle H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, (T\otimes\Lambda)^*)}\right)^\vee \to 0,$

(ii) *For any class* $c \in H^1(k, T \otimes \Lambda)$,

$$0 \to \frac{H^1_{\mathcal{F}_{\mathbb{L}}}(k, T\otimes\Lambda)}{\Lambda \cdot c} \xrightarrow{\mathrm{loc}_p} \frac{\mathbb{L}}{\Lambda \cdot c} \longrightarrow \left(\varinjlim_n A^{\chi}_{L_n}\right)^{\vee} \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, (T\otimes\Lambda)^*)\right)^{\vee} \longrightarrow 0.$$

Proof. We only give a sketch since similar versions of this Proposition are already available in the literature (c.f., [Rub00, Theorem I.7.3 and III.2.10], [dS87, §III.1.7]).

Thanks to the argument of Proposition 2.21 and [Rub00, Proposition B.1.1], there is an exact sequence

$$0 \longrightarrow \varprojlim_n H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) \longrightarrow \varprojlim_n \mathcal{L}_n \longrightarrow \left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n, T^*)\right)^{\vee} \longrightarrow \left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*)\right)^{\vee} \longrightarrow 0,$$

where $\mathcal{L}_n$ is the image of $\mathbb{L}$ under the natural map

$$H^1(k_p, T \otimes \Lambda) \longrightarrow H^1((k_n)_p, T).$$

By definition, $\varprojlim_n \mathcal{L}_n = \mathbb{L}$, and by [MR04, Lemma 5.3.1] (or rather by its proof) it follows that there is a canonical isomorphism

$$\varprojlim_n H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n, T) \cong H^1_{\mathcal{F}_{\mathbb{L}}}(k, T \otimes \Lambda).$$

Furthermore, by Lemma 2.14, $\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n, T^*) = \varinjlim_n A^{\chi}_{L_n}$. Finally, by Shapiro's lemma

$$H^1(k_n, T^*) = H^1(k, T^* \otimes \mathbb{Z}_p[\Gamma_n]),$$

where $\Gamma_n = \mathrm{Gal}(k_n/k)$, hence

$$\varinjlim_n H^1(k_n, T^*) = H^1(k, \varinjlim_n T^* \otimes \mathbb{Z}_p[\Gamma_n]). \tag{2.13}$$

Now, using the fact that the functors $- \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]$ and $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[\Gamma_n], -)$ are adjoint functors (we drop the subscripts below and write $\otimes$ and Hom for short), it follows that

$$\begin{aligned}
(T \otimes \Lambda)^* &:= \mathrm{Hom}(\varprojlim_n T \otimes \mathbb{Z}_p[\Gamma_n], \mathbb{Q}_p/\mathbb{Z}_p)(1) \\
&\cong \varinjlim_n \mathrm{Hom}(T, \mathrm{Hom}(\mathbb{Z}_p[\Gamma_n], \mathbb{Q}_p/\mathbb{Z}_p))(1) \\
&\cong \varinjlim_n \mathrm{Hom}(T, \mathbb{Q}_p/\mathbb{Z}_p[\Gamma_n])(1) \\
&\cong \varinjlim_n \mathrm{Hom}(T, \mathbb{Q}_p/\mathbb{Z}_p)(1) \otimes \mathbb{Z}_p[\Gamma_n] =: \varinjlim_n T^* \otimes \mathbb{Z}_p[\Gamma_n],
\end{aligned}$$

where the isomorphism of the modules in the second and the third line comes from the isomorphism

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Z}_p[\Gamma_n], \mathbb{Q}_p/\mathbb{Z}_p) &\xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p[\Gamma_n] \\
f &\longmapsto \textstyle\sum_{\gamma\in\Gamma_n} f(\gamma)\cdot\gamma
\end{aligned}$$

of $\mathbb{Z}_p[\Gamma_n]$-modules. This and (2.13) (together with its semi-local analogue) show at once that

$$\varinjlim_n H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n, T^*) = H^1_{\mathcal{F}^*_{\mathbb{L}}}(k, (T \otimes \Lambda)^*).$$

This completes the proof of (i), and (ii) follows trivially from (i). □

2.5. **Kolyvagin systems for modified Selmer groups.** This section closely follows the exposition of [Büy09a, §1.2] and [Büy09b, §2.5]. We assume (A1) and (A2) throughout this section.

Remark 2.24. It is straightforward to verify that the following hypotheses (which were introduced in [MR04, §3.5]) hold for $T = \mathbb{Z}_p(\chi)$:

(**H**.**1**) The residual $\mathbb{F}_p[[G_k]]$-representation T/pT is absolutely irreducible.
(**H**.**2**) There is a $\tau \in G_k$ such that $\tau = 1$ on μ_{p^∞} and $T/(\tau - 1)T$ is free of rank one over $\mathbb{Z}_p$.
(**H**.**3**) $H^0(k, T/pT) = H^0(k, T^*[p]) = 0$.
(**H**.**4**) $\mathrm{Hom}_{\mathbb{F}_p[[G_k]]}(T/pT, T^*[p]) = 0$.

We remark that the hypothesis (**H**.**3**) above is implied by what Mazur and Rubin call $(H.3)$ (c.f., [MR04, Lemma 3.5.2]). The hypothesis (**H**.**3**) above is sufficient for our purposes.

Let $\mathcal{P}$ denote the set whose elements are prime ideals of k which are prime to $p\mathfrak{f}_\chi$. For each positive integer m and n, let

$$\mathcal{P}_{m+n} = \{\mathfrak{q} \in \mathcal{P} : \mathfrak{q} \text{ splits completely in } L(\mu_{p^{m+n+1}})/k\}$$

be a subset of $\mathcal{P}$. Note that $\mathcal{P}_{m+n}$ is exactly the set of primes determined by [Rub00, Definition IV.1.1] when $T = \mathbb{Z}_p(\chi)$. The hypothesis (H.5) of [MR04, §3.5] holds with this choice of $\mathcal{P}$. Let $\mathcal{N} = \mathcal{N}(\mathcal{P})$ (resp., $\mathcal{N}_j = \mathcal{N}(\mathcal{P}_j) \subset \mathcal{N}$) be the square free products of primes $\mathfrak{q} \in \mathcal{P}$ (resp., in $\mathcal{P}_j$), with the convention that $1 \in \mathcal{N}_j \subset \mathcal{N}$.

Using [MR04, Lemma 3.7.1], one may also check that $\mathcal{F}_{\text{can}}$ and $\mathcal{F}_{\mathcal{L}}$ satisfy the hypothesis (H.6) of [MR04, §3.5]. We may therefore apply the main results of [MR04]. In particular, the existence of Kolyvagin systems for these Selmer structures will be decided by their *core Selmer ranks* (for a definition, c.f., [MR04, Definition 4.1.11 and 5.2.4]). Let $\mathcal{X}(T, \mathcal{F})$ denote the core Selmer rank of the Selmer structure $\mathcal{F}$, for $\mathcal{F} = \mathcal{F}_{\text{can}}$ or for $\mathcal{F} = \mathcal{F}_{\mathcal{L}}$.

Proposition 2.25. *We have* $\mathcal{X}(T, \mathcal{F}_{\text{can}}) = r\ (= [k : \mathbb{Q}])$.

Proof. This follows from [MR04, Theorem 5.2.15], applied with the base field k (instead of $\mathbb{Q}$; we therefore have r real places instead of one) and using our assumption that χ is totally odd. □

Proposition 2.26. *The core Selmer rank* $\mathcal{X}(T, \mathcal{F}_{\mathcal{L}})$ *of the Selmer structure* $\mathcal{F}_{\mathcal{L}}$ *on* T *is one.*

Proof. The proof of this Proposition is identical to the proof of [Büy09a, Proposition 1.8]. □

2.5.1. *Kolyvagin systems over* k. We recall the definition of the (generalized) module of Kolyvagin systems (introduced in [MR04]) for the Selmer triple $(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$:

Definition 2.27.[Compare to [MR04] Definition 3.1.6] Define the (generalized) module of Kolyvagin systems

$$\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P}) := \varprojlim_{s} \varinjlim_{j} \mathbf{KS}(T/p^sT, \mathcal{F}_{\mathcal{L}}, \mathcal{P}_j),$$

where $\mathbf{KS}(T/p^sT, \mathcal{F}_{\mathcal{L}}, \mathcal{P}_j)$ is the module of Kolyvagin systems for the Selmer structure $\mathcal{F}_{\mathcal{L}}$ on the representation T/p^sT, as in [MR04, Definition 3.1.3].

We will call an element of $\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ an $\mathcal{L}$*-restricted Kolyvagin system* for T.

Remark 2.28. In order to define Kolyvagin systems, one needs to first define the "transverse local condition" (see [MR04, Definition 1.1.6(iv)]). In this remark, we briefly recall this definition. Let F be any local field, and fix once and for all an abelian extension F'/F which is

totally and tamely ramified, and moreover is a maximal such extension. When $F = \mathbb{Q}_\ell$, then there is a natural choice for F', namely $F' = \mathbb{Q}_\ell(\boldsymbol{\mu}_\ell)$. In general, we simply fix an extension F' as above and define the *transverse local condition* to be

$$H^1_{\text{tr}}(F,T) = \ker\{H^1(F,T) \longrightarrow H^1(F',T)\}.$$

Proposition 2.29. *The $\mathbb{Z}_p$-module $\overline{\mathbf{KS}}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ is free of rank* one. *Furthermore, it is generated by a Kolyvagin system $\kappa \in \overline{\mathbf{KS}}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ whose image* (*under the canonical map induced from reduction mod p*) *inside $\mathbf{KS}(T/pT,\mathcal{F}_\mathcal{L},\mathcal{P})$ is nonzero.*

A generator of the cyclic module $\overline{\mathbf{KS}}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ will be called a *primitive* Kolyvagin system.

Proof. This is immediate after Proposition 2.26 and [MR04, Theorem 5.2.10]. To apply Theorem 5.2.10 of loc.cit., one needs to verify that the hypotheses (H.1)-(H.6) of [MR04, §3.5] hold true for the triple $(T,\mathcal{F}_\mathcal{L},\mathcal{P})$. □

Remark 2.30. Using Proposition 2.26 and [MR04, Proposition 5.2.9], the generalized module of Kolyvagin systems $\overline{\mathbf{KS}}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ may be identified by the module of of Kolyvagin systems $\mathbf{KS}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ (defined as in [MR04, Definition 3.1.3]). We will use this identification without warning.

We record here the main application of a Kolyvagin system for the Selmer triple $(T,\mathcal{F}_\mathcal{L},\mathcal{P})$. Suppose $\{\{\kappa_\tau(s)\}_{\tau\in\mathcal{N}_s}\}_s = \kappa \in \overline{\mathbf{KS}}(T,\mathcal{F}_\mathcal{L},\mathcal{P})$ is any Kolyvagin system. See [MR04, §3] for an explanation of our notation. We loosely say here that $\kappa_\tau(s) \in H^1(k,T/p^sT)$, and by definition, there is a well defined element

$$\kappa_1 = \{\kappa_1(s)\}_s \in \varprojlim_s H^1_{\mathcal{F}_\mathcal{L}}(k,T/p^sT) = H^1_{\mathcal{F}_\mathcal{L}}(k,T).$$

Theorem 2.31. [MR04, Theorem 5.2.13 and 5.2.14] *Under our running hypotheses*

(i) $\text{length}(H^1_{\mathcal{F}^*_\mathcal{L}}(k,T^*)) \leq \text{length}(H^1_{\mathcal{F}_\mathcal{L}}(k,T)/\mathbb{Z}_p\cdot\kappa_1)$,
(ii) *the inequality in* (i) *is an equality if and only if κ is primitive.*

Remark 2.32. Note that the *choice* of a rank one direct summand $\mathcal{L} \subset H^1(k_p,T)$ makes our approach somewhat unnatural. We address this issue in this remark. Put

$$H^1(k_p,T) = \bigoplus_{i=1}^r \mathcal{L}_i \tag{2.14}$$

(where each $\mathcal{L}_i$ is a free $\mathbb{Z}_p$-submodule of $H^1(k_p,T)$ of rank one) and consider

$$\sum_{i=1}^r \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i},\mathcal{P}) \subset \mathbf{KS}(T,\mathcal{F}_{\text{can}},\mathcal{P}). \tag{2.15}$$

Claim. *The sum in* (*2.15*) *is a direct sum.*

Proof. Assume contrary: Suppose $0 \neq \boldsymbol{\kappa}^i \in \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i},\mathcal{P})$ (for $i = 1,\ldots,r$) is such that

$$\sum_{i=1}^r a_i\boldsymbol{\kappa}^i = 0$$

for some $a_i \in \mathbb{Z}_p$, and $a_{i_0} \neq 0$ for a certain $1 \leq i_0 \leq r$. This means

$$a_{i_0}\boldsymbol{\kappa}^{i_0} = -\sum_{\substack{i=1\\ i\neq i_0}}^{r} a_i\boldsymbol{\kappa}^{i} \in \sum_{\substack{i=1\\ i\neq i_0}}^{r} \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}). \tag{2.16}$$

Write $\boldsymbol{\kappa}^{i_0} = \{\kappa_n^{i_0}\}$ (see [MR04, §3] for a precise definition of a Kolyvagin system to clarify this notation, see also Remark 2.28 below). Equation (2.16) therefore shows that

$$\mathrm{loc}_p(a_{i_0}\kappa_1^{i_0}) \in \bigoplus_{\substack{i=1\\ i\neq i_0}}^{r} \mathcal{L}_i. \tag{2.17}$$

Also, by definition, $\mathrm{loc}_p(a_{i_0}\kappa_1^{i_0}) \in \mathcal{L}_{i_0}$; using this together with (2.17) we conclude that $\mathrm{loc}_p(a_{i_0}\kappa_1^{i_0}) = 0$. The injectivity of loc_p (which we checked in §2.4.1) gives $a_{i_0}\kappa_1^{i_0} = 0$.

On the other hand, Proposition 2.21 (applied with $\mathcal{L} = \mathcal{L}_{i_0}$) shows that $H^1_{\mathcal{F}^*_{\mathcal{L}_{i_0}}}(k, T^*)$ is finite (as the finite group $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)^\vee = (A_L^\chi)^\vee$ surjects onto its Pontryagin dual). This in return shows, using [MR04, 5.2.12(v)], that for any $0 \neq \boldsymbol{\kappa} = \{\kappa_n\} \in \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_{i_0}}, \mathcal{P})$, we have $\kappa_1 \neq 0$. Therefore, $a_{i_0}\kappa_1^{i_0} = 0$ implies that $a_{i_0}\kappa^{i_0} = 0$, a contradiction. □

Note that, in order to prove the Claim above, we used the facts that loc_p is injective (on $H^1_{\mathcal{F}_{\mathcal{L}_{i_0}}}(k, T)$) and that $H^1_{\mathcal{F}^*_{\mathcal{L}_{i_0}}}(k, T^*)$ is finite in our current setting. With a bit more work, it is possible to prove this Claim without having either of these conditions. We leave the more general proof aside not to digress from the main point of our paper any further.

It would be very interesting to have an answer for the following:
Question: Is the direct sum

$$\bigoplus_{i=1}^{r} \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}) \subset \mathbf{KS}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

independent of the choice of the decomposition (2.14)?

When the answer to this question is affirmative, we would have a *canonically* defined rank-r submodule of $\mathbf{KS}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$. It would be even more tempting to inquire whether this rank-r submodule descends from Euler systems. In §3 below, we construct a rank r submodule of $\mathbf{KS}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ out of Stickelberger elements; which still does depend on the decomposition (2.14).

These questions seem to be out of reach in the current state of the art. We may, however, prove the following weaker (yet still interesting) statement. First, we recall some terminology from [MR04].

Define the module of L-values

$$\begin{aligned}\mathbf{LV} = &\mathbf{LV}(T; \{\mathcal{L}_i\}_{i=1}^{r}) := \\ &\mathrm{span}_{\mathbb{Z}_p}(\kappa_1 : \boldsymbol{\kappa} \in \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}) \text{ for some } i) \subset H^1_{\mathcal{F}_{\mathrm{can}}}(k, T).\end{aligned}$$

(Compare this definition with [MR04, Definition 3.1.5].) Note that the $\mathbb{Z}_p$-module $\mathbf{LV}$ depends a priori on the choice of the decomposition (2.14).

Theorem 2.33. *The module of L-values $\mathbf{LV}$ is independent of the choice of the decomposition (2.14).*

Proof. Fix any generator ℓ_i of the free $\mathbb{Z}_p$-module $\mathcal{L}_i$ of rank one. Suppose $\mathcal{L} \subset H^1(k_p, T)$ is any rank one direct summand (not necessarily one of $\mathcal{L}_i$ which appear in (2.14)). Let $\boldsymbol{\kappa}^{\mathcal{L}} = \{\kappa_n^{\mathcal{L}}\}$ be any generator of the cyclic $\mathbb{Z}_p$-module $\mathbf{KS}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$. To prove the Theorem, it suffices to show that $\kappa_1^{\mathcal{L}} \in \mathbf{LV}$. We may write

$$\mathrm{loc}_p(\kappa_1^{\mathcal{L}}) = \sum_{i=1}^{r} a_i \ell_i$$

with $a_i \in \mathbb{Z}_p$.

Claim. *Let a_i be as above. Then*

$$\mathrm{ord}_p(a_i) \geq \mathrm{ord}_p(\#H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*))$$

for all $1 \leq i \leq r$.

Proof of the Claim. Let $d = \gcd(a_1, \ldots, a_r)$ and set $\alpha_i := \frac{a_i}{d} \in \mathbb{Z}_p$. By definition, at least one of the α_i is a p-adic unit. We also set

$$x := \mathrm{loc}_p(\kappa_1^{\mathcal{L}}) = \sum_i^r a_i \ell_i, \text{ and, } y = \frac{x}{d} = \sum_i^r \alpha_i \ell_i.$$

(1) Since $d \cdot y = x \in \mathcal{L}$ and $H^1(k_p, T)/\mathcal{L}$ is $\mathbb{Z}_p$-torsion free, it follows that $y \in \mathcal{L}$.
(2) $H^1(k_p, T)/\mathbb{Z}_p y$ is $\mathbb{Z}_p$-torsion free; indeed suppose

$$z = \sum_i^r \beta_i \ell_i \in \bigoplus_i^r \mathcal{L}_i = H^1(k_p, T)$$

is such that $m \cdot z \in \mathbb{Z}_p y$ for some $m \in \mathbb{Z}_p$. This means

$$\sum_i^r m\beta_i \ell_i = sy = \sum_i^r s\alpha_i \ell_i$$

for some $s \in \mathbb{Z}_p$, hence $m\beta_i = s\alpha_i$, in particular $m | s\alpha_i$ for every $1 \leq i \leq r$. Since $\gcd(\alpha_1, \ldots, \alpha_r) = 1$, it follows that $m|s$, hence $z = \frac{s}{m} y \in \mathbb{Z}_p y$.
(3) The items (1) and (2) together show that $\mathcal{L} = \mathbb{Z}_p y$.

We may now conclude that

$$\#\mathcal{L}/\mathbb{Z}_p \mathrm{loc}_p(\kappa_1^{\mathcal{L}}) = \#\left(\mathbb{Z}_p y/\mathbb{Z}_p x\right) = p^{\mathrm{ord}_p(d)},$$

with d as above. On the other hand, Corollary 2.22 shows that

$$\#\frac{\mathcal{L}}{\mathbb{Z}_p \mathrm{loc}_p(\kappa_1^{\mathcal{L}})} \geq \#H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \iff \#\frac{H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)}{\mathbb{Z}_p \kappa_1^{\mathcal{L}}} \geq \#H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*).$$

The latter statement is the main application of the Kolyvagin system $\kappa^{\mathcal{L}}$ (c.f., Theorem 2.31 above). We therefore conclude that

$$p^{\mathrm{ord}_p(d)} \geq \#H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*),$$

which is our Claim. □

We now prove that Theorem 2.33 follows from this Claim. As in the final paragraph of the proof of the Claim above, it follows from Corollary 2.22 and [MR04, Theorem 5.2.10 and 5.2.14] that there exists a Kolyvagin system $\tilde{\boldsymbol{\kappa}}^i \in \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P})$ such that

$$\mathrm{loc}_p(\tilde{\kappa}_1^i) = \#H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \cdot \ell_i \in \mathcal{L}_i,$$

for every $i = 1, \dots, r$. By the Claim above, there is a Kolyvagin system $\boldsymbol{\kappa}^i \in \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P})$ such that $\mathrm{loc}_p(\kappa_1^i) = a_i \ell_i$ (just set $\boldsymbol{\kappa}^i = \frac{a_i}{\# H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*)} \tilde{\boldsymbol{\kappa}}^i$). We therefore have

$$\mathrm{loc}_p(\kappa_1^{\mathcal{L}}) = \sum_i^r \mathrm{loc}_p(\kappa_1^i),$$

and since the map loc_p is injective in our setting, it follows that

$$\kappa_1^{\mathcal{L}} = \sum_i^r \kappa_1^i \in \mathbf{LV},$$

as desired. □

We close our Remark noting that all this discussion applies equally well in the setting of [Büy09a, Büy09b] as long as we assume Leopoldt's conjecture (i.e., the injectivity of loc_p in the setting of loc.cit.).

2.5.2. *Kolyvagin systems over k_∞.* We start with the observation that the following versions of the hypotheses $\mathbb{H}.\mathbf{T}$ and $\mathbb{H}.\mathbf{sEZ}$ of [Büy07, §2.2] hold for T:

($\mathbb{H}.\mathbf{T}_{/k}$) $(T \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{I}_\lambda}$ is divisible for every prime $\lambda \nmid p$, $\lambda \subset k$.
($\mathbb{H}.\mathbf{sEZ}_{/k}$) $H^0(k_\wp, T^*) = 0$ for primes $\wp | p$.

We define a Selmer structure $\mathcal{F}^\Lambda_{\mathrm{can}}$ on certain quotients of $T \otimes \Lambda$. This is the Selmer structure $\mathcal{F}_{\mathrm{can}}$ of [Büy07, Definition 2.2]:

Definition 2.34. Suppose $f \in \Lambda$ is any *distinguished* polynomial, in the sense that the quotient $\Lambda/(f)$ is a free $\mathbb{Z}_p$-module of finite rank. Let $\mathcal{F}^\Lambda_{\mathrm{can}}$ be the following Selmer structure on $T_f := T \otimes \Lambda/(f)$:

- $\Sigma(\mathcal{F}^\Lambda_{\mathrm{can}}) = \Sigma(\mathcal{F}_\Lambda)$,
- The local conditions are given by

$$H^1_{\mathcal{F}^\Lambda_{\mathrm{can}}}(k_\lambda, T_f) = \begin{cases} H^1(k_\lambda, T_f) & , \quad \text{if } \lambda | p \\ \\ H^1_{\mathrm{f}}(k_\lambda, T_f) & , \quad \text{if } \lambda \in \Sigma(\mathcal{F}_{\mathrm{can}}) \text{ and } \lambda \nmid p \end{cases}$$

with

$$H^1_{\mathrm{f}}(k_\lambda, T_f) = \ker\{H^1(k_\lambda, T_f) \longrightarrow H^1(k_\lambda^{\mathrm{unr}}, T_f \otimes \mathbb{Q}_p)\},$$

where k_λ^{unr} is the maximal unramified extension of k_λ.

The induced Selmer structure on the quotients $T \otimes \Lambda/(p^s, f)$, which is obtained by *propagating* $\mathcal{F}^\Lambda_{\mathrm{can}}$ (in the sense of Definition 2.1) will also be denoted by $\mathcal{F}^\Lambda_{\mathrm{can}}$.

Let $T_{s,m} := T \otimes \Lambda/(p^s, \mathbf{X}^m)$, where $\mathbf{X}$ is as in §2.3.2.

Remark 2.35. By the definition of $\mathcal{F}_{\mathbb{L}}$, the local conditions on $T_{s,m}$ at primes $\lambda \nmid p$ propagated from $\mathcal{F}_{\mathbb{L}}$ coincide with the local conditions propagated from $\mathcal{F}_\Lambda$; and thanks to [Büy07, Corollary 2.8 and 2.9], they also coincide with the local conditions determined by $\mathcal{F}^\Lambda_{\mathrm{can}}$, since $\mathbb{H}.\mathbf{T}_{/k}$ holds true. Indeed, it is proved in loc.cit. that all these local conditions coincide with

$$H^1_{\mathrm{unr}}(k_\lambda, T_{s,m}) := \ker\left\{H^1(k_\lambda, T_{s,m}) \longrightarrow H^1(k_\lambda^{\mathrm{unr}}, T_{s,m})\right\},$$

as long as the hypothesis $\mathbb{H}.\mathbf{T}_{/k}$ holds true. We note further that $\mathcal{F}_{\mathbb{L}}$ propagates to the Selmer structure $\mathcal{F}_{\mathcal{L}}$ on $T = T \otimes \Lambda/(\mathbf{X})$.

Definition 2.36.(Compare with [MR04, Definition 3.1.6]) We define the module of $\mathbb{L}$-*restricted* Λ-*adic Kolyvagin systems* to be

$$\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}},\mathcal{P}):=\varprojlim_{s,m}\varinjlim_{j}\mathbf{KS}(T_{s,m},\mathcal{F}_{\mathbb{L}},\mathcal{P}_j),$$

where $\mathbf{KS}(T_{s,m},\mathcal{F}_{\mathbb{L}},\mathcal{P}_j)$ is the module of Kolyvagin systems for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on the representation $T_{s,m}$.

Theorem 2.37. *Suppose* $\mathbb{H}.\mathbf{T}_{/k}$ *and* $\mathbb{H}.\mathbf{sEZ}_{/k}$ *hold true. Then* Λ*-module* $\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}},\mathcal{P})$ *is free of rank one, and the canonical map*

$$\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}},\mathcal{P})\longrightarrow\overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$$

is surjective.

Note that, thanks to our assumption (A1), both $\mathbb{H}.\mathbf{T}_{/k}$ and $\mathbb{H}.\mathbf{sEZ}_{/k}$ are true for the particular Galois representation T we are interested in. Any generator of the cyclic Λ-module $\overline{\mathbf{KS}}(T\otimes\Lambda,\mathcal{F}_{\mathbb{L}},\mathcal{P})$ will be called a *primitive* Λ-adic Kolyvagin system.

Proof. The proof of this theorem is very similar to the proof of [Büy09b, Theorem 2.19] and we refer the reader to loc.cit. for details. We only remark here that the proof follows from an appropriate variant of [Büy07, Theorem 3.23]. Note that Theorem 3.23 of loc.cit. applies (with the base field $\mathbb{Q}$ replaced by k, and the Selmer structure $\mathcal{F}_{\text{can}}$ replaced by $\mathcal{F}_{\mathbb{L}}$) thanks to Proposition 2.26 and the truth of the hypotheses $\mathbf{H.1}$-$\mathbf{H.4}$, $\mathbb{H}.\mathbf{T}_{/k}$, $\mathbb{H}.\mathbf{sEZ}_{/k}$.

□

In §4.2 below, we explain how to obtain these Kolyvagin systems out of the Stickelberger elements, assuming a weak version of *Brumer's conjecture.* Note, however, that the existence of Λ-adic Kolyvagin systems does not rely on Brumer's conjecture.

We record here the main application of a Λ-adic Kolyvagin system $\boldsymbol{\kappa}=\left\{\{\kappa_\tau(s,m)\}_{\tau\in\mathcal{N}_{s+m}}\right\}_{s,m}$. For an explanation of our notation, see [MR04, §3]. Here we only note that $\kappa_\tau(s,m)\in H^1(k,T_{s,m})$, and by definition, there is a well defined element

$$\kappa_1=\{\kappa_1(s,m)\}_{s,m}\in\varprojlim_{s,m}H^1_{\mathcal{F}_{\mathbb{L}}}(k,T_{s,m})=H^1_{\mathcal{F}_{\mathbb{L}}}(k,T\otimes\Lambda).$$

For notational simplicity, we write $\mathbb{T}=T\otimes\Lambda$. Recall that $\mathrm{char}(\mathbb{A})$ denotes the characteristic ideal of a finitely generated Λ-module $\mathbb{A}$, with the convention that $\mathrm{char}(\mathbb{A})=0$ unless $\mathbb{A}$ is Λ-torsion.

Theorem 2.38. *Under the assumptions* (A1)-(A2)*,*

$$\mathrm{char}(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)^\vee)\mid\mathrm{char}(H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})/\Lambda\cdot\kappa_1).$$

Proof. This is [MR04, Theorem 5.3.10(iii)] applied in our setting. We remark that all the hypotheses of Theorem 5.3.10(iii) of loc.cit. hold thanks to (A1)-(A2) (as we have already demonstrated above). □

3. Euler systems from Stickelberger elements

We begin with recalling the definition of Stickelberger elements. We first set our notation. Assume $k, \chi, \mathfrak{f} = \mathfrak{f}_\chi$ and L are as above. For a (square free) cycle $\tau = \mathfrak{q}_1 \dots \mathfrak{q}_m$ of the number field k, let $k(\tau)$ be the compositum

$$k(\tau) = k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_m),$$

where $k(\mathfrak{q})$ denotes the maximal p-extension inside the ray class field of k modulo the prime ideal $\mathfrak{q}$. For any field K, define $K(\tau)$ as the composite of $k(\tau)$ and K. Let

$$\mathcal{K} = \{L_n(\tau) : \tau \in \mathcal{N};\ n \geq 0\},$$
$$\mathcal{K}_0 = \{k_n(\tau) : \tau \in \mathcal{N};\ n \geq 0\}$$

be two collection of abelian extensions of k. Note that any field $L_n(\tau) \in \mathcal{K}$ is CM and abelian over the totally real field k. Let S be the set of places of k, consisting of all places above p, all places dividing $\mathfrak{f}$ and all infinite places. For any $K \in \mathcal{K}$, write S_K for the set of all places of the field K lying above the places in S. When there is no confusion, we will simply write S for S_K.

For any $K \in \mathcal{K}$, the partial zeta function for $\sigma \in \mathrm{Gal}(K/k)$ is defined as usual by

$$\zeta_S(s, \sigma) := \sum_{\substack{(\mathfrak{a}, K/k) = \sigma \\ \mathfrak{a} \text{ is prime to } S}} \mathbf{N}\mathfrak{a}^{-s}$$

for $\mathrm{Re}(s) > 1$. Here $\mathbf{N}\mathfrak{a}$ is the absolute norm of the ideal $\mathfrak{a} \in k$, and $(\mathfrak{a}, K/k)$ is the Artin symbol. The partial zeta functions admit a meromorphic continuation to the whole complex plane, and holomorphic everywhere except at $s = 1$. We may therefore set

$$\theta_K = \theta_{K,S} := \sum_{\sigma \in \mathrm{Gal}(K/k)} \zeta_S(0, \sigma)\sigma^{-1} \in \mathbb{C}[\mathrm{Gal}(K/k)].$$

Thanks to [Sie70], θ_K is an element of $\mathbb{Q}[\mathrm{Gal}(K/k)]$. Further, we know for the χ-part θ_K^χ of θ_K, thanks to [DR80], that $\theta_K^\chi \in \mathbb{Z}_p[\mathrm{Gal}(K/k)]^\chi$, since we assumed $\chi \neq \omega$.

Lemma 3.1. *For any $L_n(\tau) = K \subset K' = L_{n'}(\tau')$ inside $\mathcal{K}$,*

$$\theta_{K'}|_K = \prod_{\mathfrak{q} | \tau', \mathfrak{q} \nmid \tau} (1 - \mathrm{Fr}_\mathfrak{q}^{-1})\theta_K.$$

Proof. This follows from [Tat84, Proposition IV.1.8]. □

As before, let A_K denote the p-part of the ideal class group of $K \in \mathcal{K}$, and A_K^χ its χ-isotypic part. Until the end of this section we suppose that the χ-part of the Brumer's conjecture (Assumption 1.1) holds true.

Remark 3.2. Greither [Gre04, Corollary 2.7] and Kurihara [Kur03, Corollary 2.4] have deduced Assumption 1.1 from Iwasawa's main conjecture in this setting (which holds thanks to [Wil90b]) and the vanishing of the Iwasawa μ-invariant for K. However, we wish not to assume the truth of the main conjecture; in fact we rather assume in this paper Assumption 1.1 and deduce the main conjecture itself.

Having referred the reader to [Kur03], we warn about one minor point: If a prime $\wp \subset k$ above p is unramified in K/k, then Kurihara's Stickelberger element $\tilde{\theta}_K^\chi$ differs from our θ_K^χ by a factor of $(1 - \mathrm{Fr}_\wp)^\chi$, where $\mathrm{Fr}_\wp$ is the Frobenius at $\wp$ for the unramified extension K/k. If (A1) holds, it follows that $(1 - \mathrm{Fr}_\wp)^\chi$ is a unit inside $\mathbb{Z}_p[\mathrm{Gal}(K/k)]^\chi$. Therefore, the statement

of Assumption 1.1 is still equivalent to the statement $\tilde{\theta}_K^{\chi} \cdot A_K^{\chi} = 0$, which is the assertion deduced from the main conjecture in [Kur03].

Suppose F is any finite abelian extension of k, and $K = FL$. Then by the inflation-restriction sequence and class field theory one has

$$H^1(F, \mathbb{Z}_p(\chi)) \cong H^1(K, \mathbb{Z}_p)^{\chi^{-1}} = \mathrm{Hom}(\mathbb{A}_K^{\times}/K^{\times}, \mathbb{Z}_p)^{\chi^{-1}}, \tag{3.1}$$

where $\mathbb{A}_K^{\times}$ denotes the ideles of K. Since any continuous homomorphism of $\mathbb{A}_K^{\times}$ into $\mathbb{Z}_p$ should vanish on

$$B_K := \prod_{w|\infty} K_w^{\times} \times \prod_{w|p} \{1\} \times \prod_{w\nmid p\infty} \mathcal{O}_{K_w}^{\times} \subset \mathbb{A}_K^{\times},$$

Equation (3.1) gives

$$H^1(F, \mathbb{Z}_p(\chi)) \cong \mathrm{Hom}(\mathbb{A}_K^{\times}/K^{\times}B_K, \mathbb{Z}_p)^{\chi^{-1}} = \mathrm{Hom}(\left(\mathbb{A}_K^{\times}/K^{\times}B_K\right)^{\chi}, \mathbb{Z}_p). \tag{3.2}$$

Further, there is an exact sequence

$$0 \longrightarrow U_K/\overline{\mathcal{O}_K^{\times}} \longrightarrow \mathbb{A}_K^{\times}/K^{\times}B_K \longrightarrow A_K \longrightarrow 0,$$

which is induced from the map that sends an idèle to the corresponding ideal class. Here $\overline{\mathcal{O}_K^{\times}}$ is the closure of the global units $\mathcal{O}_K^{\times}$ inside the local units $U_K \subset K \otimes \mathbb{Q}_p$. Since taking χ-parts is exact (as the order of χ is prime to p), we obtain an exact sequence

$$0 \longrightarrow U_K^{\chi}/(\overline{\mathcal{O}_K^{\times}})^{\chi} \longrightarrow (\mathbb{A}_K^{\times}/K^{\times}B_K)^{\chi} \longrightarrow A_K^{\chi} \longrightarrow 0.$$

Thus, by Assumption 1.1, multiplication by θ_K^{χ} gives a map

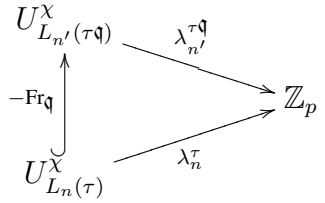
$$(\mathbb{A}_K^{\times}/K^{\times}B_K)^{\chi} \xrightarrow{\theta_K^{\chi}} U_K^{\chi}/(\overline{\mathcal{O}_K^{\times}})^{\chi}.$$

Since we assumed χ is totally odd, $(\overline{\mathcal{O}_K^{\times}})^{\chi}$ is finite (see the final paragraph of the proof of Proposition 2.13), and we therefore have an induced map

$$(\mathbb{A}_K^{\times}/K^{\times}B_K)^{\chi} \xrightarrow{\theta_K^{\chi}} U_K^{\chi}/(U_K^{\chi})_{\mathrm{tors}}. \tag{3.3}$$

Suppose we are given a collection of homomorphisms $\boldsymbol{\lambda} = \{\lambda_n^{\tau}\}$ with $\lambda_n^{\tau} \in \mathrm{Hom}(U_{L_n(\tau)}^{\chi}, \mathbb{Z}_p)$ which satisfies the following properties:

(1) For all $L_n(\tau), L_{n'}(\tau\mathfrak{q}) \in \mathcal{K}$, the following diagram commutes:



(2) For $n' \geq n$, we have $\lambda_{n'}^{\tau}|_{U^{\chi}_{L_n(\tau)}} = \lambda_n^{\tau}$.

Define

$$\tilde{c}_{k_n(\tau)} \in \mathrm{Hom}\left(\left(\mathbb{A}_{L_n(\tau)}^{\times}/(L_n(\tau))^{\times}B_{L_n(\tau)}\right)^{\chi}, \mathbb{Z}_p\right)$$

(which we view also as an element of $H^1(k_n(\tau), T)$ via the identification (3.2) above) as the composition[3]

$$\tilde{c}_{k_n(\tau)} : \left(\mathbb{A}^{\times}_{L_n(\tau)}/(L_n(\tau))^{\times} B_{L_n(\tau)}\right)^{\chi} \xrightarrow{\theta^{\chi}_{L_n(\tau)}} U^{\chi}_{L_n(\tau)}/(U_{L_n(\tau)})^{\chi}_{\text{tors}} \xrightarrow{\lambda^{\tau}_n} \mathbb{Z}_p. \tag{3.4}$$

Set $\tilde{\mathbf{c}} = \{\tilde{c}_{k_n(\tau)}\}$.

Theorem 3.3. *Fix a collection of homomorphisms $\boldsymbol{\lambda} = \{\lambda^{\tau}_n\}$ as above (if it exists). Then there is an Euler system $\mathbf{c} = \{c_{k_n(\tau)}\}$ (which depends on the choice of $\boldsymbol{\lambda}$) for the Galois representation T (in the sense of* [Rub00, Definition II.1.1 and Remark II.1.4]*) such that $c_{k_n} = \tilde{c}_{k_n}$ for all n.*

In §4.1 below, we construct a collection $\boldsymbol{\lambda}$ which satisfies the desired properties, and hence conclude with the existence of an Euler system for T, assuming the truth of Assumption 1.1.

Proof. Since the proof of this theorem very closely follows the proof of [Rub00, Proposition III.3.4], we only give a sketch. All the references in this proof are to [Rub00]. First, one checks (mimicking the proof of Proposition III.3.4) that the collection $\tilde{\mathbf{c}}$ (which should be compared with the collection $\tilde{\mathbf{c}}'$ of Rubin) satisfies a distribution relation with *wrong* Euler factors. This could be remedied, as in the paragraph following Remark III.4.4, using Lemma IX.6.1 to obtain a new collection $\mathbf{c}$ (which corresponds to what Rubin calls $\tilde{\mathbf{c}}$) as desired. □

We close this section with a final remark which we will refer to in what follows:

Remark 3.4. The argument of Remark 2.16 shows that, under the hypothesis (A1),

$$H^1(k_n(\tau)_p, T) \cong \text{Hom}(U^{\chi}_{L_n(\tau)}, \mathbb{Z}_p).$$

Remark 3.5. In this remark we discuss the main differences between the cases when the base field k is a general totally real field (i.e., the case we study in this article) and the particular case $k = \mathbb{Q}$ (i.e., the case Rubin studies in [Rub00, §III.4]).

(i) The first difference is in regard of the core Selmer ranks: The core rank $\mathcal{X}(T, \mathcal{F}_{\text{can}})$ of the canonical Selmer structure (c.f., Example 2.6 above) on the G_k-representation T is $[k : \mathbb{Q}] = r$. Rubin treats the case $r = 1$ (i.e., the case $k = \mathbb{Q}$). In this paper, we study the case $r > 1$, adapting the work of Mazur and Rubin [MR04] to the general case when $\mathcal{X}(T, \mathcal{F}_{\text{can}}) > 1$ via what we call $\boldsymbol{\mathcal{L}}$-restricted Euler systems. Although Kurihara [Kur03] successfully applies the classical Euler system argument to Stickelberger elements to prove Theorem A and Theorem B above, our approach via $\boldsymbol{\mathcal{L}}$-restricted Euler systems yields in addition a comparison between the Stickelberger elements and Rubin-Stark elements (see Theorem 5.16 below). Furthermore, our approach here fits well into the framework developed in [MR04] which was later enhanced by the author in [Büy09a, Büy09b, Büy08].

(ii) The second difference is the manner the collection $\boldsymbol{\lambda} = \{\lambda^{\tau}_n\}$ of homomorphisms is chosen. In [Rub00, Appendix D], Rubin constructs these homomorphisms explicitly when $k = \mathbb{Q}$. This construction is not available when $k \neq \mathbb{Q}$; that is why we prove "abstractly" in §4.1 that a collection $\boldsymbol{\lambda}$ exists with the desired properties.

[3]We remark that any homomorphism $\lambda \in \text{Hom}(U^{\chi}_{L_n(\tau)}, \mathbb{Z}_p)$ necessarily factors through the quotient $U^{\chi}_{L_n(\tau)}/(U_{L_n(\tau)})^{\chi}_{\text{tors}}$; this is how we make sense of the right most map in (3.4).

4. EULER SYSTEMS TO KOLYVAGIN SYSTEMS MAP

We first recall what Mazur and Rubin call the *Euler system to Kolyvagin system map*. Suppose T, $\mathcal{K}$ and $\mathcal{P}$ are as above. Let $\mathbf{ES}(T) = \mathbf{ES}(T, \mathcal{K})$ denote the collection of Euler systems for $(T, \mathcal{K})$ in the sense of [Rub00, §3]. Recall also the generalized module of Kolyvagin systems $\overline{\mathbf{KS}}(T, \mathcal{F}, \mathcal{P})$ and $\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}, \mathcal{P})$ for various choice of Selmer structures $\mathcal{F}$.

Theorem 4.1 (Mazur-Rubin)**.** *There are canonical maps*

- $\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$,
- $\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\Lambda}, \mathcal{P})$

with the following properties:

(1) *If* $\mathbf{c} \in \mathbf{ES}(T)$ *maps to* $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$*, then* $\kappa_1 = c_k$*,*
(2) *If* $\mathbf{c} \in \mathbf{ES}(T)$ *maps to* $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\Lambda}, \mathcal{P})$*, then*

$$\kappa_1 = \{c_{k_n}\} \in \varprojlim_n H^1(k_n, T) = H^1(k, T \otimes \Lambda).$$

Remark 4.2. Mazur and Rubin's definition of the generalized module of Λ-adic Kolyvagin systems $\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\Lambda}, \mathcal{P})$ slightly differs from our definition of this module (Definition 2.36). It is not hard to see that these two definitions give rise to isomorphic modules; see also Remark 4.13 below.

We would like to apply this map on the Euler systems we have constructed[4] in §3. Note however that Theorem 4.1 will give rise to Kolyvagin systems only for the coarser Selmer structures $\mathcal{F}_{\Lambda}$ and $\mathcal{F}_{\text{can}}$ (rather than finer Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$). To be able to obtain Kolyvagin systems for the modified Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$, we need to analyze the structure of the semi-local cohomology groups for $T \otimes \Lambda$ and T, over various ray class fields of k. This is carried out in §4.1. We then apply the results of §4.1 to construct the desired Kolyvagin systems for the modified Selmer structures in §4.2.

Remark 4.3. In effect, one only needs a *weak Kolyvagin system* (in the sense of [MR04, Definition 3.1.8]) for the main application of the Euler system/Kolyvagin system machinery, i.e., for bounding the dual Selmer group. Weak Kolyvagin systems are essentially the derivative classes of Kolyvagin (cf. [Rub00, §IV]) which are obtained directly applying the derivative operators, without the need of the alterations carried out in [MR04, Appendix A].

4.1. A good choice of homomorphisms.

Recall that k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k, and $\Gamma = \text{Gal}(k_\infty/k)$. Let k_n denote the unique sub-extension of k_∞/k with $[k_n : k] = p^n$ and set $\Gamma_n := \text{Gal}(k_n/k)$. Recall also that $\Delta_\tau := \text{Gal}(k(\tau)/k)$.

Lemma 4.4. *For every* $n \in \mathbb{Z}_{\geq 0}$ *and* $\tau \in \mathcal{N}(\mathcal{P})$*, the following corestriction maps on the semi-local cohomology*

(i) $H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T)$,

(ii) $H^1(k(\tau)_p, T) \longrightarrow H^1(k_p, T)$,

(iii) $H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_p, T)$

are surjective.

[4]Modulo the existence of a family of homomorphisms $\boldsymbol{\lambda}$.

Proof. The cokernel of the map

$$H^1(k(\tau), T\otimes\Lambda) = \varprojlim_n H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T)$$

is given by $H^2(k(\tau)_p, T\otimes\Lambda)[\gamma-1]$, where γ is any topological generator of $\Gamma = \mathrm{Gal}(k_\infty/k)$. Since it is known that (cf. [PR94]) $H^2(k(\tau)_p, T\otimes\Lambda)$ is a finitely generated $\mathbb{Z}_p$-module, it follows that

$$H^2(k(\tau)_p, T\otimes\Lambda)[\gamma-1]=0 \iff H^2(k(\tau)_p, T\otimes\Lambda)/(\gamma-1)=0.$$

Since the cohomological dimension of the absolute Galois group of any local field is 2,

$$H^2(k(\tau)_p, T\otimes\Lambda)/(\gamma-1) \cong H^2(k(\tau)_p, T\otimes\Lambda/(\gamma-1)) = H^2(k(\tau)_p, T).$$

It therefore suffices to check that

$$H^2(k(\tau)_p, T) := \bigoplus_{v|p} H^2(k(\tau)_v, T) = 0,$$

which, via local duality is equivalent to checking that $(T^*)^{G_{k(\tau)_v}}=0$ for each $v|p$.

Write $\mathcal{D}_v$ for the decomposition group at $v\mid p$ inside $\mathrm{Gal}(k(\tau)/k) := \Delta_\tau$. We may identify $\mathcal{D}_v\subset\Delta_\tau$ by the local Galois group $\mathrm{Gal}(k(\tau)_v/k_\wp)$ where $\wp\subset k$ is the prime below v. Since Δ_τ is generated by inertia groups at the primes dividing τ, all of which act trivially on T^* (by the choice of τ's). Hence, it follows that

$$(T^*)^{G_{k(\tau)_v}} = (T^*)^{G_{k_\wp}}.$$

Note that $T^* = \boldsymbol{\mu}_{p^\infty}\otimes\chi^{-1}$, so it follows at once that$(T^*)^{G_{k_\wp}}=0$, and thus (i) is proved.

Set $T_\tau := \mathrm{Ind}_{k(\tau)}^k T$. The semi-local version of Shapiro's lemma (which is explained in [Rub00, §A.5]) shows that

$$H^1(k(\tau)_p, T) \cong H^1(k_p, T_\tau).$$

The corestriction map

$$\mathbf{N}_\tau:\ H^1(k_p, T_\tau)\cong H^1(k(\tau)_p, T)\longrightarrow H^1(k_p, T)$$

is simply induced from the augmentation sequence

$$0\longrightarrow \mathcal{A}_\tau\cdot T_\tau\longrightarrow T_\tau\longrightarrow T\longrightarrow 0,$$

where $\mathcal{A}_\tau$ is the augmentation ideal of the local ring $\mathbb{Z}_p[\Delta_\tau]$. The argument above shows that the cokernel of $\mathbf{N}_\tau$ is dual to $H^0(k_p, (\mathcal{A}_\tau\cdot T_\tau)^*)$. Furthermore,

$$(\mathcal{A}_\tau\cdot T_\tau)^* := \mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau, \boldsymbol{\mu}_{p^\infty}) = \mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau, \mathbb{Q}_p/\mathbb{Z}_p)\otimes\mathbb{Z}_p(1),$$

and $\mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau, \mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{A}_\tau\cdot\mathrm{Hom}(T_\tau, \mathbb{Q}_p/\mathbb{Z}_p)$, we thence see that

$$H^0(k_p, (\mathcal{A}_\tau\cdot T_\tau)^*)\hookrightarrow H^0(k_p, T_\tau^*).$$

It therefore suffices to show that $H^0(k_p, T_\tau^*)=0$. By local duality, this is equivalent to proving $H^2(k_p, T_\tau)=0$, which, by the semi-local version of Shapiro's lemma, is equivalent to checking $H^2(k(\tau)_p, T)=0$. This final statement is equivalent to the assertion that $H^0(k(\tau)_p, T^*)=0$ by local duality. This, however, has been already verified in the third paragraph of this proof. This completes the proof of (ii).

(iii) clearly follows from (i) and (ii). □

Proposition 4.5. *For every $\tau\in\mathcal{N}(\mathcal{P})$,*

(i) *the semi-local cohomology group $H^1(k(\tau)_p, T)$ is a free $\mathbb{Z}_p[\Delta_\tau]$-module of rank r,*
(ii) *for every $n\in\mathbb{Z}_{\geq 0}$, the $\mathbb{Z}_p[\Gamma_n\times\Delta_\tau]$-module $H^1(k_n(\tau)_p, T)$ is free of rank r.*

Proof. We start with the remark that $H^1(k(\tau)_p, T)$ is a free $\mathbb{Z}_p$-module of rank $r \cdot |\Delta_\tau|$. Indeed, this may be proved by the argument of Lemma 2.15 (or alternatively, and more directly, following the argument of Remark 2.16). Further, we know thanks to Lemma 4.4 that the map

$$H^1(k(\tau)_p, T) \longrightarrow H^1(k_p, T)$$

(which could be thought of as the reduction modulo the augmentation ideal $\mathcal{A}_\tau$) is surjective. Nakayama's lemma and Lemma 2.15 therefore imply that $H^1(k(\tau)_p, T)$ is generated by (at most) r elements over $\mathbb{Z}_p[\Delta_\tau]$. Let $\mathfrak{B} = \{x_1, x_2, \ldots, x_r\}$ be any set of such generators. To prove (i), it suffices to check that the x_i's do not admit any $\mathbb{Z}_p[\Delta_\tau]$-linear relation. Assume contrary, and suppose there is a relation

$$\sum_{i=1}^{r} \alpha_i x_i = 0, \quad \alpha_i \in \mathbb{Z}_p[\Delta_\tau]. \tag{4.1}$$

Define

$$S = \{\delta x_j : \delta \in \Delta_\tau, 1 \leq j \leq r\},$$

and note that S generates (as a $\mathbb{Z}_p$-module) $H^1(k(\tau)_p, T)$ by our assumption on $\mathfrak{B}$, and also that $|S| = r \cdot |\Delta_\tau| = \text{rank}_{\mathbb{Z}_p}(H^1(k(\tau)_p, T))$. Equation (4.1) can be rewritten as

$$\sum_{\delta, j} a_{\delta,j} \cdot \delta x_j = 0$$

with $a_{\delta,j} \in \mathbb{Z}_p$. Since we already know that $H^1(k(\tau)_p, T)$ is $\mathbb{Z}_p$-torsion free, we may assume without loss of generality that $a_{\delta_0, j_0} \in \mathbb{Z}_p^\times$ for some δ_0, j_0. This in return implies that

$$\delta_0 x_{j_0} \in \text{span}_{\mathbb{Z}_p}(S - \{\delta_0 x_{j_0}\}),$$

hence $H^1(k(\tau)_p, T)$ is generated by $S - \{\delta_0 x_{j_0}\}$. This, however, is a contradiction since we already know that the $\mathbb{Z}_p$-rank of $H^1(k(\tau)_p, T)$ is $r \cdot |\Delta_\tau| = |S|$, hence it cannot be generated by $|S| - 1$ elements over $\mathbb{Z}_p$. The proof of (i) now follows.

One proves (ii) in an identical fashion, now considering the *augmentation map*

$$H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T),$$

which is surjective thanks to Lemma 4.4. □

Define the field $\mathfrak{F}$ as the compositum of the fields $k(\tau)$,

$$\mathfrak{F} = \bigcup_{\tau \in \mathcal{N}(\mathcal{P})} k(\tau),$$

as τ runs through the set $\mathcal{N}$. We set $\mathbf{\Delta} := \text{Gal}(\mathfrak{F}/k)$.

Corollary 4.6. *The $\mathbb{Z}_p[[\Gamma \times \mathbf{\Delta}]]$-module $\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$ is free of rank r, and the natural projection*

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_m(\eta)_p, T)$$

is surjective for every $m \in \mathbb{Z}_{\geq 0}$ and $\eta \in \mathcal{N}$.

Proof. Immediate after Proposition 4.5. □

Definition 4.7. Fix a $\mathbb{Z}_p[[\Gamma\times\Delta]]$-rank-one direct summand $\boldsymbol{\mathcal{L}}$ of $\varprojlim_{n,\tau} H^1(k_n(\tau)_p,T)$. Denote its image under the (surjective) map

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p,T)\longrightarrow H^1(k_m(\eta)_p,T)$$

by $\mathcal{L}_m^\eta$. When $\eta=1$, we simply write $\mathcal{L}_m$ instead of $\mathcal{L}_m^1$; and when $m=0$ we write $\mathcal{L}$ for $\mathcal{L}_0$. Finally, let $\mathbb{L}$ denote the image of $\boldsymbol{\mathcal{L}}$ under the projection

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p,T)\longrightarrow \varprojlim_{n} H^1((k_n)_p,T)=H^1(k_p,T\otimes\Lambda).$$

We fix generators $\boldsymbol{\varphi},\varphi_m^\eta,\varphi_m,\varphi$ and Φ of $\boldsymbol{\mathcal{L}},\mathcal{L}_m^\eta,\mathcal{L}_m,\mathcal{L}$ and $\mathbb{L}$, respectively; such that

$$\boldsymbol{\varphi}\mapsto\varphi_m^\eta\mapsto\varphi_m,\ \text{and}$$
$$\boldsymbol{\varphi}\mapsto\Phi\mapsto\varphi$$

under the projection maps we mentioned above.

As in Definition 4.7, we could start with a choice of $\boldsymbol{\mathcal{L}}$, which in return fixes $\mathbb{L}$ and $\mathcal{L}$. Alternatively, we could start with an arbitrary $\mathcal{L}$ (and $\mathbb{L}$) as we did in §2.3 and show (using linear algebra) that there is a rank one direct summand $\boldsymbol{\mathcal{L}}\subset\varprojlim_{n,\tau} H^1(k_n(\tau)_p,T)$ which projects down to $\mathcal{L}$ (and $\mathbb{L}$), as in Definition 4.7.

Remark 4.8. In Definition 2.27 (resp., Definition 2.36) above, we give a definition of a $\mathcal{L}$-restricted Kolyvagin system (resp., $\mathbb{L}$-restricted Λ-adic Kolyvagin system). When $\boldsymbol{\mathcal{L}}$ is above, so that the line $\boldsymbol{\mathcal{L}}$ projects down to $\mathcal{L}$ (and respectively, down to $\mathbb{L}$), we also call these $\boldsymbol{\mathcal{L}}$-restricted Kolyvagin systems.

By Remark 3.4, we may identify $\varprojlim_{n,\tau} H^1(k_n(\tau)_p,T)$ by the module $\varprojlim_{n,\tau}\mathrm{Hom}\left(U^\chi_{L_n(\tau)},\mathbb{Z}_p\right)$, where we recall that $U_{L_n(\tau)}$ stands for the local units inside $L_n(\tau)\otimes\mathbb{Q}_p$. We define, for each $m\geq 0$ and $\eta\in\mathcal{N}$, a homomorphism $\lambda_m^\eta\in\mathrm{Hom}\left(U^\chi_{L_m(\eta)},\mathbb{Z}_p\right)$ as the composite

$$\lambda_m^\eta:=\varphi_m^\eta\circ\prod_{\mathfrak{q}|\eta}(-\mathrm{Fr}_\mathfrak{q}).$$

We further set $\beth_m^\eta$ for the (free of rank one) $\mathbb{Z}_p[\Gamma_m\times\Delta_\eta]$-module generated by λ_m^η. Clearly,

$$\beth_m^\eta=\prod_{\mathfrak{q}|\eta}(-\mathrm{Fr}_\mathfrak{q}^{-1})\mathcal{L}_m^\eta=\mathcal{L}_m^\eta, \tag{4.2}$$

where the final equality is because $\mathcal{L}_m^\eta$ is a $\mathbb{Z}_p[\Gamma_m\times\Delta_\eta]$-stable submodule of

$$H^1(k_m(\eta),T)\cong\mathrm{Hom}\left(U^\chi_{L_m(\eta)},\mathbb{Z}_p\right). \tag{4.3}$$

When η is fixed and m varies, note that the collection $\{\lambda_m^\eta\}_m$ forms a projective system with respect to norm maps[5]. When $\eta=1$, we write λ_m (resp., $\beth_m$) instead of λ_m^η (resp., $\beth_m^\eta$). Also when $m=0$, we simply write λ (resp., $\beth$) for λ_0 (resp., $\beth_0$).

We finally remark that $\lambda_m=\varphi_m$ for all m, by definition.

Proposition 4.9. *For $\eta,\eta\mathfrak{q}\in\mathcal{N}$, and any $m'\geq m$,*

(i) $\lambda_{m'}^{\eta\mathfrak{q}}\big|_{U^\chi_{L_m(\eta)}}=\lambda_m^\eta\circ(-\mathrm{Fr}_\mathfrak{q})$,

[5]Under the identification (4.3) above, the norm maps on the cohomology are induced from the inclusions $U^\chi_{L_m(\eta)}\hookrightarrow U^\chi_{L_{m'}(\eta)}$, $m'\geq m$.

(ii) $\lambda^{\eta}_{m'}\big|_{U^{\chi}_{Lm(\eta)}} = \lambda^{\eta}_{m}$.

Proof. This is evident, since by construction

$$\begin{aligned}\lambda^{\eta\mathfrak{q}}_{m'}\big|_{U^{\chi}_{Lm(\eta)}} &= \varphi^{\eta\mathfrak{q}}_{m'} \circ \prod_{\varpi|\eta}(-\mathrm{Fr}_{\varpi})(-\mathrm{Fr}_{\mathfrak{q}})\big|_{U^{\chi}_{Lm(\eta)}} \\ &= \varphi^{\eta}_{m} \circ \prod_{\varpi|\eta}(-\mathrm{Fr}_{\varpi})(-\mathrm{Fr}_{\mathfrak{q}})\big|_{U^{\chi}_{Lm(\eta)}} \\ &= \lambda^{\eta}_{m} \circ (-\mathrm{Fr}_{\mathfrak{q}})\big|_{U^{\chi}_{Lm(\eta)}},\end{aligned}$$

where the second equality is because

$$\varphi^{\eta\mathfrak{q}}_{m'}\big|_{U^{\chi}_{Lm(\eta)}} = \varphi^{\eta}_{m}\big|_{U^{\chi}_{Lm(\eta)}}$$

by the norm coherence property of the collection $\{\varphi^{\eta}_{m}\}_{\eta,m}$. This completes (i), and (ii) is proved similarly.

□

Let $\mathbf{c}^{\mathrm{St}} = \left\{c^{\mathrm{St}}_{k_n(\tau)}\right\} \in \mathbf{ES}(T)$ be the Euler system constructed via Theorem 3.3 using the Stickelberger elements and the collection $\{\lambda^{\eta}_{m}\}$ we defined above. In the next section, we will use $\mathbf{c}^{\mathrm{St}}$ to construct a Kolyvagin system for the Selmer triple $(T, \mathcal{F}_{\mathcal{L}}, P)$ (resp., for the triple $(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$).

Remark/Definition 4.10. Let M be any G_k-representation which is free of finite rank as a $\mathbb{Z}_p$-module and which is unramified outside a finite set of places of k. Let $\mathcal{K}_M$ be a large abelian extension of k defined as in [Rub00, Definition 1.1]. Suppose $\mathfrak{S} \subset \varprojlim_{K \subset \mathcal{K}_M} H^1(K_p, M)$ is any submodule. Let $\mathfrak{S}_K \subset H^1(K_p, M)$ denote the image of $\mathfrak{S}$ under the obvious projection map. We say that an Euler system

$$\{c_K\}_{K \subset \mathcal{K}_M} = \mathbf{c} \in \mathbf{ES}(M, \mathcal{K}_M) = \mathbf{ES}(M)$$

is $\mathfrak{S}$-*restricted* if $\mathrm{loc}_p(c_K) \in \mathfrak{S}_K$ for any finite extension $K \subset \mathcal{K}_M$ of k. The collection of $\mathfrak{S}$-restricted Euler systems for the pair $(M, \mathcal{K}_M)$ will be denoted by $\mathbf{ES}_{\mathfrak{S}}(M) = \mathbf{ES}_{\mathfrak{S}}(M, \mathcal{K}_M)$.

Example 4.11.

(1) Let $\boldsymbol{\mathcal{L}} \subset \varprojlim_{n,\tau} H^1(k_n(\tau), T)$ be as in Definition 4.7. It is easy to see that the Euler system $\mathbf{c}^{\mathrm{St}}$ we construct above is an $\boldsymbol{\mathcal{L}}$-restricted Euler system.

(2) Consider the even, non-trivial character $\omega\chi^{-1} := \psi$ of G_k, and set $T' := \mathbb{Z}_p(1) \otimes \psi^{-1}$. Only in this example, we let $\mathbb{L}$ denote a fixed $\mathbb{Z}_p[[\Gamma \times \boldsymbol{\Delta}]]$-rank-one direct summand of $\varprojlim_{n,\tau} H^1(k_n(\tau), T')$. The author [Büy09b, §3] has constructed an $\mathbb{L}$-restricted Euler system for the pair $(T', \mathcal{K})$, starting from the conjectural Rubin-Stark elements.

Later in §5.3, we will construct another $\mathbb{L}$-restricted Euler system for $(T', \mathcal{K})$, applying a formal twisting argument on the Euler system $\mathbf{c}^{\mathrm{St}}$. We will also compare this Euler system with the one coming from the Rubin-Stark elements, using the "rigidity" of the collection of Λ-adic Kolyvagin systems.

4.2. **Kolyvagin systems for modified Selmer groups (bis).** Recall the sets $\mathcal{P}_j \subset \mathcal{P}$ and $\mathcal{N}_j \subset \mathcal{N}$. For notational simplicity, we write $\mathbb{T} := T \otimes \Lambda$ from now on, and for a fixed topological generator $\gamma \in \Gamma$, we set $\gamma_n = \gamma^{p^n}$. Finally, let $\mathcal{M}$ be the maximal ideal of the ring Λ.

Definition 4.12. For $\mathbb{F} = \mathcal{F}_\Lambda$ or $\mathcal{F}_\mathbb{L}$, we set

$$\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P}) := \varprojlim_{m,n} \varinjlim_{j} \mathbf{KS}(\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}, \mathbb{F}, \mathcal{P}_j),$$

where $\mathbf{KS}(\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}, \mathbb{F}, \mathcal{P}_j)$ is the $\Lambda/(p^m, \gamma_n - 1)$-module of Kolyvagin systems (in the sense of [MR04, Definition 3.1.3]) for the propagated Selmer structure $\mathbb{F}$ on the quotient $\mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}$.

Remark 4.13. We introduced the module $\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P})$ above because, after applying Kolyvagin's descent procedure [Rub00, §IV], one directly obtains elements of the module $\overline{\mathbf{KS}}'(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P})$. On the other hand, it is not hard to see that the module $\overline{\mathbf{KS}}'(\mathbb{T}, \mathbb{F}, \mathcal{P})$ defined above is naturally isomorphic to the module $\overline{\mathbf{KS}}(\mathbb{T}, \mathbb{F}, \mathcal{P})$ of Definition 2.36, using the fact that each of the collections $\{p^m, \gamma_n - 1\}_{m,n}$ and $\{p^m, \mathbf{X}^n\}_{m,n}$ forms a base of neighborhoods at $zero$. Furthermore, using the fact that the collection $\{\mathcal{M}^\alpha\}_{\alpha \in \mathbb{Z}^+}$ also forms a base of neighborhoods at $zero$, one may identify these two modules Kolyvagin systems by the generalized module of Kolyvagin systems defined in [MR04, Definition 3.1.6]. By slight abuse, we will write $\overline{\mathbf{KS}}(\mathbb{T}, \mathbb{F}, \mathcal{P})$ for any of the three modules of Kolyvagin systems given by three different definitions (i.e., by Definition 2.36 and 4.12 here; and [MR04, Definition 3.1.6]). For our purposes in this section, we will use Definition 4.12 to define this module.

Write

$$\left\{ \left\{ \kappa^{\mathrm{St}}_{\tau,m} \right\}_{\tau \in \mathcal{N}_m} \right\}_m = \boldsymbol{\kappa}^{\mathrm{St}} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

(resp.,

$$\left\{ \left\{ \kappa^{\mathrm{St}_\infty}_{\tau}(m,n) \right\}_{\tau \in \mathcal{N}_{m+n}} \right\}_{m,n} = \boldsymbol{\kappa}^{\mathrm{St}_\infty} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P}))$$

for the Kolyvagin systems obtained via the descent procedure of [Rub00, §4] applied on the Euler system $\mathbf{c}^{\mathrm{St}} = \{c^{\mathrm{St}}_{k_n(\tau)}\}$. We know that

$$\begin{array}{ccc} \kappa^{\mathrm{St}}_1 & \overset{\text{def}}{=\!=} & \varprojlim_m \kappa^{\mathrm{St}}_{1,m} \in \varprojlim_m H^1(k, T/p^m T) = H^1(k, T) \\ \| & & \\ c^{\mathrm{St}}_k & \overset{\text{def}}{=} \lambda \circ \theta^\chi_L = \varphi \circ \theta^\chi_L \in \mathrm{Hom}\left(\left(\mathbb{A}^\times_L / L^\times \right)^\chi, \mathbb{Z}_p \right) = H^1(k, T) & \end{array}$$

and

$$\begin{array}{ccc} \kappa^{\mathrm{St}_\infty}_1 & \overset{\text{def}}{=\!=} & \varprojlim_{m,n} \kappa_1(m,n) \in \varprojlim_{m,n} H^1(k, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) = H^1(k, \mathbb{T}) \\ \| & & \\ \{c^{\mathrm{St}}_{k_n}\}_n & \overset{\text{def}}{=\!=} \{\lambda_n \circ \theta^\chi_{L_n}\}_n = \{\varphi_n \circ \theta^\chi_{L_n}\}_n \in \varprojlim_n H^1(k_n, T) = H^1(k, \mathbb{T}). & \end{array}$$

Remark 4.14. For every (rational) prime ℓ, Shapiro's lemma shows that

$$H^1(k(\tau), \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) \cong H^1(k_n(\tau), T/p^m T) \text{ and,} \tag{4.4}$$

$$H^1(k(\tau)_\ell, \mathbb{T}/(p^m, \gamma_n - 1)\mathbb{T}) \cong H^1(k_n(\tau)_\ell, T/p^m T). \tag{4.5}$$

See [Col98, Proposition II.1.1] for (4.4) and [Rub00, Appendix B.5] for (4.5). Thanks to these identifications, we may talk about the propagation of a local condition $H^1_{\mathcal{F}}(k_\ell,\mathbb{T})\subset H^1(k_\ell,\mathbb{T})$ at ℓ to a local condition

$$H^1_{\mathcal{F}}((k_n)_\ell,T/p^mT)\subset H^1((k_n)_\ell,T/p^mT)\cong H^1(k_\ell,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T}).$$

Namely, we define $H^1_{\mathcal{F}}((k_n)_\ell,T/p^mT)$ as the isomorphic copy of the module $H^1_{\mathcal{F}}(k_\ell,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T})$ under the isomorphism (4.5) of Shapiro's lemma.

Theorem 4.15. (i) $\boldsymbol{\kappa}^{\mathrm{St}}\in\overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$.
(ii) $\boldsymbol{\kappa}^{\mathrm{St}_\infty}\in\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}},\mathcal{P})$.

Proof. Identical to the proofs of [Büy09a, Theorem 2.19] and [Büy09b, Theorem 3.23]. We remark that the only essential point beyond [Rub00, MR04] is to verify that

$$\mathrm{loc}_p\left(\kappa^{\mathrm{St}}_{\tau,m}\right)\in H^1_{\mathcal{F}_{\mathcal{L}}}(k_p,T/p^mT)\cong\mathcal{L}/p^m\mathcal{L} \tag{4.6}$$

for each $\tau\in\mathcal{N}_m$ and $m\in\mathbb{Z}^+$; and that

$$\mathrm{loc}_p\left(\kappa^{\mathrm{St}_\infty}_{\tau}(m,n)\right)\in H^1_{\mathcal{F}_{\mathbb{L}}}(k_p,\mathbb{T}/(p^m,\gamma_n-1)\mathbb{T}) \tag{4.7}$$

$$\cong\mathbb{L}/(p^m,\gamma_n-1)\mathbb{L}:=\mathcal{L}_n/p^m\mathcal{L}_n \tag{4.8}$$

for every $\tau\in\mathcal{N}_{m+n}$ and every $m,n\in\mathbb{Z}^+$. As in [Büy09a, Büy09b], the key point in proving the assertions (4.6) and (4.7) is the fact that $\mathbf{c}^{\mathrm{St}}$ is an $\boldsymbol{\mathcal{L}}$-restricted Euler system. □

We give the main applications of our construction in Section 5. This will be twofold: The first application is somewhat standard; we will bound the dual Selmer groups. As the second application, we will relate the Stickelberger elements, making use of the first application, to the conjectural Rubin-Stark elements. Among other things, this will enable us to control the local behavior of Rubin-Stark elements.

Remark 4.16. As remarked earlier, one only needs a *weak Kolyvagin system* in order to deduce the main applications of the $\boldsymbol{\mathcal{L}}$-restricted Euler system $\mathbf{c}^{\mathrm{St}}$. See [MR04, Definition 3.1.8] for a definition of a weak Kolyvagin system. We remark that Kolyvagin's descent [Rub00, §4] applied on an Euler system gives rise to a weak Kolyvagin system. A weak Kolyvagin system can be used following the formalism of [Rub00, §5 and §7] with slight alterations, to obtain the same results which we present below.

5. Applications

Before we state our main applications of the $\boldsymbol{\mathcal{L}}$-restricted Euler system $\mathbf{c}^{\mathrm{St}}$, we recall our running hypotheses. We fix a totally odd character χ of $G_k:=\mathrm{Gal}(\overline{k}/k)$ which is *not* the Teichmüller character ω (giving the action of G_k on the p-th roots of unity $\boldsymbol{\mu}_p$). Throughout Section 5, we assume that (A1) holds. Suppose also that Assumption 1.1 is true.

5.1. Main theorem over k.

We first prove a bound on the size of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)$. We use this bound to obtain bounds on the classical (dual) Selmer groups, via the comparison Theorem established in §2.4.1.

Theorem 5.1. *Under our running hypotheses,*

(i) $\mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*))\le\mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)/\mathbb{Z}_p\cdot\kappa^{\mathrm{St}}_1)$,
(ii) $\mathrm{length}_{\mathbb{Z}_p}(H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k,T^*))\le\mathrm{length}_{\mathbb{Z}_p}(\mathcal{L}/\mathbb{Z}_p\cdot c^{\mathrm{St}}_k)$.

Proof. (i) is Theorem 2.31. (ii) follows from (i) and Corollary 2.22 applied with $c = c_k^{\mathrm{St}} = \kappa_1^{\mathrm{St}}$. □

Let $\theta_L^\chi \in \mathbb{Z}_p[\Delta]^\chi$ be as in §3. The evaluation map

$$\chi : \mathbb{Z}_p[\Delta]^\chi \longrightarrow \mathbb{Z}_p$$

induces an isomorphism and we write $\chi(\theta_L)$ for the image of θ_L^χ under this map. Recall the definition of φ and λ, which we used in §3 and §4 to define $\mathbf{c}_k^{\mathrm{St}}$. Recall also that $\lambda = \varphi$ by definition.

Theorem 5.2. *Under the assumptions above,*

$$|A_L^\chi| \leq |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

Proof. By Proposition 2.13, $H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k, T^*) \cong A_L^\chi$, and by construction, $c_k^{\mathrm{St}} = \chi(\theta_L)\lambda = \chi(\theta_L)\varphi$. Since φ is a $\mathbb{Z}_p$-generator of $\mathcal{L}$ (by definition), it follows that $\mathcal{L}/\mathbb{Z}_p \cdot c_k^{\mathrm{St}} = \mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p$. The proof now follows from Theorem 5.1(ii). □

The inequality of Theorem 5.2 may be strengthened to an equality:

Theorem 5.3. *As in Theorem 5.2 above, assume that (A1) holds and Assumption 1.1 is true. Suppose in addition*

(i) *either that $\boldsymbol{\mu}_p \not\subset L$, or*
(ii) *the statement of Theorem 5.2 is true for $\chi = \omega$.*

Then

$$|A_L^\chi| = |\mathbb{Z}_p/\chi(\theta_L)\mathbb{Z}_p|.$$

Proof. Unless $\boldsymbol{\mu}_p \subset L$ (i.e., if we are in the case (i) above), the claimed equality follows from the inequality of Theorem 5.2 using a standard argument involving the class number formula; see [Rub92, §5] and [Büy09a, §3] for details. Note that we need the assumption $\boldsymbol{\mu}_p \not\subset L$ for this portion since otherwise, we would need the inequality of Theorem 5.2 also for the Teichmüller character ω, and this escapes the methods of the current paper.

When $\boldsymbol{\mu}_p \subset L$, Theorem 5.2, used along with our assumption (ii), gives again the desired equality utilizing the class number formula. □

We note that Wiles' work [Wil90a, Theorem 3] (*only* for the case $\chi = \omega$) is exactly the statement of Theorem 5.2 in the case $\chi = \omega$. Therefore, the condition (ii) that appears in the statement of Theorem 5.3 is true if we assume Wiles' result in this particular case. *We henceforth assume* [Wil90a, Theorem 3] *(only for the case $\chi = \omega$)* if $\boldsymbol{\mu}_p \subset L$.

Remark 5.4. Note that all the Euler factors at primes $\wp | p$ are excluded in the definition of θ_K^χ, for any $K \in \mathcal{K}$ (recall the collection $\mathcal{K}$ from §3), contrary to the standard definition of Stickelberger elements when K/k is unramified at a certain prime above p. Theorem 5.3 is still equivalent to [Wil90a, Theorem 3], since our assumption (A1) assures that our Stickelberger elements agree with that of [Wil90a, Kur03] up to units.

5.2. **Main theorem over k_∞.** Along with the assumptions above, suppose also that (A2) holds. Write $\mathrm{char}(M)$ for the characteristic ideal of a torsion Λ-module M.

We proceed as in the previous section: First, we prove a bound for the characteristic ideal of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)^\vee$. We then use this bound, together with Proposition 2.23, to obtain a bound on the characteristic ideal of (the Pontryagin dual of) the classical (dual) Selmer groups.

Theorem 5.5. *Under the running assumptions,*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})/\Lambda\cdot\kappa_1^{\mathrm{St}_\infty}\right).$$

Proof. This is Theorem 2.38. □

Set $c^{\mathrm{St}}_{k_\infty} := \{c^{\mathrm{St}}_{k_n}\}_n \in \varprojlim_n H^1(k_n,T) = H^1(k,\mathbb{T})$.

Corollary 5.6. $\mathrm{char}\left((\varinjlim_n A^\chi_{L_n})^\vee\right) \mid \mathrm{char}\left(\mathbb{L}/\Lambda\cdot c^{\mathrm{St}_\infty}_{k_\infty}\right)$.

Proof. This follows from Theorem 5.5 and Proposition 2.23(ii) applied with $c = c^{\mathrm{St}_\infty}_{k_\infty}$; together with the fact that $\kappa_1^{\mathrm{St}_\infty} = c^{\mathrm{St}_\infty}_{k_\infty}$. □

Recall the element $\theta^\chi_{L_n} \in \mathbb{Z}_p[\Delta\times\Gamma_n]^\chi = \mathbb{Z}_p[\Delta]^\chi[\Gamma_n]$ from §3. We denote the image of $\theta^\chi_{L_n}$ under the map

$$\chi_\Lambda : \mathbb{Z}_p[\Delta]^\chi[\Gamma_n] \longrightarrow \mathbb{Z}_p[\Gamma_n],$$

(which extends $\chi : \mathbb{Z}_p[\Delta]^\chi \to \mathbb{Z}_p$ from the previous section to Γ_n by letting $\chi_\Lambda(\gamma)=\gamma$ for $\gamma\in\Gamma_n$) by $\chi_\Lambda(\theta_{L_n})$. Lemma 3.1 shows that $\{\chi_\Lambda(\theta_{L_n})\}$ is a projective system with respect to natural surjections $\mathbb{Z}_p[\Gamma_{n'}]\to\mathbb{Z}_p[\Gamma_n]$, $n'\geq n$. We define

$$\chi_\Lambda(\Theta_{L_\infty}) := \{\chi_\Lambda(\theta_{L_n})\} \in \varprojlim_n \mathbb{Z}_p[\Gamma_n] = \Lambda.$$

Finally, let $x\mapsto x^\bullet$ be the involution on Λ induced from $\gamma\mapsto\gamma^{-1}$ for $\gamma\in\Gamma$.

Theorem 5.7. *Under the running hypotheses of this section,*

$$\mathrm{char}\left((\varinjlim_n A^\chi_{L_n})^\vee\right) \mid \chi_\Lambda(\Theta_{L_\infty})^\bullet.$$

Proof. By the construction of $c^{\mathrm{St}}_{k_n}$ and λ_n,

$$c^{\mathrm{St}}_{k_n} = \lambda_n\circ\theta^\chi_{L_n} = \chi_\Lambda(\theta_{L_n})^\bullet\lambda_n = \chi_\Lambda(\theta_{L_n})^\bullet\varphi_n.$$

It follows that

$$c^{\mathrm{St}}_{k_\infty} = \{\chi_\Lambda(\theta_{L_n})^\bullet\varphi_n\}_n = \chi_\Lambda(\Theta_{L_\infty})^\bullet\Phi, \tag{5.1}$$

with $\Phi=\{\varphi_n\}$ as in §4.1. Since Φ is a generator of $\mathbb{L}$ (by definition), Theorem follows from Corollary 5.6. □

Once again, making use of a standard class number argument (and yet again the case $\boldsymbol{\mu}_p\subset L$ requires more care as above) shows that this equality may be turned into an equality:

Theorem 5.8. *Under the hypotheses above,*

$$\mathrm{char}\left((\varinjlim_n A^\chi_{L_n})^\vee\right) = \chi_\Lambda(\Theta_{L_\infty})^\bullet.$$

Let $\rho_{\text{cyc}} : G_k \to \mathbb{Z}_p^\times$ be the cyclotomic character (giving the action of G_k on $\boldsymbol{\mu}_{p^\infty}$), and set $\langle\rho_{\text{cyc}}\rangle = \omega^{-1}\rho_{\text{cyc}} : \Gamma \to 1 + p\mathbb{Z}_p$. We define a twisting map $\text{Tw}_{\langle\rho_{\text{cyc}}\rangle} : \Lambda \to \Lambda$ by setting

$$\text{Tw}_{\langle\rho_{\text{cyc}}\rangle}(\gamma) = \langle\rho_{\text{cyc}}\rangle(\gamma)\gamma \text{ for } \gamma \in \Gamma,$$

and extending to Λ by linearity and continuity. Finally, let $\mathcal{L}_{\omega\chi^{-1}} \in \Lambda$ denote the *Deligne-Ribet p-adic L-function* for the character $\omega\chi^{-1}$. We will loosely say here that $\mathcal{L}_{\omega\chi^{-1}}$ is characterized by the following interpolation property:

$$\langle\rho_{\text{cyc}}\rangle^k\xi(\mathcal{L}_{\omega\chi^{-1}}) = \prod_{\wp|p}(1-\omega^{-k}\xi\chi(\wp)\mathbf{N}\wp^{k-1})L(1-k,\omega^{-k}\xi\chi), \tag{5.2}$$

for every $k \geq 1$ and every character ξ of Γ of finite order. Here, $L(s,\varrho)$ is the (abelian) Artin L-function attached to a character ϱ of G_k which is of finite order.

Lemma 5.9. $\chi_\Lambda(\Theta_{L_\infty})^\bullet = \text{Tw}_{\langle\rho_{\text{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}})$.

Proof. For every character ξ of Γ of finite order, it follows from the definitions that

$$\begin{aligned}\xi(\chi_\Lambda(\Theta_{L_\infty})^\bullet) = \xi^{-1}\left(\chi_\Lambda(\Theta_{L_\infty})\right) &= \prod_{\wp|p}(1-\chi^{-1}\xi(\wp))L(0,\chi^{-1}\xi)\\ &= \langle\rho_{\text{cyc}}\rangle\xi(\mathcal{L}_{\omega\chi^{-1}}) = \xi(\langle\rho_{\text{cyc}}\rangle\mathcal{L}_{\omega\chi^{-1}}).\end{aligned}$$

Since this is true for every ξ, Lemma follows. □

Corollary 5.10. *Under the assumptions of Theorem 5.8,*

$$\text{char}\left((\varinjlim_n A^\chi_{L_n})^\vee\right) = \text{Tw}_{\langle\rho_{\text{cyc}}\rangle}(\mathcal{L}_{\omega\chi^{-1}}).$$

5.3. **Twisting and local Iwasawa theory of Stark elements.** The goal of this section is to establish a connection between the Stickelberger elements and the conjectural Rubin-Stark elements. To achieve this, we will employ the following two ingredients:

(1) The twisting formalism developed in [Rub00, §VI].
(2) The rigidity statement [Büy09b, Theorem 2.19(ii)] for the module of Λ-adic Kolyvagin systems.

Write $\rho := \langle\rho_{\text{cyc}}\rangle$ for notational simplicity. Let $\psi = \omega\chi^{-1}$ as in the introduction and set

$$T' = \mathbb{Z}_p(1)\otimes\psi^{-1} = T\otimes\rho.$$

We also write $\mathbb{T}' = T'\otimes\Lambda$.

5.3.1. *Twisting argument.*

Lemma 5.11. *Suppose ρ is as above.*

(i) *There is a commutative diagram*

$$\begin{array}{ccc} \varprojlim_n H^1(k_n,T)\otimes\rho & \xrightarrow{\sim} & \varprojlim_n H^1(k_n,T') \\ \downarrow{\scriptstyle \text{loc}_p} & & \downarrow{\scriptstyle \text{loc}_p} \\ \varprojlim_n H^1((k_n)_p,T)\otimes\rho & \xrightarrow{\sim} & \varprojlim_n H^1((k_n)_p,T') \end{array}$$

such that the horizontal arrows are natural isomorphisms.

(ii) *There is a commutative diagram*

$$\begin{array}{ccc} \varprojlim_{K\in\mathcal{K}_0} H^1(K,T)\otimes\rho & \xrightarrow{\sim} & \varprojlim_{K\in\mathcal{K}_0} H^1(K,T') \\ \Big\downarrow{\scriptstyle \mathrm{loc}_p} & & \Big\downarrow{\scriptstyle \mathrm{loc}_p} \\ \varprojlim_{K\in\mathcal{K}_0} H^1(K_p,T)\otimes\rho & \xrightarrow{\sim} & \varprojlim_{K\in\mathcal{K}_0} H^1(K_p,T') \end{array}$$

such that the horizontal arrows are natural isomorphisms,

(iii) $\left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,T^*)\right)^\vee\otimes\rho \xrightarrow{\sim} \left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,(T')^*)\right)^\vee$.

Proof. This is [Rub00, Proposition VI.2.1]. We note that, $\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,T^*)$ here coincides with what Rubin calls $\mathcal{S}_{\Sigma_p}(k_\infty,W)$, thanks to Remark 2.12, where $W=T^*$ and Σ_p is the set of places of k which lie above p. □

Let $\boldsymbol{\mathcal{L}}\subset\varprojlim_{K\in\mathcal{K}_0}H^1(K_p,T)$ be as in §4.7, and let $\boldsymbol{\mathcal{L}}_\rho$ denote the image of $\boldsymbol{\mathcal{L}}$ under the isomorphism of Lemma 5.11(ii). Recall the $\boldsymbol{\mathcal{L}}$-restricted Euler system $\mathbf{c}^{\mathrm{St}}$ that we constructed at the end of §4.1. Let $\mathbf{c}^{\mathrm{St},\rho}$ denote the twist of the Euler system $\mathbf{c}^{\mathrm{St}}$ defined via [Rub00, Theorem VI.3.5]; this means $\mathbf{c}^{\mathrm{St},\rho}\in\mathbf{ES}(T',\mathcal{K})$. Furthermore, one can see without difficulty that the following Lemma is true:

Lemma 5.12. *The twisted Euler system $\mathbf{c}^{\mathrm{St},\rho}$ is $\boldsymbol{\mathcal{L}}_\rho$-restricted.*

Let $\mathbb{L}_\rho\in H^1(k_p,\mathbb{T}')$ denote the image of $\boldsymbol{\mathcal{L}}_\rho$ under the obvious projection map. Recall the element $c^{\mathrm{St}}_{k_\infty}=\{c^{\mathrm{St}}_{k_n}\}_n\in H^1(k,\mathbb{T})$, and set $c^{\mathrm{St},\rho}_{k_\infty}=\{c^{\mathrm{St},\rho}_{k_n}\}_n\in H^1(k,\mathbb{T}')$. Note that $\mathrm{loc}_p(c^{\mathrm{St}}_{k_\infty})\in\mathbb{L}\subset H^1(k_p,\mathbb{T})$ and $\mathrm{loc}_p(c^{\mathrm{St},\rho}_{k_\infty})\in\mathbb{L}_\rho\subset H^1(k_p,\mathbb{T}')$, by construction. As before, we drop loc_p from notation and we denote $\mathrm{loc}_p(c^{\mathrm{St}}_{k_\infty})$ (resp., $\mathrm{loc}_p(c^{\mathrm{St},\rho}_{k_\infty})$) simply by $c^{\mathrm{St}}_{k_\infty}$ (resp., by $c^{\mathrm{St},\rho}_{k_\infty}$). Lemma 5.11(i) induces an isomorphism

$$\mathbb{L}/\Lambda\cdot c^{\mathrm{St}}_{k_\infty}\otimes\rho\xrightarrow{\sim}\mathbb{L}_\rho/\Lambda\cdot c^{\mathrm{St},\rho}_{k_\infty}. \tag{5.3}$$

To simplify notation, set $X_\infty(T):=\left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,T^*)\right)^\vee$ and set similarly $X_\infty(T'):=\left(\varinjlim_n H^1_{\mathcal{F}^*_{\mathrm{cl}}}(k_n,(T')^*)\right)^\vee$. Until the end of this paper, assume (A1) and (A2) both hold true.

Proposition 5.13. *Let* $\mathrm{Tw}_\rho:\Lambda\to\Lambda$ *be the twisting operator as above.*

(i) $\mathrm{Tw}_\rho\left(\mathrm{char}\left(X_\infty(T')\right)\right)=\mathrm{char}\left(X_\infty(T)\right)$.

(ii) $\mathrm{Tw}_\rho\left(\mathrm{char}\left(\mathbb{L}_\rho/\Lambda\cdot c^{\mathrm{St},\rho}_{k_\infty}\right)\right)=\mathrm{char}\left(\mathbb{L}/\Lambda\cdot c^{\mathrm{St}}_{k_\infty}\right)$.

(iii) $\mathrm{char}\left(X_\infty(T')\right)=\mathrm{char}\left(\mathbb{L}_\rho/\Lambda\cdot c^{\mathrm{St},\rho}_{k_\infty}\right)$.

Proof. The assertion (i) follows from Lemma 5.11(iii) together with [Rub00, Lemma VI.1.2(i)], and (ii) from (5.3) applied with [Rub00, Lemma VI.1.2(i)].

Thanks to (i) and (ii), the assertion (iii) is equivalent to verifying that

$$\mathrm{Tw}_{\rho^{-1}}\left(\mathrm{char}\left(X_\infty(T)\right)\right)=\mathrm{Tw}_{\rho^{-1}}\left(\mathrm{char}\left(\mathbb{L}/\Lambda\cdot c^{\mathrm{St}}_{k_\infty}\right)\right). \tag{5.4}$$

This, however, is the statement of Theorem 5.8 put together with (5.1), and twisted by $\mathrm{Tw}_{\rho^{-1}}$. □

5.3.2. *Stickelberger elements vs. Rubin-Stark elements.* Following Definition 2.10 of [Büy09b], we define the $\mathbb{L}_\rho$-modified Selmer structure $\mathcal{F}_{\mathbb{L}_\rho}$ on $\mathbb{T}'$, for $\mathbb{L}_\rho \subset H^1(k_p, \mathbb{T}')$ as above. The argument of [Büy09b, Theorem 3.23] shows that the $\boldsymbol{\mathcal{L}}_\rho$-restricted Euler system $\mathbf{c}^{\mathrm{St},\rho}$ gives rise to a Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho} \in \overline{\mathbf{KS}}(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$.

Proposition 5.14. *We have*

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}^*_\rho}}(k, (\mathbb{T}')^*)^\vee\right) = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}_\rho}}(k, \mathbb{T}')/\Lambda \cdot \kappa_1^{\mathrm{St}_\infty,\rho}\right).$$

Proof. This follows from Proposition 5.13 and the exact sequence of Proposition 2.23(ii) rewritten for $\mathbb{T}'$ instead of $\mathbb{T}$. See also [Büy09b, Proposition 2.12] for this version of the exact sequence of Proposition 2.23(ii). □

Assume until the end of this paper that the character ψ satisfies the hypothesis (A1) as well, that is, $\psi(\wp) \neq 1$ for any prime $\wp$ of k lying above p. According to [Büy09b, Theorem 2.19(ii)], the Λ-module of Kolyvagin systems $\overline{\mathbf{KS}}(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$ is free of rank one.

Corollary 5.15. *The Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho}$ generates the cyclic module $\overline{\mathbf{KS}}(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$.*

Proof. We know by [Büy09b, Theorem 2.19(ii) and Proposition 4.2] that the free Λ-module $\overline{\mathbf{KS}}(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$ is generated by a Λ-primitive Kolyvagin system $\boldsymbol{\kappa}$ (in the sense of [MR04, Definition 5.3.9]). In particular, we may write

$$\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho} = u \cdot \boldsymbol{\kappa},$$

with $u \in \Lambda$.

The main application of the Λ-primitive Kolyvagin system $\boldsymbol{\kappa}$ for the Selmer triple $(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$ is the following (see [Büy09b, Theorem 2.20]):

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}^*_\rho}}(k, (\mathbb{T}')^*)^\vee\right) = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}_\rho}}(k, \mathbb{T}')/\Lambda \cdot \kappa_1\right). \tag{5.5}$$

The assertion (5.5) together with Proposition 5.14 shows that $u \in \Lambda^\times$. This completes the proof. □

Let $\boldsymbol{\epsilon}^{\mathrm{Stark}} = \{\epsilon^{\mathrm{Stark}}\}_{K \in \mathcal{K}_0}$ be the Euler system $\left\{\varepsilon^{\psi}_{K,\Phi_0^{(\infty)}}\right\}_{K \in \mathcal{K}_0}$ defined in [Büy09b, Proposition 3.14]. This Euler system is obtained from the Rubin-Stark elements that Rubin [Rub96] conjectured to exist, and therefore the existence of $\boldsymbol{\epsilon}^{\mathrm{Stark}}$ is implicitly assumed[6] here. Note that our ψ here is denoted by χ in [Büy09b]. Let

$$c^{\mathrm{Stark}}_{k_\infty} \in \wedge^r H^1(k_p, \mathbb{T}')$$

be the element defined as in [Büy09b, Remark 4.5]. Finally, let $\boldsymbol{\kappa}^{\mathrm{Stark}}$ be the Λ-primitive Kolyvagin system for the Selmer triple $(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$, which is obtained from the Euler system $\boldsymbol{\epsilon}^{\mathrm{Stark}}$. We remark that what we call $\mathbb{L}_\rho$ here is denoted by $\mathbb{L}_\infty$ in [Büy09b], and $\boldsymbol{\kappa}^{\mathrm{Stark}}$ here is denoted by $\boldsymbol{\kappa}^{\Phi_0^{(\infty)}}$ in loc.cit. Recall the Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho}$ which is obtained from the $\boldsymbol{\mathcal{L}}_\rho$-restricted Euler system $\mathbf{c}^{\mathrm{St},\rho}$ of Stickelberger elements.

The theorem below draws a connection between the Stickelberger elements and the Rubin-Stark elements.

[6]Having said that, note that we will recover in Theorem 5.16(i) below the Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{Stark}}$ (which descends from $\boldsymbol{\epsilon}^{\mathrm{Stark}}$) directly from the Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty}$, which is obtained in this article out of Stickelberger elements.

Theorem 5.16. *The following holds under the running assumptions:*

(i) *There is a unit* $u \in \Lambda^\times$ *such that* $\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho} = u \cdot \boldsymbol{\kappa}^{\mathrm{Stark}}$,

(ii) $\mathrm{char}\left(\mathbb{L}_\rho/\Lambda \cdot \kappa_1^{\mathrm{St}_\infty,\rho}\right) = \mathrm{char}\left(\mathbb{L}_\rho/\kappa_1^{\mathrm{Stark}}\right)$,

(iii) $\mathrm{char}\left(\wedge^r H^1(k_p,\mathbb{T}')/\Lambda \cdot c_{k_\infty}^{\mathrm{Stark}}\right) = \mathcal{L}_\psi$.

Here, $\mathcal{L}_\psi$ *is the Deligne-Ribet* p*-adic* L*-function attached to the totally even character* ψ.

Proof. The assertion (i) follows from Corollary 5.15 and the fact that $\boldsymbol{\kappa}^{\mathrm{Stark}}$ generates the free rank-one Λ-module $\overline{\mathbf{KS}}(\mathbb{T}', \mathcal{F}_{\mathbb{L}_\rho}, \mathcal{P})$. The assertion (ii) is immediate from (i).

The discussion preceding the statement of [Büy09b, Corollary 4.6] shows that

$$\wedge^r H^1(k_p,\mathbb{T}')/\Lambda \cdot c_{k_\infty}^{\mathrm{Stark}} \cong \mathbb{L}_\rho/\Lambda \cdot \kappa_1^{\mathrm{Stark}}.$$

Hence, it follows from (ii) that

$$\mathrm{char}\left(\wedge^r H^1(k_p,\mathbb{T}')/\Lambda \cdot c_{k_\infty}^{\mathrm{Stark}}\right) = \mathrm{char}\left(\mathbb{L}_\rho/\Lambda \cdot \kappa_1^{\mathrm{St}_\infty,\rho}\right).$$

By the construction of the Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{St}_\infty,\rho}$ out of the Euler system $\mathbf{c}^{\mathrm{St},\rho}$, it follows that $\kappa_1^{\mathrm{St}_\infty,\rho} = c_{k_\infty}^{\mathrm{St},\rho}$, which in return implies that

$$\mathrm{char}\left(\wedge^r H^1(k_p,\mathbb{T}')/\Lambda \cdot c_{k_\infty}^{\mathrm{Stark}}\right) = \mathrm{char}\left(\mathbb{L}_\rho/\Lambda \cdot c_{k_\infty}^{\mathrm{St},\rho}\right).$$

Using Proposition 5.13(ii), Equation (5.1) and Lemma 5.9, we see that

$$\mathrm{Tw}_\rho\left(\mathrm{char}\left(\mathbb{L}_\rho/\Lambda \cdot c_{k_\infty}^{\mathrm{St},\rho}\right)\right) = \mathrm{char}\left(\mathbb{L}/\Lambda \cdot c_{k_\infty}^{\mathrm{St}}\right) = \mathrm{Tw}_\rho(\mathcal{L}_\psi).$$

This completes the proof of the theorem. □

The author [Büy09b, Theorem 4.7] has previously deduced Theorem 5.16(iii) from Iwasawa's main conjecture. Here, we need to assume slightly less, namely, the χ-part of the Brumer's conjecture (Assumption 1.1) to prove this statement.

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