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Exactness of the First Born Approximation in Electromagnetic Scattering

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For the scattering of plane electromagnetic waves by a general possibly anisotropic stationary linear medium in three dimensions, we give a condition on the permittivity and permeability tensors of the medium under which the first Born approximation yields the exact expression for the scattered wave whenever the incident wavenumber k does not exceed a preassigned value α. We also show that under this condition the medium is omnidirectionally invisible for $k \leq \alpha/2$, i.e. it displays broadband invisibility regardless of the polarization of the incident wave.

Subject Index A01

1. Introduction

Since its inception in 1926 [1], the Born approximation [2–4] has been the principal approximation scheme for performing scattering calculations [5–13]. Yet the search for scattering systems for which the Born approximation is exact did not succeed until 2019 when the first examples of complex potentials possessing this property could be constructed within the context of potential scattering of scalar waves in two dimensions [14]. The key ingredient leading to the discovery of these potentials is a recently proposed dynamical formulation of stationary scattering [15–17]. The purpose of the present article is to employ the dynamical formulation of electromagnetic scattering developed in Ref. [18] to address the problem of the exactness of the first Born approximation in electromagnetic scattering.

Consider the scattering of plane electromagnetic waves by a general stationary linear medium. The electric field $\mathbf{E}$ of the wave, which together with its magnetic field $\mathbf{H}$ satisfy Maxwell's equations, admits the asymptotic expression: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_{\rm i}(\mathbf{r},t) + \mathbf{E}_{\rm s}(\mathbf{r},t)$ for $r \to \infty$, where $\mathbf{E}_{\rm i}$ and $\mathbf{E}_{\rm s}$ are respectively the electric fields of the incident and scattered waves, $\mathbf{r}$ is the position of the detector observing the wave, and $r := |\mathbf{r}|$. These fields have the form [4,7]:

$$\mathbf{E}_{\rm i}(\mathbf{r},t) = \mathsf{E}_0\, e^{i(\mathbf{k}_{\rm i}\cdot\mathbf{r}-\omega t)}\,\mathbf{e}_{\rm i}, \tag{1}$$

$$\mathbf{E}_{\rm s}(\mathbf{r},t) = \frac{\mathsf{E}_0\, e^{i(kr-i\omega t)}}{r}\,\mathbf{F}(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i}), \tag{2}$$

where E_0 is a complex amplitude; $\mathbf{k}_{\rm i}$, ω, and $\mathbf{e}_{\rm i}$ are respectively the wave vector, angular frequency, and polarization vector of the incident wave; $k := \omega/c = |\mathbf{k}_{\rm i}|$ is the wavenumber; $\mathbf{F}$ is a vector-valued function; $\mathbf{k}_{\rm s} := k\hat{\mathbf{r}}$; and $\hat{\mathbf{r}} := r^{-1}\mathbf{r}$.

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The electric field of the scattered wave turns out to admit a perturbative series expansion known as the Born series [2,4,7]. We can express it as

$$\mathbf{E}_{\rm s}(\mathbf{r},t)=\frac{\mathsf{E}_0\,e^{i(kr-i\omega t)}}{r}\sum_{n=1}^{\infty}\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_i),\tag{3}$$

where $\mathbf{F}_n$ are vector-valued functions [19]. The N-th order Born approximation amounts to neglecting all but the first N terms of the series in Eq. (3).

To reveal the perturbative nature of the Born series, we introduce: $\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}(\mathbf{r}):=\hat{\boldsymbol{\varepsilon}}(\mathbf{r})-\mathbf{I}$ and $\boldsymbol{\eta}_{\boldsymbol{\mu}}(\mathbf{r}):=\hat{\boldsymbol{\mu}}(\mathbf{r})-\mathbf{I}$, where $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$ are respectively the relative permittivity and permeability tensors[1] of the medium, and $\mathbf{I}$ is the 3×3 identity matrix. Let ς be a positive real number. Then under the scaling transformation,

$$\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}(\mathbf{r})\to\varsigma\,\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}(\mathbf{r}),\qquad \boldsymbol{\eta}_{\boldsymbol{\mu}}(\mathbf{r})\to\varsigma\,\boldsymbol{\eta}_{\boldsymbol{\mu}}(\mathbf{r}),\tag{4}$$

the vector-valued functions $\mathbf{F}_n$, which determine the terms of the Born series (3), transform as[2]

$$\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i})\to\varsigma^n\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i}).\tag{5}$$

Because $\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}(\mathbf{r})$ and $\boldsymbol{\eta}_{\boldsymbol{\mu}}(\mathbf{r})$ quantify the scattering properties of the medium, the transformations (4) with $\varsigma<1$ correspond to a medium with a weaker scattering response. This in turn shows that for such a medium, $|\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i})|$ becomes increasingly small as n grows, and terminating the Born series yields a reliable approximation. The principal example is the first Born approximation, which involves neglecting all but the first term of the Born series [2,4,7]. This approximation is exact if

$$\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i})=\mathbf{0}\quad\text{for}\quad n\geq 2,\tag{6}$$

equivalently

$$\mathbf{E}_{\rm s}(\mathbf{r},t)=\frac{\mathsf{E}_0\,e^{i(kr-i\omega t)}}{r}\,\mathbf{F}_1(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i}).\tag{7}$$

We can also express this condition in terms of the scaling rule (5); we state it as a theorem for later reference.

Theorem 1. *The first Born approximation is exact if and only if under the scaling transformation (4), the electric field of the scattered wave transforms as* $\mathbf{E}_{\rm s}\to\varsigma\,\mathbf{E}_{\rm s}$.

Given the difficulties associated with finding explicit formulas for $\mathbf{F}_n(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i})$ and the fact that Eq. (6) corresponds to an infinite system of complicated integral equations (constraints) for $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$, it is practically impossible to use Eq. (6) for the purpose of determining the permittivity and permeability profiles for which the first Born approximation is exact.[3] This is the main reason why identifying the explicit conditions for the exactness of the first Born approximation has been an open problem for close to a century.[4] Motivated by our results on the scattering of scalar waves [14], we pursue a different route toward a solution of this problem. This is based

[1] Recall that $\hat{\boldsymbol{\varepsilon}}:=\varepsilon_0^{-1}\boldsymbol{\varepsilon}$ and $\hat{\boldsymbol{\mu}}:=\mu_0^{-1}\boldsymbol{\mu}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ denote the permittivity and permeability tensors, respectively, and ε_0 and μ_0 are the permittivity and permeability of the vacuum, respectively.

[2] This follows from the recurrence relation for $\mathbf{F}_n$ which has its root in the structure of the electromagnetic Lippmann–Schwinger equation [19].

[3] The transverse vector nature of the electromagnetic waves and the tensorial nature of the corresponding interaction potentials $\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\eta}_{\boldsymbol{\mu}}$ make this a considerably more elaborate task than addressing the same problem for scalar waves.

[4] There have been extensive studies of the Born series and its convergence properties in quantum scattering theory of scalar waves [20–25] as well as the scattering theory of electromagnetic waves [19]. None of these, however, provide conditions for the truncation of this series.

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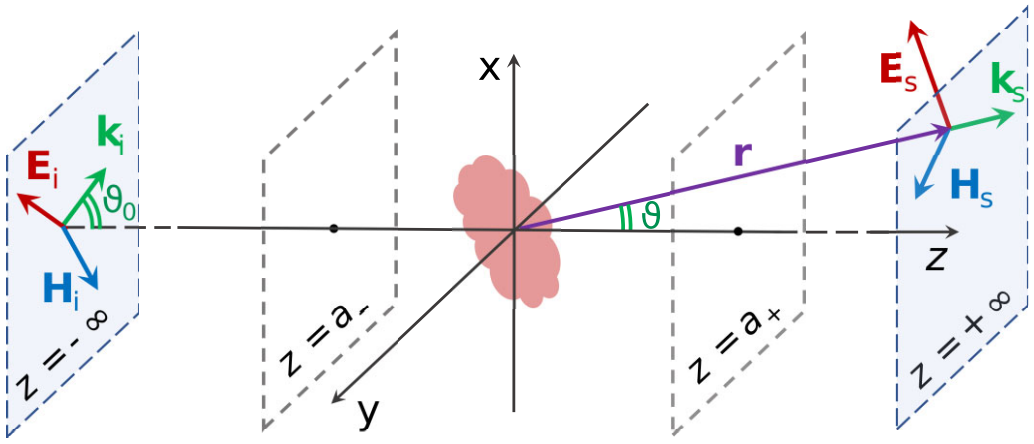


Fig. 1. Schematic view of scattering setup where the source of the incident wave lies on the plane $z=-\infty$. The region colored in pink represents the scatterer that is confined between the planes $z=a_\pm$. $\mathbf{k}_\mathrm{i}$, $\mathbf{E}_\mathrm{i}$, and $\mathbf{H}_\mathrm{i}$ are respectively the wave vector, the electric field, and the magnetic field of the incident wave, while $\mathbf{k}_\mathrm{s}$, $\mathbf{E}_\mathrm{s}$, and $\mathbf{H}_\mathrm{s}$ are respectively the wave vector, the electric field, and the magnetic field of the scattered wave detected by a detector placed on the plane $z=+\infty$.

on a dynamical formulation of the stationary electromagnetic scattering [18] whose main ingredient is the notion of a fundamental transfer matrix. This is a linear operator acting in an infinite-dimensional function space that similarly to the traditional numerical transfer matrices [26–32] stores the information about the scattering properties of the medium but unlike the latter allows for analytic calculations. In this article, we use the fundamental transfer matrix to obtain a sufficient condition for the exactness of the first Born approximation in electromagnetic scattering.

The outline of this article is as follows. In Sect. 2 we present our main results as well as specific examples of scattering media for which the first Born approximation is exact. In Sect. 3 we offer a concise review of the dynamical formulation of stationary electromagnetic scattering. In Sect. 4, we discuss the application of this formulation in addressing the problem of finding conditions for the exactness of the first Born approximation. In Sect. 5, we present a summary of our findings and our concluding remarks.

2. Main results

We begin our analysis by considering the scattering setup where the source of the incident wave and the detectors detecting the scattered wave are, without loss of generality, placed on the planes $z=\pm\infty$ in a Cartesian coordinate system with coordinates x, y, and z, as depicted in Fig. 1. We also suppose that the space outside the region bounded by a pair of normal planes to the z axis is empty, i.e. there is an interval (a_-, a_+) on the z axis such that

$$\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}(x,y,z)=\boldsymbol{\eta}_{\boldsymbol{\mu}}(x,y,z)=\mathbf{0}\quad\text{for}\quad z\notin(a_-,a_+). \tag{8}$$

Furthermore, we assume that the Fourier transform of all functions of real variables that enter our analysis exists.

Throughout this article we employ the following notations:

- For each vector $\mathbf{v}\in\mathbb{R}^3$, v_x, v_y, and v_z label the x, y, and z components of $\mathbf{v}$, and $\vec{v}$ stands for (v_x, v_y) so that $\mathbf{v}=(\vec{v}, v_z)$. In particular, $\vec{r}:=(x,y)$ and $\mathbf{r}=(\vec{r}, z)$.

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- Given a scalar, vector-valued, or matrix-valued function g of $\mathbf{r}$, we use $\tilde{g}(\vec{p}, z)$ to denote the 2D Fourier transform of $g(\vec{r}, z)$ with respect to $\vec{r}$, i.e.

$$\tilde{g}(\vec{p}, z) := \int_{\mathbb{R}^2} d^2r\, e^{-i\vec{p}\cdot\vec{r}} g(\vec{r}, z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-i(xp_x+yp_y)} g(x, y, z), \tag{9}$$

  where $\vec{p} := (p_x, p_y)$. For example, $\tilde{\boldsymbol{\eta}}_\varepsilon$ and $\tilde{\boldsymbol{\eta}}_\mu$ are respectively the 2D Fourier transforms of $\boldsymbol{\eta}_\varepsilon$ and $\boldsymbol{\eta}_\mu$ with respect to $\vec{r}$.
- We use $\hat{\varepsilon}_{ij}$ and $\hat{\mu}_{ij}$ to mark the entries of $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$, respectively.

The following is the main result of this article which we prove in Sect. 4.

Theorem 2. *Consider the electromagnetic scattering problem for a time-harmonic plane wave propagating in a stationary linear medium with relative permittivity and permeability, $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$. Suppose that $\boldsymbol{\eta}_\varepsilon := \hat{\boldsymbol{\varepsilon}} - \mathbf{I}$ and $\boldsymbol{\eta}_\mu := \hat{\boldsymbol{\mu}} - \mathbf{I}$ satisfy Eq. (8) for some $a_\pm \in \mathbb{R}$ with $a_- < a_+$ and that the following conditions hold.*

(1) $\hat{\varepsilon}_{33}$ and $\hat{\mu}_{33}$ are bounded functions whose real part has a positive lower bound, i.e. there are real numbers m and M such that for all $\mathbf{r} \in \mathbb{R}^3$,

$$0 < m \le \mathrm{Re}[\hat{\varepsilon}_{33}(\mathbf{r})] \le |\hat{\varepsilon}_{33}(\mathbf{r})| \le M, \qquad 0 < m \le \mathrm{Re}[\hat{\mu}_{33}(\mathbf{r})] \le |\hat{\mu}_{33}(\mathbf{r})| \le M, \tag{10}$$

*where "*Re*" stands for the real part of its argument.*

(2) There are a positive real number α and a unit vector $\vec{e}$ lying on the x-y plane such that

$$\tilde{\boldsymbol{\eta}}_{\boldsymbol{\varepsilon}}(\vec{p}, z) = \tilde{\boldsymbol{\eta}}_{\boldsymbol{\mu}}(\vec{p}, z) = \mathbf{0} \quad \text{for} \quad \vec{p}\cdot\vec{e} \le \alpha. \tag{11}$$

Then the first Born approximation provides the exact solution of the scattering problem, if the wavenumber k for the incident wave does not exceed α, i.e. $k \le \alpha$. Moreover, the medium does not scatter the incident waves with wavenumber $k \le \alpha/2$, i.e. it displays broadband invisibility in the wavenumber spectrum $(0, \alpha/2]$.

Notice that Condition 1 of this theorem holds for all nonexotic isotropic media. We can also satisfy it for realistic anisotropic media by an appropriate choice of the z axis. Furthermore, if Condition 2 holds, we can perform a rotation about the z axis to align $\vec{e}$ and the x axis, in which case Eq. (11) takes the form

$$\tilde{\boldsymbol{\eta}}_{\boldsymbol{\varepsilon}}(p_x, p_y, z) = \tilde{\boldsymbol{\eta}}_{\boldsymbol{\mu}}(p_x, p_y, z) = \mathbf{0} \quad \text{for} \quad p_x \le \alpha. \tag{12}$$

Since such a rotation will not affect the 2D Fourier transform of a function with respect to x and y, Condition 2 is equivalent to Eq. (12).

To provide concrete examples of linear media satisfying Eq. (12), we confine our attention to nonmagnetic isotropic media, where $\hat{\boldsymbol{\mu}} = \mathbf{I}$, $\hat{\boldsymbol{\varepsilon}} = \hat{\varepsilon}(\mathbf{r})\mathbf{I}$, and $\hat{\varepsilon}$ is the scalar relative permittivity. Then $\boldsymbol{\eta}_{\boldsymbol{\mu}} = \mathbf{0}$ and $\boldsymbol{\eta}_{\boldsymbol{\varepsilon}} = \eta_\varepsilon(\mathbf{r})\mathbf{I}$, where $\eta_\varepsilon = \hat{\varepsilon} - 1$, and Eq. (12) reduces to $\tilde{\eta}_\varepsilon(p_x, p_y, z) = 0$ for $p_x \le \alpha$. We can identify this with the condition that the Fourier transform of $e^{-i\alpha x}\eta_\varepsilon(x, y, z)$ with respect to x vanishes on the negative p_x axis.[5] This means that there is a function $u : \mathbb{R}^3 \to \mathbb{C}$ such that[6]

$$\hat{\varepsilon}(x, y, z) = 1 + e^{i\alpha x} \int_0^{\infty} d\mathfrak{K}\, e^{i\mathfrak{K}x} \mathrm{u}(\mathfrak{K}, y, z). \tag{13}$$

[5] In one dimension, scattering potentials with this property are known to be unidirectionally invisible for all wavenumbers [33–36].

[6] To ensure the existence of the 2D Fourier transform of $\hat{\varepsilon} - 1$ with respect to $\vec{r}$, we can require that $\int_0^\infty d\mathfrak{K} \int_{-\infty}^\infty dy\, |\mathrm{u}(\mathfrak{K}, y, z)|^2 < \infty$ for all $z \in \mathbb{R}$.

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A class of possible choices for u, which allow for the analytic evaluation of the integral in Eq. (13), is given by

$$\mathrm{u}(\mathfrak{K}, y, z) = \frac{a}{m!}\,(a\mathfrak{K})^m e^{-a\mathfrak{K}}\mathrm{f}(y, z), \tag{14}$$

where a is a positive real parameter, m is a positive integer, and $f : \mathbb{R}^2 \to \mathbb{C}$ is a function.[7] Substituting Eq. (14) into Eq. (13), we find

$$\hat{\varepsilon}(x, y, z) = 1 + \frac{e^{i\alpha x}\mathrm{f}(y, z)}{\left(1 - \frac{ix}{a}\right)^{m+1}}. \tag{15}$$

It is easy to show that Eq. (15) satisfies the first relation in Eq. (10), if there is a real number b such that[8]

$$|\mathrm{f}(y, z)| \leq b < 1 \quad \text{for all} \quad (y, z) \in \mathbb{R}^2. \tag{16}$$

Suppose that this condition holds. Then according to Theorem 2, the first Born approximation provides the exact solution of the scattering problem for the permittivity profile (15), if the incident wave has a wavenumber k not greater than α. Furthermore, the medium is invisible if $k \leq \alpha/2$. Another example for such a permittivity profile is

$$\hat{\varepsilon}(x, y, z) = 1 + \sqrt{\pi}\, e^{i\alpha x} e^{-\frac{x^2}{a^2}} \left[1 + \mathrm{erf}\left(\frac{ix}{a}\right)\right] \mathrm{f}(y, z), \tag{17}$$

where erf stands for the error function, and f is a function such that $|\mathrm{f}(y, z)| \leq b < 1/\sqrt{\pi} \approx 0.564$ for some $b \in \mathbb{R}^+$. Equation (17) corresponds to setting $\mathrm{u}(\mathfrak{K}, y, z) = a\, e^{-a^2\mathfrak{K}^2/4}\mathrm{f}(y, z)$ in Eq. (13).

Consider the following choice for the function f appearing in Eqs. (15) and (17):

$$\mathrm{f}(y, z) := \begin{cases} \mathfrak{z} & \text{for } |y| \leq \frac{\ell_y}{2} \text{ and } |z| \leq \frac{\ell_z}{2}, \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

where $\mathfrak{z}$ is a nonzero real or complex number, and ℓ_y and ℓ_z are positive real parameters. Then Eqs. (15) and (17) correspond to situations where the inhomogeneity of the medium that is responsible for the scattering of waves is confined to an infinite box with a finite rectangular base of side lengths ℓ_y and ℓ_z. The amplitude of the inhomogeneity decays to zero as $|x| \to \infty$ and we can approximate the box by the finite box given by $|x| \leq \frac{\ell_x}{2}$, $|y| \leq \frac{\ell_y}{2}$, and $|z| \leq \frac{\ell_z}{2}$, where ℓ_x is a positive real parameter much larger than a. Figure 2 provides a schematic demonstration of this box and plots of real and imaginary parts of η_ε inside the box for the permittivity profile (15) with

$$\mathfrak{z} = 0.01, \qquad m = 1, \qquad a = 2\alpha^{-1}, \qquad \ell_y = 3\alpha^{-1}, \qquad \ell_z = 4\alpha^{-1}. \tag{19}$$

Using these numerical values, we find that $|\eta_\varepsilon(x, y, z)| < 10^{-4}$ for $|x| \geq \ell_x = 10a$. Notice also that the broadband invisibility of the permittivity profile given by Eqs. (15) and (19) for $k \leq \alpha/2$ remains intact for all real and complex values of $\mathfrak{z}$ such that $|\mathfrak{z}| < 1$.

We close this section by drawing attention to the following points:

- The hypothesis of Theorem 2 does not prohibit the presence of dispersion, i.e. it also applies to situations where the relative permittivity and permeability tensors depend on the

[7] Requiring $\int_{\infty}^{\infty} dy|\mathrm{f}(y, z)|^2 < \infty$, we can guarantee the existence of the right-hand side of Eq. (13) and the 2D Fourier transform of $\hat{\varepsilon} - 1$ with respect to $\vec{r}$.

[8] Because $|1 - ix/a| \geq 1$, Eqs. (15) and (16) imply $|\eta_\varepsilon| = |\varepsilon - 1| \leq |\mathrm{f}| \leq b$. Using this relation together with $-|\eta_\varepsilon| \leq \mathrm{Re}(\eta_\varepsilon) \leq |\eta_\varepsilon|$, $\mathrm{Re}(\hat{\varepsilon}) = 1 + \mathrm{Re}(\eta_\varepsilon)$, and $b < 1$, we have $0 < 1 - b \leq \mathrm{Re}(\hat{\varepsilon}) \leq 1 + b$, i.e. Eq. (10) holds for $m = 1 - b$ and $M = 1 + b$.

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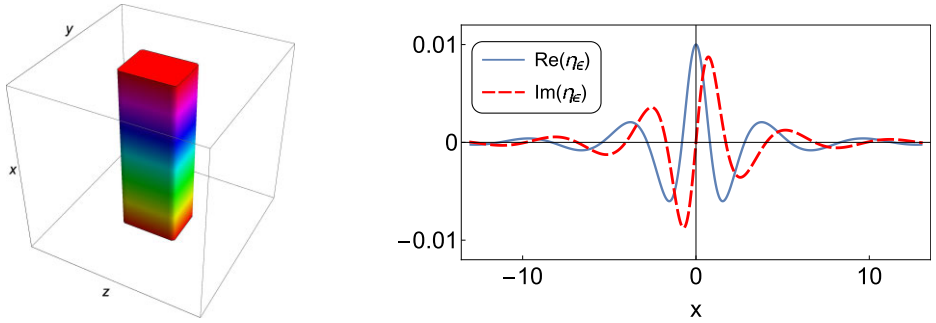


Fig. 2. Schematic view of the box confining the inhomogeneous part of the medium given by Eqs. (15), (18), and (19) (left); plots of the real and imaginary parts of η_ε as a function of x inside this box (right). Here we use units where $\alpha = 1$.

wavenumber k. If Conditions (8), (10), and (11) hold for all $k \leq \alpha$, the first Born approximation is exact for $k \leq \alpha$, and the medium is invisible for $k \leq \alpha/2$.[9] For example, the nonmagnetic isotropic media described by Eqs. (15) and (17) satisfy these conditions even if the function f has an arbitrary k-dependence.

- We can apply Theorem 2 also for situations where, similarly to the 1D setups considered in Refs. [33–35], the regions $z \leq a_-$ and $z \geq a_+$ are filled with a homogeneous and isotropic background medium. In this case we only need to define the incident wavenumber and the relative permittivity and permeability tensors relative to the background medium, i.e. set $k := \omega\sqrt{\varepsilon_{\rm B}\mu_{\rm B}}$, $\hat{\boldsymbol{\varepsilon}} := \varepsilon_{\rm B}^{-1}\boldsymbol{\varepsilon}$, and $\hat{\boldsymbol{\mu}} := \mu_{\rm B}^{-1}\boldsymbol{\mu}$, where $\varepsilon_{\rm B}$ and $\mu_{\rm B}$ are respectively the permittivity and permeability of the background.

3. Dynamical formulation of electromagnetic scattering

Consider a time-harmonic electromagnetic wave propagating in a stationary linear medium with relative permittivity and permeability tensors $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$. Then we can express the electric and magnetic fields of the wave in the form $\mathbf{E}(\mathbf{r}, t) = e^{-i\omega t}\boldsymbol{\mathcal{E}}(\mathbf{r})/\sqrt{\varepsilon_0}$ and $\mathbf{H}(\mathbf{r}, t) = e^{-i\omega t}\boldsymbol{\mathcal{H}}(\mathbf{r})/\sqrt{\mu_0}$, where ω is the angular frequency of the wave, and $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are vector-valued functions in terms of which Maxwell's equations take the form [18]:

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} = ik\hat{\boldsymbol{\mu}}\,\boldsymbol{\mathcal{H}}, \qquad \boldsymbol{\nabla}\times\boldsymbol{\mathcal{H}} = -ik\hat{\boldsymbol{\varepsilon}}\,\boldsymbol{\mathcal{E}}. \tag{20}$$

Suppose that $\hat{\varepsilon}_{33} \neq 0$ and $\hat{\mu}_{33} \neq 0$. We can then use Eq. (20) to express the z component of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ in terms of their x and y components, i.e. $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{H}_x$, and $\mathcal{H}_y$. This in turn allows for reducing Eq. (20) to a system of first-order equations which we can express in the form of the time-dependent Schrödinger equation [18],

$$i\partial_z\boldsymbol{\Phi} = \widehat{\mathbf{H}}\,\boldsymbol{\Phi}, \tag{21}$$

where z plays the role of time, $\boldsymbol{\Phi}$ is a 4-component function given by

$$\boldsymbol{\Phi} := \begin{bmatrix}\vec{\mathcal{E}}\\ \vec{\mathcal{H}}\end{bmatrix}, \qquad \vec{\mathcal{E}} := \begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{bmatrix}, \qquad \vec{\mathcal{H}} := \begin{bmatrix}\mathcal{H}_x\\ \mathcal{H}_y\end{bmatrix}, \tag{22}$$

and $\widehat{\mathbf{H}}$ is a 4×4 matrix-valued differential operator. The latter has the form,

$$\widehat{\mathbf{H}} := \begin{bmatrix}\widehat{\mathbf{H}}_{11} & \widehat{\mathbf{H}}_{12}\\ \widehat{\mathbf{H}}_{21} & \widehat{\mathbf{H}}_{22}\end{bmatrix}, \tag{23}$$

[9] This in turn implies that the medium does not scatter incident wave packets that are superpositions of plane waves with $k \leq \alpha/2$.

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where $\widehat{\mathbf{H}}_{ij}$ are the 2×2 matrix-valued operators given by

$$\widehat{\mathbf{H}}_{11} := -i\vec{\partial}\,\frac{1}{\hat{\varepsilon}_{33}}\,\vec{\varepsilon}_3^{\,T} + \frac{1}{\hat{\mu}_{33}}\,\vec{K}_{\mathcal{H}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2, \qquad \widehat{\mathbf{H}}_{12} := -\frac{i}{k}\vec{\partial}\,\frac{1}{\hat{\varepsilon}_{33}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2 + k\,\mathbf{K}_{\mathcal{H}}, \tag{24}$$

$$\widehat{\mathbf{H}}_{21} := \frac{i}{k}\vec{\partial}\,\frac{1}{\hat{\mu}_{33}}\,\vec{\partial}^{\,T}\boldsymbol{\sigma}_2 - k\,\mathbf{K}_{\mathcal{E}}, \qquad \widehat{\mathbf{H}}_{22} := -i\vec{\partial}\,\frac{1}{\hat{\mu}_{33}}\,\vec{\mu}_3^{\,T} + \frac{1}{\hat{\varepsilon}_{33}}\vec{K}_{\mathcal{E}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2, \tag{25}$$

$$\vec{\partial} := \begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix}, \qquad \boldsymbol{\sigma}_2 := \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \qquad \vec{\varepsilon}_\ell := \begin{bmatrix}\hat{\varepsilon}_{\ell 1}\\ \hat{\varepsilon}_{\ell 2}\end{bmatrix}, \qquad \vec{\mu}_\ell := \begin{bmatrix}\hat{\mu}_{\ell 1}\\ \hat{\mu}_{\ell 2}\end{bmatrix}, \tag{26}$$

$\ell\in\{1,2,3\}$, a superscript "T" stands for the transpose of a matrix or a matrix-valued operator, $\vec{\partial}$ and $\vec{\partial}^{\,T}$ act on all the terms appearing to their right,[10] and

$$\vec{K}_{\mathcal{E}} := \begin{bmatrix}-\hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{13}\end{bmatrix}, \qquad \mathbf{K}_{\mathcal{E}} := \begin{bmatrix}-\vec{\varepsilon}_2^{\,T}\\ \vec{\varepsilon}_1^{\,T}\end{bmatrix} - \frac{1}{\hat{\varepsilon}_{33}}\vec{K}_{\mathcal{E}}\vec{\varepsilon}_3^{\,T}, \tag{27}$$

$$\vec{K}_{\mathcal{H}} := \begin{bmatrix}-\hat{\mu}_{23}\\ \hat{\mu}_{13}\end{bmatrix}, \qquad \mathbf{K}_{\mathcal{H}} := \begin{bmatrix}-\vec{\mu}_2^{\,T}\\ \vec{\mu}_1^{\,T}\end{bmatrix} - \frac{1}{\hat{\mu}_{33}}\vec{K}_{\mathcal{H}}\vec{\mu}_3^{\,T}. \tag{28}$$

The time-dependent Schrödinger Eq. (21) determines the dynamics of an effective quantum system. Because z plays the role of time, we view x and y as the configuration- (position-) space variables, and identify $\boldsymbol{\Phi}$ and $\widehat{\mathbf{H}}$ respectively with the position wave function for an evolving state and the position-representation of a time-dependent Hamiltonian operator.[11] To make the z-dependence of the latter explicit, we denote it by $\widehat{H}(z)$. Employing Dirac's bracket notation, we can express the evolving state vector by $|\Phi(z)\rangle$. By definition, this solves the Schrödinger equation,

$$i\partial_z|\Phi(z)\rangle = \widehat{H}(z)|\Phi(z)\rangle. \tag{29}$$

We also have $\boldsymbol{\Phi}(x,y,z)=\boldsymbol{\Phi}(\vec{r},z)=\langle\vec{r}\,|\Phi(z)\rangle$ and $\widehat{\mathbf{H}}\boldsymbol{\Phi}(\vec{r},z)=\langle\vec{r}\,|\widehat{H}(z)|\Phi(z)\rangle$. We can obtain the explicit form of the Hamiltonian operator $\widehat{H}(z)$ by making the following changes in the expression for $\widehat{\mathbf{H}}$: $x\to\widehat{x}$, $y\to\widehat{y}$, $\partial_x\to i\widehat{p}_x$, and $\partial_y\to i\widehat{p}_y$, where $\widehat{x}$ and $\widehat{y}$ are the x and y components of the standard position operators, $\widehat{p}_x$ and $\widehat{p}_y$ are the x and y components of the standard momentum operators, and we use conventions where $\hbar=1$.

Let us consider the description of the above effective quantum system in the momentum representation. Because of our convention for the definition of the 2D Fourier transform, i.e., Eq. (9), the momentum wave function associated with the state vector $|\Phi(z)\rangle$ is given by $\langle\vec{p}\,|\Phi(z)\rangle=(2\pi)^{-1}\tilde{\boldsymbol{\Phi}}(\vec{p},z)$. Denoting the 2D Fourier transform and its inverse respectively by $\mathcal{F}$ and $\mathcal{F}^{-1}$we have $\tilde{\boldsymbol{\Phi}}=\mathcal{F}\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}=\mathcal{F}^{-1}\tilde{\boldsymbol{\Phi}}$. The momentum representation of the Hamiltonian, which we label by $\widehat{\tilde{\mathbf{H}}}(z)$, satisfies $\widehat{\tilde{\mathbf{H}}}(z)\tilde{\boldsymbol{\Phi}}(\vec{p},z)=2\pi\,\langle\vec{p}\,|\widehat{H}(z)|\Phi(z)\rangle$. This in turn shows that $\widehat{\tilde{\mathbf{H}}}(z)=\mathcal{F}\,\widehat{\mathbf{H}}\,\mathcal{F}^{-1}$. We can obtain the explicit form of $\widehat{\tilde{\mathbf{H}}}(z)$ by making the following changes in the formula for $\widehat{\mathbf{H}}$: $x\to i\partial_{p_x}$, $y\to i\partial_{p_y}$, $\partial_x\to ip_x$, $\partial_y\to ip_y$.

If the wave propagates in vacuum, $\hat{\boldsymbol{\varepsilon}}=\hat{\boldsymbol{\mu}}=\mathbf{I}$ and $\widehat{\tilde{\mathbf{H}}}(z)=\widehat{\tilde{\mathbf{H}}}_0$, where $\widehat{\tilde{\mathbf{H}}}_0\mathbf{f}(\vec{p}):=\tilde{\mathbf{H}}_0(\vec{p})\mathbf{f}(\vec{p})$ and

$$\tilde{\mathbf{H}}_0(\vec{p}) := \begin{bmatrix}\mathbf{0} & \tilde{\mathbf{L}}_0(\vec{p})\\ -\tilde{\mathbf{L}}_0(\vec{p}) & \mathbf{0}\end{bmatrix}, \qquad \tilde{\mathbf{L}}_0(\vec{p}) := \frac{1}{k}\begin{bmatrix}-p_xp_y & p_x^2-k^2\\ -p_y^2+k^2 & p_xp_y\end{bmatrix}. \tag{30}$$

[10] For example, for every test function $\mathbf{f}$, $\vec{\partial}\,\frac{1}{\hat{\varepsilon}_{33}}\,\vec{\varepsilon}_3^{\,T}\mathbf{f}$ stands for $\vec{\partial}\big(\frac{1}{\hat{\varepsilon}_{33}}\,\vec{\varepsilon}_3^{\,T}\mathbf{f}\big)$.

[11] Viewed as an operator acting in the space of 4-component wave functions equipped with the L^2-inner product, $\widehat{\mathbf{H}}$ is generally non-Hermitian. This makes the corresponding effective quantum system nonunity.

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Because $\widehat{\tilde{\mathbf{H}}}_0$ is z-independent, its evolution operator has the form, $\widehat{\tilde{\mathbf{U}}}_0(z, z_0) = e^{-i(z-z_0)\widehat{\tilde{\mathbf{H}}}_0}$, where $z, z_0 \in \mathbb{R}$ and z_0 represents an initial "time." The dynamics generated by $\widehat{\tilde{\mathbf{H}}}_0$ corresponds to the propagation of the wave in the absence of interaction with the medium, i.e. $\widehat{\tilde{\mathbf{H}}}_0$ plays the role of a free Hamiltonian in the momentum representation. This suggests that the information about the scattering effects of the medium should be contained in the corresponding interaction-picture Hamiltonian [37]. In the momentum-space representation, this has the form

$$\widehat{\mathscr{H}}(z) := e^{iz\widehat{\tilde{\mathbf{H}}}_0}\, \delta\widehat{\tilde{\mathbf{H}}}(z)\, e^{-iz\widehat{\tilde{\mathbf{H}}}_0}, \tag{31}$$

where

$$\delta\widehat{\tilde{\mathbf{H}}}(z) := \widehat{\tilde{\mathbf{H}}}(z) - \widehat{\tilde{\mathbf{H}}}_0. \tag{32}$$

Let $\mathbb{C}^{m\times n}$ denote the space of $m \times n$ complex matrices, $\mathscr{F}^4$ be the space of 4-component functions of $\vec{p}$, and $\mathscr{F}^4_k$ be the subspace of $\mathscr{F}^4$ consisting of functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{C}^{4\times 1}$ such that $\mathbf{f}(\vec{p}) = \mathbf{0}$ for $|\vec{p}| < k$. In Ref. [18], we introduce the fundamental transfer matrix $\widehat{\mathbf{M}}$ as a linear operator acting in $\mathscr{F}^4$ that is given by

$$\widehat{\mathbf{M}} = \widehat{\boldsymbol{\pi}}_k\, \widehat{\mathscr{U}}(\infty, -\infty)\widehat{\boldsymbol{\pi}}_k, \tag{33}$$

where $\widehat{\boldsymbol{\pi}}_k$ is the projection operator mapping $\mathscr{F}^4$ onto $\mathscr{F}^4_k$ according to

$$\big(\widehat{\boldsymbol{\pi}}_k\mathbf{f}\big)(\vec{p}) := \begin{cases} \mathbf{f}(\vec{p}) & \text{for} \quad |\vec{p}| < k, \\ \mathbf{0} & \text{for} \quad |\vec{p}| \geq k, \end{cases} \tag{34}$$

and $\widehat{\mathscr{U}}(z, z_0)$ is the interaction-picture evolution operator in the momentum representation.[12] Clearly, $\widehat{\mathbf{M}}$ maps $\mathscr{F}^4_k$ to $\mathscr{F}^4_k$. We can use the Dyson series expansion [37] of $\widehat{\mathscr{U}}(z, z_0)$ and Eq. (33) to express it in the form

$$\widehat{\mathbf{M}} = \widehat{\boldsymbol{\pi}}_k + \widehat{\boldsymbol{\pi}}_k \sum_{\ell=1}^{\infty}(-i)^\ell \int_{-\infty}^{\infty} dz_\ell \int_{-\infty}^{z_\ell} dz_{\ell-1} \cdots \int_{-\infty}^{z_2} dz_1\, \widehat{\mathscr{H}}(z_\ell)\widehat{\mathscr{H}}(z_{\ell-1})\cdots\widehat{\mathscr{H}}(z_1)\widehat{\boldsymbol{\pi}}_k. \tag{35}$$

Notice that if $\mathbf{f} \in \mathscr{F}^4_k$,

$$(\widehat{\mathbf{M}} - \widehat{\mathbf{I}})\mathbf{f} = (\widehat{\mathbf{M}} - \widehat{\boldsymbol{\pi}}_k)\mathbf{f}, \tag{36}$$

where $\widehat{\mathbf{I}}$ is the identity operator for $\mathscr{F}^4_k$.

In order to reveal the relationship between the fundamental transfer matrix and electromagnetic scattering, we make the following observations:

(1) In the coordinate system we have chosen, the source of the incident wave and the detectors are placed on the planes $z = \pm\infty$. The detectors reside on both of these planes, whereas the source lies on one of them. If the source is on the plane $z = -\infty$ (respectively $z = +\infty$), we say that the wave is left-incident (respectively right-incident). We can quantify these using the spherical coordinates of the incident wave vector $\mathbf{k}_{\rm i}$ which we denote by $(k, \vartheta_0, \varphi_0)$. For a left-incident wave, $\vartheta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\cos\vartheta_0 > 0$. For a right-incident wave, $\vartheta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\cos\vartheta_0 < 0$. Similarly if we use (r, ϑ, φ) for the spherical coordinate of the position $\mathbf{r}$ of a detector placed at $z = +\infty$ (respectively $z = -\infty$) we have $\cos\vartheta > 0$ (respectively $\cos\vartheta < 0$).

[12] Note that $\widehat{\mathscr{U}}(\infty, -\infty)$ coincides with the S matrix of the effective quantum system [38]. See also Ref. [39].

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(2) Let us introduce

$$\mathbf{h}_{\rm i} := \frac{1}{k}\,\mathbf{k}_{\rm i}\times\mathbf{e}_{\rm i}, \qquad \vec{\mathbf{e}}_{\rm i} := \begin{bmatrix} e_{{\rm i}\,x} \\ e_{{\rm i}\,y}\end{bmatrix}, \qquad \vec{\mathbf{h}}_{\rm i} := \begin{bmatrix} h_{{\rm i}\,x} \\ h_{{\rm i}\,y}\end{bmatrix}, \qquad \boldsymbol{\Upsilon}_{\rm i} := \begin{bmatrix} \vec{\mathbf{e}}_{\rm i} \\ \vec{\mathbf{h}}_{\rm i}\end{bmatrix}, \tag{37}$$

$$\vec{k}_{\rm i} := (k_{{\rm i}\,x}\,,\ k_{{\rm i}\,y}), \qquad \varpi(\vec{p}) := \sqrt{k^2-|\vec{p}|^2}, \qquad \mathscr{D}_k := \{\vec{p}\in\mathbb{R}^2\,|\,|\vec{p}|<k\}, \tag{38}$$

$$\boldsymbol{\Pi}_j(\vec{p}) := \frac{1}{2}\left[\mathbf{I}+\frac{(-1)^j}{\varpi(\vec{p})}\,\tilde{\mathbf{H}}_0(\vec{p})\right] = \frac{1}{2\varpi(\vec{p})}\begin{bmatrix} \varpi(\vec{p}) & (-1)^j\tilde{\mathbf{L}}_0(\vec{p}) \\ (-1)^{j+1}\tilde{\mathbf{L}}_0(\vec{p}) & \varpi(\vec{p})\end{bmatrix}, \tag{39}$$

where subscripts x and y mark the x and y components of the corresponding vector, and $j\in\{1,2\}$ and $|\vec{p}|\neq k$ in Eq. (39). Then it turns out that [18]:

$$\boldsymbol{\Pi}_1(\vec{k}_{\rm i})\boldsymbol{\Upsilon}_{\rm i} = \boldsymbol{\Upsilon}_{\rm i}\ \ \&\ \ \boldsymbol{\Pi}_2(\vec{k}_{\rm i})\boldsymbol{\Upsilon}_{\rm i} = \mathbf{0}\quad \text{for}\quad \cos\vartheta_0>0\ \ \text{(left-incident waves)}, \tag{40}$$

$$\boldsymbol{\Pi}_1(\vec{k}_{\rm i})\boldsymbol{\Upsilon}_{\rm i} = \mathbf{0}\ \ \&\ \ \boldsymbol{\Pi}_2(\vec{k}_{\rm i})\boldsymbol{\Upsilon}_{\rm i} = \boldsymbol{\Upsilon}_{\rm i}\quad \text{for}\quad \cos\vartheta_0<0\ \ \text{(right-incident waves)}. \tag{41}$$

It is easy to check that for all $\vec{p}\in\mathscr{D}_k$ and $\boldsymbol{\Gamma}\in\mathbb{C}^{4\times1}$, $\boldsymbol{\Pi}_j(\vec{p})\boldsymbol{\Gamma}$ is either zero or an eigenvector of $\mathbf{H}_0(\vec{p})$ with eigenvalue $(-1)^j\varpi(\vec{p})$. In view of Eqs. (40) and (41), $\boldsymbol{\Upsilon}_{\rm i}$ is an eigenvector of $\mathbf{H}_0(\vec{k}_{\rm i})$ with eigenvalue $-\varpi(\vec{k}_{\rm i})$ for a left-incident wave (respectively $\varpi(\vec{k}_{\rm i})$ for a right-incident wave). We can also define a pair of linear projection operators $\widehat{\boldsymbol{\Pi}}_j$ acting in $\mathscr{F}_k^4$ according to

$$(\widehat{\boldsymbol{\Pi}}_j\mathbf{f})(\vec{p}) := \boldsymbol{\Pi}_j(\vec{p})\mathbf{f}(\vec{p}), \tag{42}$$

where $\vec{p}\in\mathscr{D}_k$ and $\mathbf{f}\in\mathscr{F}_k^4$. These form an orthogonal pair of projection operators, because $\widehat{\boldsymbol{\Pi}}_i\widehat{\boldsymbol{\Pi}}_j=\delta_{ij}\widehat{\boldsymbol{\Pi}}_j$.

(3) In Ref. [40] we show that the vector-valued function $\mathbf{F}$ that enters the expression (2) for the electric field of the scattered wave is given by

$$\mathbf{F}(\mathbf{k}_{\rm s},\mathbf{k}_{\rm i}) = -\frac{ik|\cos\vartheta|}{2\pi}\,\boldsymbol{\Xi}^T\mathbf{T}_\pm(\vec{k}_{\rm s}) \qquad \text{for}\quad \pm\cos\vartheta>0, \tag{43}$$

where $\boldsymbol{\Xi}^T$ is a 1×4 matrix with entries belonging to $\mathbb{R}^3$ that is given by

$$\boldsymbol{\Xi}^T := \Big[\mathbf{e}_x \qquad \mathbf{e}_y \qquad \sin\theta\sin\varphi\,\mathbf{e}_z \qquad -\sin\theta\cos\varphi\,\mathbf{e}_z\Big], \tag{44}$$

$\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are respectively unit vectors along the x, y, and z axes, $\mathbf{T}_\pm\in\mathscr{F}_k^4$ are the 4-component functions satisfying

$$\widehat{\boldsymbol{\Pi}}_1\mathbf{T}_+=\mathbf{T}_+, \qquad \widehat{\boldsymbol{\Pi}}_2\mathbf{T}_-=\mathbf{T}_-, \qquad \widehat{\boldsymbol{\Pi}}_1\mathbf{T}_-=\widehat{\boldsymbol{\Pi}}_2\mathbf{T}_+=\mathbf{0}, \tag{45}$$

$$\widehat{\boldsymbol{\Pi}}_2\,\widehat{\mathbf{M}}\,\mathbf{T}_- = -4\pi^2\widehat{\boldsymbol{\Pi}}_2\big(\widehat{\mathbf{M}}-\mathbf{I}\big)\boldsymbol{\Upsilon}_{\rm i}\delta_{\vec{k}_{\rm i}}, \tag{46}$$

$$\mathbf{T}_+ = \widehat{\boldsymbol{\Pi}}_1\big(\widehat{\mathbf{M}}-\widehat{\mathbf{I}}\big)\big(\mathbf{T}_- + 4\pi^2\boldsymbol{\Upsilon}_{\rm i}\delta_{\vec{k}_{\rm i}}\big), \tag{47}$$

$\delta_{\vec{k}_{\rm i}}$ is the Dirac delta function in two dimensions centered at $\vec{k}_{\rm i}$, i.e. $\delta_{\vec{k}_{\rm i}}(\vec{p}) := \delta(\vec{p}-\vec{k}_{\rm i})$, and $\vec{k}_{\rm s}$ is the projection of $\mathbf{k}_{\rm s}$ onto the x-y plane. Note that

$$\vec{k}_{\rm i} := k\sin\vartheta_0(\cos\varphi_0\,\mathbf{e}_x+\sin\varphi_0\,\mathbf{e}_y), \qquad \vec{k}_{\rm s} := k\sin\vartheta(\cos\varphi\,\mathbf{e}_x+\sin\varphi\,\mathbf{e}_y).$$

Equation (47) specifies $\mathbf{T}_+$ in terms of $\widehat{\mathbf{M}}$ and $\mathbf{T}_-$. Equation (46) is a linear integral equation for $\mathbf{T}_-$. Dynamical formulation of stationary electromagnetic scattering reduces the scattering problem (finding $\mathbf{F}$) to the calculation of the fundamental transfer matrix and the

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solution of Eq. (46). Substituting the solution of this equation into Eq. (47) and using Eqs. (2) and (43), we obtain the electric field of the scattered wave. References [18,40] offer concrete applications of this approach in the study of electromagnetic point scatterers and the construction of isotropic scatterers that display broadband omnidirectional invisibility.

4. Exactness of the first Born approximation and broadband invisibility

Theorem 1 provides a necessary and sufficient condition for the exactness of the first Born approximation in electromagnetic scattering. We will prove Theorem 2 by showing that its hypothesis implies this condition. This requires some preparation. We begin by introducing some useful notation:

- Given a function $f : \mathbb{R}^3 \to \mathbb{C}$, η_f stands for f − 1, i.e. $\eta_f(\mathbf{r}) := \mathrm{f}(\mathbf{r}) - 1$.
- For each $d \in \mathbb{Z}^+$, we use $\mathfrak{F}_d$ to denote the space of functions $\phi : \mathbb{R}^d \to \mathbb{C}$, and label the d-dimensional Fourier transform of ϕ by $\check{\phi}$.
- For each $\alpha \in \mathbb{R}$, $\mathcal{S}_\alpha := \{\mathfrak{f} \in \mathfrak{F}_1 \,|\, \mathfrak{f}(p) = 0 \text{ for } p \leq \alpha\}$.
- Given $v \in \mathfrak{F}_3$ and $z \in \mathbb{R}$, we introduce the operator $\widehat{\mathscr{V}}(z) := v(\widehat{x}, \widehat{y}, z)$ which acts in the space of functions of $\vec{p} := (p_x, p_y)$ according to

$$\big(\widehat{\mathscr{V}}(z)\phi\big)(\vec{p}) = \langle \vec{p}\,|v(\widehat{x}, \widehat{y}, z)|\phi\rangle = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2q\; \tilde{v}(\vec{p} - \vec{q}, z)\phi(\vec{q}), \tag{48}$$

  where $\tilde{v} := \mathcal{F}v$, i.e. $\tilde{v}(\vec{p}, z) := \int_{\mathbb{R}^2} d^2r\, e^{-i\vec{r}\cdot\vec{p}} v(\vec{r}, z)$. Note also that because $\langle \vec{p}|\widehat{x}|\phi\rangle = i\partial_{p_x}\phi(\vec{p})$ and $\langle \vec{p}|\widehat{y}|\phi\rangle = i\partial_{p_y}\phi(\vec{p})$, we have

$$\big(\widehat{\mathscr{V}}(z)\phi\big)(\vec{p}) = v(i\partial_{p_x}, i\partial_{p_y}, z)\phi(\vec{p}).$$

- For each $k \in \mathbb{R}^+$, let $\widehat{\pi}_k : \mathfrak{F}_2 \to \mathfrak{F}_2$ be the operator defined by

$$(\widehat{\pi}_k\phi)(\vec{p}) := \begin{cases} \phi(\vec{p}) & \text{for} \quad |\vec{p}| < k, \\ 0 & \text{for} \quad |\vec{p}| \geq k. \end{cases} \tag{49}$$

The following lemma lists some of the immediate consequences of the definitions of $\mathcal{S}_\alpha$ and $\widehat{\pi}_k$.

Lemma 1. *Suppose that $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, $k \in \mathbb{R}^+$, and $\mathfrak{f}, \mathfrak{g} \in \mathfrak{F}_1$. Then*

(1) $\mathcal{S}_\beta \subseteq \mathcal{S}_\alpha$.
(2) $\widehat{\pi}_k\mathfrak{f} \in \mathcal{S}_{-k}$.
(3) If $k \leq \alpha$ and $\mathfrak{f} \in \mathcal{S}_\alpha$, $\widehat{\pi}_k\mathfrak{f} = 0$.
(4) If $\mathfrak{f} \in \mathcal{S}_\alpha$, $\mathfrak{f}\mathfrak{g} \in \mathcal{S}_\alpha$. In particular, if $\mathfrak{g} \in \mathcal{S}_\beta$, $\mathfrak{f}\mathfrak{g} \in \mathcal{S}_\beta$.

The following two lemmas reveal less obvious facts about $\mathcal{S}_\alpha$. We give their proofs in Appendix C of Ref. [40].

Lemma 2. *Let $\alpha \in \mathbb{R}$ and $\phi_1, \phi_2, \phi_3 \in \mathfrak{F}_1$ be such that $\check{\phi}_1, \check{\phi}_2 \in \mathcal{S}_\alpha$ and $\phi_3 = \phi_1\phi_2$. Then $\check{\phi}_3 \in \mathcal{S}_{2\alpha} \subseteq \mathcal{S}_\alpha$.*

Lemma 3. *Let $\alpha \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ be a bounded function whose real part is bounded below by a positive number, i.e. there are $m, M \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $0 < m \leq \mathrm{Re}[f(x)] \leq |f(x)| \leq M$. Then there is a sequence of complex numbers $\{c_n\}_1^\infty$ such that the series $\sum_{n=1}^\infty c_n\eta_f(x)^n$ converges absolutely to $\eta_{1/f}(x)$, so that $\eta_{1/f} = \sum_{n=1}^\infty c_n\eta_f^n$. Furthermore, if $\eta_f, \eta_{1/f} \in \mathfrak{F}_1$ and $\check{\eta}_f \in \mathcal{S}_\alpha$, we have $\check{\eta}_{1/f} \in \mathcal{S}_\alpha$.*

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We can use Lemmas 2 and 3 to establish:

Lemma 4. *Let f be as in Lemma 3,* $\mathrm{g} \in \mathfrak{F}_1$, $h := g/f$, *and* $\alpha \in \mathbb{R}$. *Suppose that* $\check{\eta}_f \in \mathcal{S}_\alpha$ *and* $\check{g} \in \mathcal{S}_\alpha$. *Then* $\check{h} \in \mathcal{S}_\alpha$.

Proof. Lemma 2 implies $h = \mathrm{g} + \sum_{n=1}^{\infty} c_n \eta_f^n \mathrm{g}$. This equation together with Lemma 2 and the conditions $\check{\eta}_f \in \mathcal{S}_\alpha$ and $\check{g} \in \mathcal{S}_\alpha$ imply $\check{h} \in \mathcal{S}_\alpha$. □

In Appendix B of Ref. [17], we prove the following lemma.

Lemma 5. *Let* $\phi \in \mathfrak{F}_2$, $v \in \mathfrak{F}_3$, $z \in \mathbb{R}$, $\widehat{\mathscr{V}}(z) := v(\widehat{x}, \widehat{y}, z)$, $\psi \in \mathfrak{F}_2$, *and* $\alpha, \beta \in \mathbb{R}$. *Suppose that for all* $p_y, z \in \mathbb{R}$, $\psi(\cdot, p_y) \in \mathcal{S}_\alpha$ *and* $\tilde{v}(\cdot, p_y, z) \in \mathcal{S}_\beta$. *Then* $\widehat{\mathscr{V}}(z)\psi \in \mathcal{S}_{\alpha+\beta}$.

Next, we present a variation of Lemma 4 of Appendix B of Ref. [17], which follows from the same argument.

Lemma 6. *Let* $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $k \in (0, \alpha]$, $\phi \in \mathfrak{F}_2$, *and* $n \in \mathbb{Z}^+$, *for all* $i \in \{1, 2, \cdots, n\}$, $z_i \in \mathbb{R}$, $v_i \in \mathfrak{F}_3$, *and* $\widehat{\mathscr{V}}_i(z_i) := v_i(\widehat{x}, \widehat{y}, z_i)$, *for all* $j \in \{0, 1, 2, \cdots, n\}$, $\xi_j \in \mathfrak{F}_2$ *and* $\widehat{\xi}_j := \xi_j(\widehat{p}_x, \widehat{p}_y)$, *and*

$$\phi_n := \widehat{\xi}_n \widehat{\mathscr{V}}_n(z_n) \widehat{\xi}_{n-1} \widehat{\mathscr{V}}_{n-1}(z_{n-1}) \widehat{\xi}_{n-2} \cdots \widehat{\xi}_1 \widehat{\mathscr{V}}_1(z_1) \widehat{\xi}_0 \widehat{\pi}_k \phi.$$

Suppose that for all $p_y, z \in \mathbb{R}$ *and* $i \in \{1, 2, \cdots, n\}$, $\tilde{v}_i(\cdot, p_y, z) \in \mathcal{S}_\beta$. *Then* $\phi_n(\cdot, p_y) \in \mathcal{S}_{n\beta-\alpha}$. *In particular,*

$$\widehat{\pi}_k \widehat{\xi}_n \widehat{\mathscr{V}}_n(z_n) \widehat{\xi}_{n-1} \widehat{\mathscr{V}}_{n-1}(z_{n-1}) \widehat{\xi}_{n-2} \cdots \widehat{\xi}_1 \widehat{\mathscr{V}}_1(z_1) \widehat{\xi}_0 \widehat{\pi}_k \tag{50}$$

coincides with the zero operator $\hat{0}$ *if* $\beta \geq \frac{2\alpha}{n}$.

According to this lemma, setting $\beta = 2\alpha$ and $\beta = \alpha$ we respectively obtain

$$\widehat{\pi}_k \widehat{\xi}_1 \widehat{\mathscr{V}}_1(z_1) \widehat{\xi}_0 \widehat{\pi}_k = \widehat{0} \quad \text{if} \quad \tilde{v}_1(p_x, p_y, z) = 0 \quad \text{for} \quad p_x \leq 2\alpha, \tag{51}$$

$$\widehat{\pi}_k \widehat{\xi}_2 \widehat{\mathscr{V}}_2(z_2) \widehat{\xi}_1 \widehat{\mathscr{V}}_1(z_1) \widehat{\xi}_0 \widehat{\pi}_k = \widehat{0} \quad \text{if} \quad \tilde{v}_1(p_x, p_y, z) = \tilde{v}_2(p_x, p_y, z) = 0 \quad \text{for} \quad p_x \leq \alpha. \tag{52}$$

Next, we examine the operator $\delta\widehat{\tilde{\mathbf{H}}}(z)$ of Eq. (32). Clearly, $\delta\widehat{\tilde{\mathbf{H}}}(z) = \mathcal{F}\, \delta\widehat{\mathbf{H}}(z)\mathcal{F}^{-1}$, where $\delta\widehat{\mathbf{H}}(z) := \widehat{\mathbf{H}}(z) - \widehat{\mathbf{H}}_0$. The fact that $\widehat{\mathbf{H}}_0$ is obtained from $\widehat{\mathbf{H}}(z)$ by setting $\hat{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\mu}} = \mathbf{I}$ together with Eqs. (23–28) show that we can obtain $\delta\widehat{\mathbf{H}}(z)$ from the expression for $\widehat{\mathbf{H}}(z)$ by making the following changes:

- In Eqs. (24) and (25): $\frac{1}{\hat{\varepsilon}_{33}} \to \eta_{1/\hat{\varepsilon}_{33}}$ and $\frac{1}{\hat{\mu}_{33}} \to \eta_{1/\hat{\mu}_{33}}$;
- In Eqs. (26–28): $\hat{\varepsilon}_{ij} \to \eta_{\boldsymbol{\varepsilon},ij}$ and $\hat{\mu}_{ij} \to \eta_{\boldsymbol{\mu},ij}$, where $\eta_{\boldsymbol{\varepsilon},ij} := \hat{\varepsilon}_{ij} - \delta_{ij}$, $\eta_{\boldsymbol{\mu},ij} := \hat{\mu}_{ij} - \delta_{ij}$, and δ_{ij} stands for the Kronecker delta symbol.

Employing this prescription to determine the entries of $\delta\widehat{\mathbf{H}}(z)$ and making use of Lemma 6, we find that whenever Condition (12) holds, the Fourier transform with respect to x of all the functions appearing in the expression for $\delta\widehat{\mathbf{H}}(z)$ vanishes for $p_x \leq \alpha$. Furthermore, we can use Eq. (31) to infer that the entries of $\widehat{\mathscr{H}}(z_2)\widehat{\mathscr{H}}(z_1)\widehat{\boldsymbol{\pi}}_k$ are sums of terms of the form in Eq. (50). This together with Eqs. (34), (49), and (52), and the fact that $e^{\pm iz\widehat{\tilde{\mathbf{H}}}_0}$ and $\widehat{\boldsymbol{\pi}}_k$ commute, imply that the quadratic and higher-order terms of the Dyson series (35) vanish. Therefore,

$$\widehat{\mathbf{M}} = \widehat{\boldsymbol{\pi}}_k - i\int_{-\infty}^{\infty} dz\, \widehat{\boldsymbol{\pi}}_k \widehat{\mathscr{H}}(z_1)\widehat{\boldsymbol{\pi}}_k = \widehat{\boldsymbol{\pi}}_k - i\int_{-\infty}^{\infty} dz\, e^{iz\widehat{\tilde{\mathbf{H}}}_0}\widehat{\boldsymbol{\pi}}_k \delta\widehat{\tilde{\mathbf{H}}}(z)\widehat{\boldsymbol{\pi}}_k e^{-iz\widehat{\tilde{\mathbf{H}}}_0}. \tag{53}$$

Substituting the explicit form of $\delta\widehat{\tilde{\mathbf{H}}}(z)$ into $\widehat{\boldsymbol{\pi}}_k\, \delta\widehat{\tilde{\mathbf{H}}}(z)\widehat{\boldsymbol{\pi}}_k$, we find that its entries are sums of terms of the form in Eq. (50) which vanish unless they involve one and only one of $\eta_{\boldsymbol{\varepsilon},ij}$ and

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$\eta_{\boldsymbol{\mu},ij}$. This implies that

$$\widehat{\boldsymbol{\pi}}_k\,\delta\widehat{\widetilde{\mathbf{H}}}(z)\widehat{\boldsymbol{\pi}}_k=\widehat{\boldsymbol{\pi}}_k\begin{bmatrix}\widehat{\mathbf{V}}_{11}&\widehat{\mathbf{V}}_{12}\\ \widehat{\mathbf{V}}_{21}&\widehat{\mathbf{V}}_{22}\end{bmatrix}\widehat{\boldsymbol{\pi}}_k, \tag{54}$$

where

$$\widehat{\mathbf{V}}_{11}:=\vec{P}\,\delta\widehat{\vec{\varepsilon}}_3^{\,T}+i\,\delta\widehat{\vec{K}}_{\mathcal{H}}\vec{P}^T\boldsymbol{\sigma}_2,\qquad \widehat{\mathbf{V}}_{12}:=-\frac{i}{k}\vec{P}\,\widehat{\eta}_{\boldsymbol{\varepsilon}_{33}}\vec{P}^T\boldsymbol{\sigma}_2+k\,\delta\widehat{\mathbf{K}}_{\mathcal{H}}, \tag{55}$$

$$\widehat{\mathbf{V}}_{21}:=\frac{i}{k}\vec{P}\,\widehat{\eta}_{\boldsymbol{\mu}_{33}}\vec{P}^T\boldsymbol{\sigma}_2-k\,\delta\widehat{\mathbf{K}}_{\mathcal{E}},\qquad \widehat{\mathbf{V}}_{22}:=\vec{P}\,\delta\widehat{\vec{\mu}}_3^{\,T}+i\,\delta\widehat{\vec{K}}_{\mathcal{E}}\vec{P}^T\boldsymbol{\sigma}_2, \tag{56}$$

$$\vec{P}:=\begin{bmatrix}p_x\\ p_y\end{bmatrix},\qquad \delta\widehat{\vec{\varepsilon}}_\ell:=\begin{bmatrix}\widehat{\eta}_{\boldsymbol{\varepsilon},\ell1}\\ \widehat{\eta}_{\boldsymbol{\varepsilon},\ell2}\end{bmatrix},\qquad \delta\widehat{\vec{\mu}}_\ell:=\begin{bmatrix}\widehat{\eta}_{\boldsymbol{\mu},\ell1}\\ \widehat{\eta}_{\boldsymbol{\mu},\ell2}\end{bmatrix}, \tag{57}$$

$$\delta\widehat{\vec{K}}_{\mathcal{E}}:=\begin{bmatrix}-\widehat{\eta}_{\boldsymbol{\varepsilon},23}\\ \widehat{\eta}_{\boldsymbol{\varepsilon},13}\end{bmatrix},\qquad \delta\widehat{\mathbf{K}}_{\mathcal{E}}=\begin{bmatrix}-\delta\widehat{\vec{\varepsilon}}_2^{\,T}\\ \delta\widehat{\vec{\varepsilon}}_1^{\,T}\end{bmatrix},\qquad \delta\widehat{\vec{K}}_{\mathcal{H}}:=\begin{bmatrix}-\widehat{\eta}_{\boldsymbol{\mu},23}\\ \widehat{\eta}_{\boldsymbol{\mu},13}\end{bmatrix}, \tag{58}$$

$$\delta\widehat{\mathbf{K}}_{\mathcal{H}}=\begin{bmatrix}-\delta\widehat{\vec{\mu}}_2^{\,T}\\ \delta\widehat{\vec{\mu}}_1^{\,T}\end{bmatrix},\qquad \widehat{\eta}_{\boldsymbol{\varepsilon},ij}:=\eta_{\boldsymbol{\varepsilon},ij}(i\partial_{p_x},i\partial_{p_y},z),\qquad \widehat{\eta}_{\boldsymbol{\mu},ij}:=\eta_{\boldsymbol{\mu},ij}(i\partial_{p_x},i\partial_{p_y},z), \tag{59}$$

and we have also benefitted from Lemmas 3 and 4.

In view of the argument leading to Eq. (53), Condition (12), and the fact that $\widehat{\boldsymbol{\pi}}_k$ commutes with $\widehat{\boldsymbol{\Pi}}_2$, we have

$$\big(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\big)\widehat{\boldsymbol{\Pi}}_2\big(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\big)=\widehat{\mathbf{0}}, \tag{60}$$

where $\widehat{\mathbf{0}}$ is the zero operator acting in $\mathscr{F}^4$. This identity allows us to solve Eq. (46) for $\mathbf{T}_-$. To see this, we use Eqs. (36) and (45) to write Eq. (46) in the form

$$\mathbf{T}_-=-\widehat{\boldsymbol{\Pi}}_2(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k)(\mathbf{T}_-+4\pi^2\boldsymbol{\Upsilon}_{\rm i}\delta_{\vec{k}_{\rm i}}). \tag{61}$$

Applying $\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k$ to both sides of this equation and making use of Eq. (60), we obtain $(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k)\mathbf{T}_-=\mathbf{0}$. Substituting this relation into Eqs. (46) and (61), we are led to

$$\mathbf{T}_+=4\pi^2\widehat{\boldsymbol{\Pi}}_1\big(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\big)\boldsymbol{\Upsilon}_{\rm i}\delta_{\vec{k}_{\rm i}}, \tag{62}$$

$$\mathbf{T}_-=-4\pi^2\widehat{\boldsymbol{\Pi}}_2\big(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\big)\boldsymbol{\Upsilon}_{\rm i}\delta_{\vec{k}_{\rm i}}. \tag{63}$$

Next, we examine the transformation property of $\mathbf{T}_\pm$ under Eq. (4). In view of Eqs. (53–59), (62), and (63), the scaling transformation (4) implies $\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\to\varsigma\big(\widehat{\mathbf{M}}-\widehat{\boldsymbol{\pi}}_k\big)$ and $\mathbf{T}_\pm\to\varsigma\,\mathbf{T}_\pm$. Using this in Eq. (43), we find that the electric field of the scattered wave (2) scales as $\mathbf{E}_s\to\varsigma\,\mathbf{E}_s$. By virtue of Theorem 1, this establishes the exactness of the first Born approximation.

To arrive at a direct proof of the exactness of the first Born approximation, we have substituted Eq. (53) into Eqs. (62) and (63), and used Eqs. (2), (43), and (55–59) to determine the explicit form of $\mathbf{E}_s$. After lengthy calculations we have shown that the resulting formula for $\mathbf{E}_s$ coincides with the one obtained by performing the first Born approximation, namely the one given by Eqs. 4.18 and 4.29 of Ref. [4]. This provides a highly nontrivial check on the validity of our analysis.

For incident waves with wavenumber $k\le\alpha/2$, we can use Eq. (51) to show that $\widehat{\boldsymbol{\pi}}_k\delta\widehat{\widetilde{\mathbf{H}}}(z)\widehat{\boldsymbol{\pi}}_k=\widehat{\mathbf{0}}$. Therefore, $\widehat{\mathbf{M}}=\widehat{\boldsymbol{\pi}}_k$, and Eqs. (62) and (63) give $\mathbf{T}_\pm=\mathbf{0}$. In view of Eqs. (2) and (43), this implies $\mathbf{F}=\mathbf{E}_{\rm s}=\mathbf{0}$ which means that the medium does not scatter the wave. Since this result is not sensitive to the direction of the incident wave vector, the medium is omnidirectionally invisible in the wavenumber spectrum $(0,\alpha/2)$. This extends a result of Ref. [40] to anisotropic media.

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5. Concluding remarks

The Born approximation has been an indispensable tool for performing quantum and electromagnetic scattering calculations since its introduction in 1926 [1]. It is therefore rather surprising that the discovery of conditions for its exactness had to wait until 2019 when such a condition was found in the context of the dynamical formulation of stationary scattering for scalar waves in two dimensions [14]. This condition emerged in an attempt to truncate the Dyson series for the fundamental matrix. It turned out to allow for an exact solution of the scattering problem leading to a formula that was identical to the one obtained by the first Born approximation. The extension of this condition to potential scattering in three dimensions is rather straightforward [17]. This is by no means true for its generalization to electromagnetic scattering because of the transverse vectorial nature of electromagnetic waves and tensorial nature of the interaction potentials $\boldsymbol{\eta}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\eta}_{\boldsymbol{\mu}}$. Progress in this direction required the development of a dynamical formulation of stationary electromagnetic scattering which was realized quite recently [18].

The condition for the exactness of the first Born approximation for the scattering of electromagnetic waves shares the basic features of the corresponding condition in potential scattering, and it is quite simple to state and realize. Yet establishing the fact that this condition actually implies the exactness of the first Born approximation requires overcoming serious technical difficulties.

The discovery of a sufficient condition for the exactness of the first Born approximation may be viewed as a basic but at the same time formal contribution to the vast subject of scattering theory. One must however note that systems satisfying this condition are exactly solvable. Therefore, imposing this condition yields a very large class of exactly solvable scattering problems. As it should be clear from the two examples we have provided in Sect. 2, it is possible to satisfy this condition for permittivity and permeability profiles whose expressions involve arbitrary functions of two of the coordinates, e.g. the function f(y, z) of Eqs. (15) and (17). In principle, one can choose these functions so that the system has certain desirable scattering features. Because the formula given by the first Born approximation specifies the scattered wave in terms of the 3D Fourier transform of the relative permittivity and permeability tensors [4], one can determine the specific form of $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$ by performing inverse Fourier transform of the scattering data. This corresponds to an electromagnetic analog of a well-known approximate inverse scattering scheme for scalar waves that relies on the first Born approximation [41,42]. If one manages to enforce the condition we have provided for the exactness of the first Born approximation, this scheme becomes exact. This suggests that our results may be used to develop a certain exact but conditional inverse scattering scheme. The study of the details and prospects of this scheme is the subject of a future investigation.

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Data availability

No new data were generated or analysed in support of this research.

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