FOLIATIONS OF HYPERBOLIC SPACE BY CONSTANT MEAN CURVATURE HYPERSURFACES

BARIS COSKUNUZER

ABSTRACT. We show that the constant mean curvature hypersurfaces in $\mathbf{H}^{n+1}$ spanning the boundary of a star shaped $C^{1,1}$ domain in $S^n_\infty(\mathbf{H}^{n+1})$ give a foliation of $\mathbf{H}^{n+1}$. We also show that if Γ is a closed codimension-1 $C^{2,\alpha}$ submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique constant mean curvature hypersurface Σ_H in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma_H = \Gamma$ for any $H \in (-1, 1)$, then the constant mean curvature hypersurfaces $\{\Sigma_H\}$ foliates $\mathbf{H}^{n+1}$.

1. INTRODUCTION

In this paper, we are interested in the question of the existence of a foliation of $\mathbf{H}^{n+1}$ with constant mean curvature (CMC) hypersurfaces asymptotic to the given codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. CMC hypersurfaces in hyperbolic space became an area of active research after the progress on minimal hypersurfaces in hyperbolic space. In [A1], Anderson showed the existence of area minimizing hypersurfaces in $\mathbf{H}^{n+1}$ for any given codimension-1 closed submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. He also showed that for any mean convex asymptotic boundary, there exists a unique minimal hypersurface in hyperbolic space. Then, Hardt and Lin studied the regularity of these area minimizing hypersurfaces, and generalized Anderson's uniqueness result to star-shaped domains in $S^n_\infty(\mathbf{H}^{n+1})$ in [HL].

In the following decade, there have been many important generalizations of these results to CMC hypersurfaces in hyperbolic space. In [To], Tonegawa generalized Anderson's existence, and Hardt and Lin's regularity results for CMC hypersurfaces by using geometric measure theory methods. In the same year, by using similar techniques, Alencar and Rosenberg got a similar existence result in [AR]. By using analytical techniques, Nelli and Spruck generalized Anderson's result on uniqueness of minimal hypersurfaces for mean convex domains to CMC context in [NS]. Then, Guan and Spruck extended Hardt and Lin's result of uniqueness of minimal hypersurfaces for star-shaped domains to CMC hypersurfaces case in [GS].

After these uniqueness results, the question of how these CMC surfaces live together in $\mathbf{H}^3$ became interesting. Indeed, in [CV], Chopp and Velling studied this problem by using computational methods, and had an interesting result that for many different type of curves in $S^2_\infty(\mathbf{H}^3)$, CMC surfaces give a foliation of $\mathbf{H}^3$. However, they do not classify any type of curves to give a foliation of $\mathbf{H}^3$,

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but for some chosen special curves, they showed the existence of a foliation with computer aid. In this paper, we show that for some specific classes of codimension-1 submanifolds in $S^n_\infty(\mathbf{H}^{n+1})$, the foliation of $\mathbf{H}^{n+1}$ by CMC surfaces exist.

Theorem 3.3. Let Γ be the boundary of a star-shaped $C^{1,1}$ domain in $S^n_\infty(\mathbf{H}^{n+1})$. Then, there exists a foliation of $\mathbf{H}^{n+1}$ by CMC hypersurfaces $\{\Sigma_H\}$ where Σ_H is a CMC hypersurface with mean curvature $H \in (-1,1)$ and $\partial_\infty \Sigma_H = \Gamma$.

We also show that a similar result is true for any closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ bounding a unique CMC hypersurface in $\mathbf{H}^{n+1}$ for any given $H \in (-1,1)$.

Theorem 4.2. Let Γ be a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. Also assume that for any $H \in (-1,1)$, there exists a unique CMC hypersurface Σ_H with $\partial_\infty \Sigma_H = \Gamma$. Then, the collection of CMC hypersurfaces $\{\Sigma_H\}$ with $H \in (-1,1)$ foliates $\mathbf{H}^{n+1}$.

The organization of the paper as follows. In Section 2, we give the basic definitions and results which we use throughout the paper. In Section 3, we show the star-shaped case. In Section 4, we prove the general case. Finally in Section 5, we have some concluding remarks.

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2. PRELIMINARIES

In this section, we will overview the basic results which we use in the following sections. Note that the results and the notions in this section can also be found in the survey article [Co3]. Let Σ^n be a compact hypersurface bounding a domain Ω^{n+1} in some ambient Riemannian manifold X. Let A be the area of Σ, and V be the volume of Ω. Let's vary Σ through a one parameter family Σ_t, with the corresponding area $A(t)$ and volume $V(t)$. If f is the normal component of the variation, and H is the mean curvature of Σ, then we get $A'(0) = -\int_\Sigma nHf$, and $V'(0) = \int_\Sigma f$ where n is the dimension of Σ, and H is the mean curvature.

Let Σ be a hypersurface with boundary Γ in X. Assume also that there exists a hypersurface M with $\partial M = \Gamma$ such that $M \cup \Sigma = \partial\Omega$ where Ω is a codimension-0 submanifold in X, and mean curvature of M is greater than $H_0 > 0$ everywhere (inward direction of Ω). Define $V(t)$ to be the volume of the domain Ω_t bounded by M and Σ_t. Now, we define a new functional as a combination of A and V. Let $I_H(t) = A(t) + nHV(t)$ for $H < H_0$. Note that $I_0(t) = A(t)$. If Σ is a critical point of the functional I_H for any variation f, then this will imply that Σ has constant mean curvature H [Gu], [AR]. Note that the critical points of the functional I_H are independent of the choice of the hypersurface M since if $\widehat{I}_H$ is the functional which is defined with a different hypersurface $\widehat{M}$, then $I_H - \widehat{I}_H = C$

for some constant C. In particular, $H = 0$ is the special case of minimal surfaces, for which the theory is very well developed [Ni], [CM].

Definition 2.1. Let Σ be a hypersurface in a Riemannian manifold X. Σ is called *minimal hypersurface* if it is critical point of I_0 (Area Functional) for any variation. Equivalently, Σ has constant mean curvature 0 at every point. Σ is called an *area minimizing hypersurface* if Σ is the absolute minimum of the functional I_0 (having the smallest area) among hypersurfaces with the same boundary.

For general H, the theory is similar, and called the constant mean curvature (CMC) case.

Definition 2.2. Let Σ be a hypersurface in a Riemannian manifold X. Σ is called *CMC hypersurface* if it is a critical point of I_H for any variation. Equivalently, Σ has constant mean curvature H at every point. Σ is a *minimizing CMC hypersurface* if Σ is the absolute minimum of the functional I_H among hypersurfaces with the same boundary.

Notation: From now on, we will call CMC hypersurfaces with mean curvature H as *H-hypersurfaces*.

We will call any complete noncompact hypersurface Σ_H as *minimizing H-hypersurface (area minimizing hypersurface)* if any compact codimension-0 submanifold with boundary of Σ_H is a minimizing H-hypersurface (area minimizing hypersurface).

After these general definitions of minimal and H-hypersurfaces, we will quote some basic facts about the H-hypersurfaces in hyperbolic space.

First, we fix a codimension-1 closed submanifold Γ in $S^n_\infty(\mathbf{H}^{n+1})$. Γ separates $S^n_\infty(\mathbf{H}^{n+1})$ into two parts, say Ω^+ and Ω^-. By using these domains, we will give orientation to hypersurfaces in $\mathbf{H}^{n+1}$ asymptotic to Γ. With this orientation, mean curvature H is positive if the mean curvature vector points towards positive side of the hypersurface, negative otherwise. The following fact is known as maximum principle.

Lemma 2.1. [*Maximum Principle*] *Let Σ_1 and Σ_2 be two hypersurfaces in a Riemannian manifold which intersect at a common point tangentially. If Σ_2 lies in positive side (mean curvature vector direction) of Σ_1 around the common point, then H_1 is less than or equal to H_2 ($H_1 \leq H_2$) where H_i is the mean curvature of Σ_i at the common point. If they do not coincide in a neighborhood of the common point, then H_1 is strictly less than H_2 ($H_1 < H_2$).*

With a simple application of this maximum principle, by using horospheres in $\mathbf{H}^{n+1}$, it is easy to show that if Σ is a complete H-hypersurface in $\mathbf{H}^{n+1}$ asymptotic to a codimension-1 submanifold Γ of $S^n_\infty(\mathbf{H}^{n+1})$, then $|H| < 1$ [Co1].

The following existence theorem for minimizing H-hypersurfaces in $\mathbf{H}^{n+1}$ asymptotic to Γ for a given codimension-1 closed submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ was proved by Tonegawa [To], and Alencar-Rosenberg [AR] independently by using geometric measure theory methods.

Theorem 2.2. [To], [AR] *Let Γ be a codimension-1 closed submanifold in $S^n_\infty(\mathbf{H}^{n+1})$, and let $|H| < 1$. Then there exists a minimizing H-hypersurface Σ in $\mathbf{H}^{n+1}$ where $\partial_\infty \Sigma = \Gamma$. Moreover, any such minimizing H-hypersurface is smooth except a closed singularity set of dimension at most $n-7$.*

Now, we define the convex hull of a subset A of $S^n_\infty(\mathbf{H}^{n+1})$ in $\mathbf{H}^{n+1}$. If γ is a round $n-1$-sphere in $S^n_\infty(\mathbf{H}^{n+1})$, then there is a unique geodesic plane P in $\mathbf{H}^{n+1}$ asymptotic to γ. γ separates $S^n_\infty(\mathbf{H}^{n+1})$ into two parts Δ^+ and Δ^-. Similarly, P divides $\mathbf{H}^{n+1}$ into two halfspaces D^+ and D^- with $\partial_\infty D^\pm = \Delta^\pm$. We will call the halfspace whose asymptotic boundary contains A as *supporting halfspace*. i.e. if $A \subset \Delta^+$, then D^+ is a supporting halfspace.

Definition 2.3. Let A be a subset of $S^n_\infty(\mathbf{H}^{n+1})$. Then the *convex hull* of A, $CH(A)$, is the smallest closed convex subset of $\mathbf{H}^{n+1}$ which is asymptotic to A. Equivalently, $CH(A)$ can be defined as the intersection of all supporting closed half-spaces of $\mathbf{H}^{n+1}$ [EM].

Note that the asymptotic boundary of the convex hull of a subset of $S^n_\infty(\mathbf{H}^{n+1})$ is the subset itself, i.e. $\partial_\infty(CH(A)) = A$. In general, we say the hypersurface Σ has the convex hull property if it is in the convex hull of its boundary, i.e. $\Sigma \subset CH(\partial\Sigma)$. In special case, if Σ is a complete and noncompact hypersurface in $\mathbf{H}^{n+1}$, then we say Σ has convex hull property if it is in the convex hull of its asymptotic boundary, i.e. $\Sigma \subset CH(\partial_\infty\Sigma)$. The minimal hypersurfaces in $\mathbf{H}^{n+1}$ have convex hull property [A1]. There is also a generalization of this property to H-hypersurfaces in $\mathbf{H}^{n+1}$ [Co1].

Now, fix Γ and orient all spheres accordingly. If T is a round $n-1$-sphere in $S^n_\infty(\mathbf{H}^{n+1})$, then there is a unique H-hypersurface P_H in $\mathbf{H}^{n+1}$ asymptotic to T for $-1 < H < 1$ [NS]. T separates $S^n_\infty(\mathbf{H}^{n+1})$ into two parts Δ^+ and Δ^-. Similarly, P_H divides $\mathbf{H}^{n+1}$ into two domains D^+_H and D^-_H with $\partial_\infty D^\pm_H = \Delta^\pm$. We will call these regions as *H-shifted halfspaces*. If the asymptotic boundary of a H-shifted halfspace contains Γ, then we will call this H-shifted halfspace as *supporting H-shifted halfspace*. i.e. if $A \subset \Delta^+$, then D^+_H is a supporting H-shifted halfspace.

Definition 2.4. Let Γ be a codimension-1 submanifold of $S^n_\infty(\mathbf{H}^{n+1})$. Then the *$H$-shifted convex hull* of Γ, $CH_H(\Gamma)$ is defined as the intersection of all supporting closed H-shifted halfspaces of $\mathbf{H}^{n+1}$.

Now, the generalization of convex hull property of minimal hypersurfaces in $\mathbf{H}^{n+1}$ to H-hypersurfaces in $\mathbf{H}^{n+1}$ is as follows [Co1]. Similar versions of this result have been proved by Alencar-Rosenberg in [AR], and by Tonegawa in [To].

Lemma 2.3. [To], [AR], [Co1] *Let Σ be a H-hypersurface in $\mathbf{H}^{n+1}$ where $\partial_\infty\Sigma = \Gamma$ and $|H| < 1$. Then Σ is in the H-shifted convex hull of Γ, i.e. $\Sigma \subset CH_H(\Gamma)$.*

The next result which we give in this section is also a generalization of a result on area minimizing hypersurfaces to H-hypersurfaces.

Lemma 2.4. [Co1] *Let Γ_1 and Γ_2 be two disjoint codimension-1 closed submanifolds in $S^n_\infty(\mathbf{H}^{n+1})$. If Σ_1 and Σ_2 are minimizing H-hypersurfaces in $\mathbf{H}^{n+1}$ where $\partial_\infty \Sigma_i = \Gamma_i$, then Σ_1 and Σ_2 are disjoint, too.*

Now, we give a regularity result for H-hypersurfaces near infinity due to Tonegawa.

Lemma 2.5. [To] *Let Γ be $C^{k,\alpha}$ codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ where $1 \leq k \leq n-1$ and $0 \leq \alpha \leq 1$ or $k = n$ and $0 \leq \alpha < 1$. If Σ is a complete CMC hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma = \Gamma$, then $\Sigma \cup \Gamma$ is a $C^{k,\alpha}$ submanifold with boundary in $\overline{\mathbf{H}^{n+1}}$ near Γ.*

We finish this section with the strong maximum principle due to Simon [Si]. The maximum principle we mentioned above, Lemma 2.1, is true for smooth hypersurfaces. Since we are working with codimension-1 rectifiable currents (Theorem 2.2), by the regularity results of the geometric measure theory, they might have $n-7$ dimensional singular set. Hence, we need a stronger version of the maximum principle which applies to codimension-1 rectifiable currents. Simon originally proved the following result for area minimizing rectifiable currents, and it naturally extends to minimizing H-hypersurfaces, too. Other versions of strong maximum principle for more general settings can also be found in [SW] and [Il].

Lemma 2.6. [Si] [*Strong Maximum Principle*] *Let T_1 and T_2 be two minimizing H-hypersurfaces in a Riemannian manifold. Let U be an open subset of N such that $\partial T_1 = \partial T_2 = 0$ in U, and reg $T_1 \cap$ reg $T_2 \cap U = \emptyset$. Then spt $T_1 \cap$ spt $T_2 \cap U = \emptyset$.*

In the proof of the main theorem of [Si], T_i is described as the boundary of a codimension-0 rectifiable current E_i. By defining E_i as in the description of minimizing H-hypersurfaces in Section 1 of [AR] (or Section 2 of [To]), the whole proof of [Si] goes through for minimizing H-hypersurfaces case, too.

3. CMC Foliation for Star Shaped Boundary

In this section, we will show that if Γ is the boundary of a star-shaped $C^{1,1}$ domain in $S^n_\infty(\mathbf{H}^{n+1})$, then there exists a foliation of $\mathbf{H}^{n+1}$ by CMC hypersurfaces $\{\Sigma_H\}$ where Σ_H is a H-hypersurface with $H \in (-1,1)$ and $\partial_\infty \Sigma_H = \Gamma$.

First, we need to show that the CMC hypersurfaces $\{\Sigma_H\}$ sharing ideal boundary Γ are disjoint.

Lemma 3.1. *Let Γ be the boundary of a star-shaped domain in $S^n_\infty(\mathbf{H}^{n+1})$. If Σ_{H_1} and Σ_{H_2} are minimizing CMC hypersurfaces in $\mathbf{H}^{n+1}$ with $-1 < H_1 < H_2 < 1$, then Σ_{H_1} and Σ_{H_2} are disjoint.*

Proof: Let's take upper half space model for $\mathbf{H}^{n+1}$. Let origin be the *center* point of the star shaped domain Ω^+ which Γ bounds in $S^n_\infty(\mathbf{H}^{n+1})$. Let γ be the unique geodesic in $\mathbf{H}^{n+1}$ connecting the origin and ∞-point (the vertical line at 0). Let $\varphi_t(x) = tx$ be the hyperbolic isometry which is a translation along the geodesic γ. Then, $\Gamma_t = \varphi_t(\Gamma)$ be also a boundary of a star shaped domain in $S^n_\infty(\mathbf{H}^{n+1})$. Note that as Γ is boundary of a star shaped domain, $\Gamma_t \cap \Gamma_s = \emptyset$ for any $t \neq$

s. Hence, the family of closed codimension-1 submanifolds $\{\Gamma_t\}$ in $S^n_\infty(\mathbf{H}^{n+1})$ foliates $S^n_\infty(\mathbf{H}^{n+1}) - \{0,\infty\}$, i.e. $S^n_\infty(\mathbf{H}^{n+1}) - \{0,\infty\} = \bigcup_{t\in(0,\infty)} \Gamma_t$.

Let Σ_{H_i} be the minimizing H_i-hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma_{H_i} = \Gamma$. We suppose the mean curvature vectors of Σ_{H_1} and Σ_{H_2} point into the domains of H^{n+1} containing Ω_+ ($0 \leq H_1 < H_2$). The same argument adapts to the other cases. Let $\Sigma^t_{H_1} = \varphi_t(\Sigma_{H_1})$. Then $\partial_\infty \Sigma^t_{H_1} = \Gamma_t$. By Lemma 2.4, $\Sigma^t_{H_1} \cap \Sigma^s_{H_1} = \emptyset$ where $t \neq s$. As φ_t is continuous family of isometries, this implies the family of minimizing H_1-hypersurfaces $\{\Sigma^t_{H_1}\}$ foliates $\mathbf{H}^{n+1}$, i.e. $\mathbf{H}^{n+1} = \bigcup_{t\in(0,\infty)} \Sigma^t_{H_1}$.

If $\Sigma_{H_1} \cap \Sigma_{H_2} \neq \emptyset$, then let $t_0 = \inf_{t\in(1,\infty)}\{t \,:\, \Sigma^t_{H_1} \cap \Sigma_{H_2} = \emptyset\}$. By H-shifted convex hull property (Lemma 2.3), there exists an infimum $1 < t_0 < \infty$. This implies $\Sigma^{t_0}_{H_1}$ intersects Σ_{H_2} tangentially and $\Sigma^{t_0}_{H_1}$ lies in positive side of the Σ_{H_2}. However, by maximum principle (Lemma 2.1), this is a contradiction as $H_1 < H_2$. □

The more general case of the lemma above will be proved in the following section. The following result is essential for the proof of the main result of this section.

Lemma 3.2. [GS] *Let Γ be the boundary of a star-shaped $C^{1,1}$ domain in $S^n_\infty(\mathbf{H}^{n+1})$ and $|H| < 1$. Then there exists a unique H-hypersurface Σ_H in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma_H = \Gamma$.*

Now, we show that the unique H-hypersurfaces in the above result indeed foliates $\mathbf{H}^{n+1}$.

Theorem 3.3. *Let Γ be the boundary of a star-shaped $C^{1,1}$ domain in $S^n_\infty(\mathbf{H}^{n+1})$ and $|H| < 1$. Then there exists a foliation of $\mathbf{H}^{n+1}$ by CMC hypersurfaces $\{\Sigma_H\}$ where Σ_H is a H-hypersurface in $\mathbf{H}^{n+1}$ and $\partial_\infty \Sigma_H = \Gamma$.*

Proof: By Lemma 3.2, for any $H \in (-1,1)$, there exists a minimizing H-hypersurface Σ_H with $\partial_\infty \Sigma_H = \Gamma$. By Lemma 3.1, for $H_1 \neq H_2$, $\Sigma_{H_1} \cap \Sigma_{H_2} = \emptyset$. To prove the theorem, first we will show that there is no gap between two H-hypersurface in the collection $\{\Sigma_H\}$. Then, we prove that $\bigcup_{-1<H<1} \Sigma_H$ fills the whole $\mathbf{H}^{n+1}$. Observe that by Lemma 2.5, $\overline{\Sigma}_H$ is a C^1 submanifold in $\overline{\mathbf{H}^{n+1}}$, and hence Σ_H is separating in $\mathbf{H}^{n+1}$. Let $S^n_\infty(\mathbf{H}^{n+1}) - \Gamma = \Omega^+ \cup \Omega^-$ and $\mathbf{H}^{n+1} - \Sigma_H = D^+_H \cup D^-_H$ with $\partial_\infty D^\pm_H = \Omega^\pm$.

Step 1: There is no gap between two H-hypersurface in the collection $\{\Sigma_H\}$. i.e. $D^-_{H_0} = \bigcup_{H<H_0} D^-_H$ and $D^+_{H_0} = \bigcup_{H>H_0} D^+_H$.

Proof: Assume that there is a gap in the collection $\{\Sigma_H\}$. i.e. $D^-_{H_0} \supsetneq \bigcup_{H<H_0} D^-_H$ (The other case is similar). Let $\{H_i \,:\, H_i \in (H_0-\epsilon, H_0),\ i \in \mathbf{Z}^+\}$ be an increasing sequence with $H_i \to H_0$. Our aim is to build a sequence $\{S_i\}$ of compact H_i-hypersurfaces with $S_i \subset \Sigma_{H_i}$ such that a subsequence S_{i_j} converges to an H_0-hypersurface $\widehat{\Sigma}_{H_0}$ with $\partial_\infty \widehat{\Sigma}_{H_0} = \Gamma$. Then, by using $\widehat{\Sigma}_{H_0}$ and star-shapedness of Γ (Lemma 3.2), we get a contradiction and prove that there cannot be any gap between in the collection $\{\Sigma_H\}$.

Now, we construct the sequence $\{S_i\}$ of compact H-hypersurfaces with $S_i \subset \Sigma_{H_i}$. Let x be a point in the H-shifted convex hull of Γ, $CH_{H_0}(\Gamma)$. By Lemma 2.5, the regularity theorem for H-hypersurfaces near infinity [To], for sufficiently small $\rho > 0$, $\Sigma_H \cap \{x_{n+1} < \rho\}$ is a graph over $\Gamma \times (0, \rho)$ in upper half space model for $\mathbf{H}^{n+1}$, i.e. $\Sigma_H \cap \{x_{n+1} < \rho\} = u_H(\Gamma \times (0, \rho))$ for some function u_H. Let R_0 be sufficiently large so that $\partial(B_{R_0}(x) \cap \Sigma_H) \subset \{x_{n+1} < \rho\}$ in upper half space model for $\mathbf{H}^{n+1}$ (Note that $\partial B_{R_0}(x)$ has mean curvature $\coth R_0 > 1 > H_0$). Let $\{R_i\}$ be an increasing sequence with $R_i > R_0$ for any i, and $R_i \to \infty$ as $i \to \infty$. Then, $\partial(B_{R_i}(x) \cap \Sigma_{H_i}) = \gamma_i = u_{H_i}(\alpha_i)$ where α_i is an (n-1)-sphere in $\Gamma_0 \times (0, \rho)$. Hence, define $S_i = B_{R_i}(x) \cap \Sigma_{H_i}$. Then, $\partial S_i = \gamma_i$ and $\gamma_i \to \Gamma$ as $i \to \infty$. Also, for some sufficiently large K, let $N_K(CH_{H_0}(\Gamma))$ be the K-neighborhood of the H_0-shifted convex hull of Γ. Since for any H_i, $\Sigma_{H_i} \subset CH_{H_i}(\Gamma)$, by replacing ϵ if necessary, we can assume that $S_i \subset N_K(CH_{H_0}(\Gamma))$ for any i.

Now, for any i, S_i is the minimizer of the functional $I_{H_i}(t) = A(t) + nH_iV(t)$ for fixed boundary $\gamma_i \subset \partial B_{R_i}(x)$, where A is the area of the codimension-1 submanifold in $B_{R_i}(x)$, and V is the volume of the component which the submanifold separates in $B_{R_i}(x)$ by [To], [AR]. Observe that $H_i \to H_0$ and $S_i \subset N_K(CH_{H_0}(\Gamma))$ with $\partial S_i = \gamma_i \to \Gamma$. By adapting the proof of theorem 3 in [AR], we get a subsequence $S_{i_j} \to \widehat{\Sigma}_{H_0}$ where $\widehat{\Sigma}_{H_0}$ is a minimizing H_0-hypersurface with $\partial_\infty \widehat{\Sigma}_{H_0} = \Gamma$ as follows. Let $\Delta_n = B_n(x)$ where $n \in \mathbf{N}$ with $n > N_0$ where N_0 is sufficiently large that $N_K(CH_{H_0}(\Gamma))$ separates Δ_{N_0} into two parts. We claim that for any $n > N_0$, we can find uniform area bounds $c_n, C_n > 0$ such that $c_n < |S_i \cap \Delta_n| < C_n$. Then by using the compactness theorem for integral currents [Fe], we get the desired limit $\widehat{\Sigma}_{H_0}$. Fix $n > N_0$. For any $i > n$, γ_i is not nullhomologous in $N_K(CH_{H_0}(\Gamma)) - \Delta_n$ by construction. Hence, $S_i \cap \Delta_n \neq \emptyset$. However, since S_i is a minimizing H_i-hypersurface, $|S_i \cap \Delta_n| < |\partial \Delta_n|$ by Corollary 1.1 of [AR], and let $C_n = |\partial \Delta_n|$.

Now, we define c_n. Let α be a round (n-1)-sphere in Ω^+ and P be the unique H-hypersurface which is a hyperplane (an equidistant sphere) with $\partial_\infty P = \alpha$ and $P \cap N_K(CH_{H_0}(\Gamma)) = \emptyset$. Further assume that, $P \cap \Delta_{N_0} \neq \emptyset$, by modifying N_0 if necessary. Recall that $N_K(CH_{H_0}(\Gamma))$ separates Δ_n into two components. Fix a point q_n in the component not intersecting P. For any point p in $P \cap \Delta_n$, let l_p be the unique geodesic segment connecting p and q_n. We claim that any geodesic segment l_p starting at $p \in P \cap \Delta_n$ intersects $S_i \cap \Delta_n$ for any i. We can see this as follows. $N_K(CH_{H_0}(\Gamma))$ separates Δ_n into two parts. Any l_p connects these two parts in Δ_n (Note that since Δ_n is convex, $l_p \subset \Delta_n$). Since l_p connects different components of $\mathbf{H}^{n+1} - N_K(CH_{H_0}(\Gamma))$ in Δ_n, and γ_i is not nullhomologous in $N_K(CH_{H_0}(\Gamma)) - \Delta_n$, γ_i cannot be nullhomologous in $N_K(CH_{H_0}(\Gamma)) - l_p$ either. Hence, for any i, $S_i \cap l_p \neq \emptyset$. Since $\Delta_n \cap N_K(CH_{H_0}(\Gamma))$ is compact, the geodesic projection of $S_i \cap \Delta_n$ from q_n onto P has bounded distortion. Then, there exists $\lambda_n > 0$ such that $|S_i \cap \Delta_n| \geq \lambda_n |P \cap \Delta_n|$ and define $c_n = \lambda_n |P \cap \Delta_n|$. Hence, we found uniform area bounds $c_n, C_n > 0$ such that $c_n < |S_i \cap \Delta_n| < C_n$. Then by using the compactness theorem for integral currents ([Fe], Theorem 4.2.17) and isoperimetric inequality, for each fixed n, a subsequence $S_{i_j} \cap \Delta_n$ converges

to a minimizing H_0-minimizing hypersurface $\widehat{\Sigma}^n_{H_0}$ in Δ_n (by the lower bound c_n, $\widehat{\Sigma}^n_{H_0}$ is nonempty for each n). By repeating the argument for each $n > N_0$ for the new sequence, the diagonal sequence argument gives a subsequence S_{i_j} converging to an integral n-current $\widehat{\Sigma}_{H_0}$ on any compact set, in the weak topology. By construction, $\widehat{\Sigma}_{H_0}$ is a minimizing H_0-hypersurface in $\mathbf{H}^{n+1}$ with $\partial_\infty \widehat{\Sigma}_{H_0} = \Gamma$. Note also that by construction, $\widehat{\Sigma}_{H_0}$ may not be smooth everywhere, and it might have at most $n-7$ dimensional singular set by the regularity theorems of the geometric measure theory [Fe].

Now, because of the gap, $\widehat{\Sigma}_{H_0}$ cannot be same with Σ_{H_0}. Hence, we get two different minimizing H_0-hypersurface with asymptotic boundary Γ. We cannot directly use the uniqueness statement in Lemma 3.2, and get the contradiction because the uniqueness statement is for the smooth hypersurfaces while $\widehat{\Sigma}_{H_0}$ may not be smooth everywhere. Hence, we get the contradiction as follows. We know that Γ is star shaped. Let φ_t, Γ_t, and $\Sigma^t_{H_0} = \varphi_t(\Sigma_{H_0})$ be as in the proof of previous lemma. Since $\Gamma_t \cap \Gamma_s = \emptyset$ when $t \neq s$, by Lemma 2.4 and Lemma 2.6, $\widehat{\Sigma}_{H_0} \cap \Sigma^t_{H_0} = \emptyset$ for any $t \neq 1$. Since $\Sigma^t_{H_0}$ foliates whole $\mathbf{H}^{n+1}$ by construction, this implies $\widehat{\Sigma}_{H_0} = \Sigma^1_{H_0} = \Sigma_{H_0}$, which is a contradiction. The other cases are similar. Hence, this shows that there is no gap between two H-hypersurface in the collection $\{\Sigma_H\}$. □

Step 2: The collection $\{\Sigma_H\}$ fills $\mathbf{H}^{n+1}$, i.e. $\mathbf{H}^{n+1} = \bigcup_{-1<H<1} \Sigma_H$.

Proof: Assume that $\mathbf{H}^{n+1} \neq \bigcup_{-1<H<1} \Sigma_H$. Recall that $S^n_\infty(\mathbf{H}^{n+1}) - \Gamma = \Omega^+ \cup \Omega^-$ and $\mathbf{H}^{n+1} - \Sigma_H = D^+_H \cup D^-_H$ with $\partial_\infty D^\pm_H = \Omega^\pm$. If $\mathbf{H}^{n+1} \neq \bigcup_{-1<H<1} \Sigma_H$, then either $\bigcap_{-1<H<1} D^+_H \neq \emptyset$ or $\bigcap_{-1<H<1} D^-_H \neq \emptyset$. Note that $D^+_{H_1} \subsetneq D^+_{H_2}$ when $H_1 < H_2$, and likewise $D^-_{H_1} \subsetneq D^-_{H_2}$ when $H_1 > H_2$. Without loss of generality, assume that $Z = \bigcap_{-1<H<1} D^-_H \neq \emptyset$. Now, the idea is to construct a minimizing 1-hypersurface $\widehat{\Sigma}_1$ with $\partial_\infty \widehat{\Sigma}_1 = \Gamma$, by using the construction in previous part. In the previous construction, we used $N_K(CH_{H_0}(\Gamma))$ as barrier, however we cannot define 1-shifted convex hull $CH_1(\Gamma)$ by definition. Instead of $CH_1(\Gamma)$, we define $\widehat{C} = N_K(X)$ where $X = D^-_{1-\epsilon} - Z$. Then for any $H_i \subset (1-\epsilon, 1)$, $\Sigma_{H_i} \subset \widehat{C}$. Let $H_i \subset (1-\epsilon, 1)$ and $H_i \nearrow 1$, and define the sequence $\{S_i\}$ of compact H_i-hypersurfaces with $S_i = B_{R_i}(x) \cap \Sigma_{H_i}$ as before. Hence, we can get the upper bounds C_n such that $|\Delta_n \cap S_i| < C_n = |\partial B_n(x)|$ as before. For lower bound c_n, since we cannot define 1-hypersurface P with $\partial_\infty P = \alpha$ in that construction, we use $(1-\epsilon)$-hypersurface P' with $\partial_\infty P' = \alpha$ to define the lower area bounds c_n to ensure that the limit is nonempty. An alternative way to show that we get nonempty limit is as follows. Let z be a point in Z, let y be a point in $\Sigma_{1-\epsilon}$, and let β be a path between z and y. Since $D^-_{H_1} \subsetneq D^-_{H_2}$ when $H_1 > H_2$, for any i, $\beta \cap S_i \neq \emptyset$ and let w_i be a point in $\beta \cap S_i$. As β compact, any subsequence of $\{w_i\}$ will have a limit point in $\mathbf{H}^{n+1}$. Hence $\{S_i\}$ cannot escape to the infinity, and we have a nonempty limit. Hence, the proof in previous part goes

through, and a subsequence S_{i_j} converges to a minimizing 1-hypersurface $\widehat{\Sigma}_1$ with $\partial_\infty \widehat{\Sigma}_1 = \Gamma$.

Now, this gives a contradiction as there cannot be a 1-hypersurface in $\mathbf{H}^{n+1}$ because of the horospheres (± 1-hypersurfaces), and the maximum principle. In particular, let p be a point in $\Omega^+ \subset S^n_\infty(\mathbf{H}^{n+1})$. Let H^p_R be the horospheres in $\mathbf{H}^{n+1}$ with $\partial_\infty H^p_R = \{p\}$ and R is the radius of the horosphere in Euclidean metric on $\overline{\mathbf{H}^{n+1}}$ in Poincare ball model (Hence, $0 < R < 1$). Let $c = \sup\{0 < R < 1 \mid H^p_R \cap \widehat{\Sigma}_1 = \emptyset\}$. Then, H^p_c is a minimizing 1-hypersurface which intersects $\widehat{\Sigma}_1$ at a point q (the point of first touch) and completely lie in one side of $\widehat{\Sigma}_1$. If q is a regular (smooth) point of $\widehat{\Sigma}_1$, this is a contradiction by the maximum principle (Lemma 2.1). If q is not a regular point of $\widehat{\Sigma}_1$ then the contradiction comes from the strict maximum principle in [Si] (Lemma 2.6). □

By Step 1 and Step 2, the collection of minimizing H-hypersurfaces $\{\Sigma_H\}$ gives a foliation of $\mathbf{H}^{n+1}$, hence the proof follows. □

4. CMC Foliation for General Case

In this section, we will show that if Γ is a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$ such that for any $H \in (-1, 1)$, there exists a unique CMC hypersurface Σ_H with $\partial_\infty \Sigma_H = \Gamma$, then the collection of CMC hypersurfaces $\{\Sigma_H\}$ foliates $\mathbf{H}^{n+1}$. In other words, in this section, we will show that the star-shapedness condition on Γ in previous section is not essential, but bounding a unique CMC hypersurface for any $H \in (-1, 1)$ is the key condition.

First, we will generalize Lemma 3.1, the pairwise disjointness result in previous section to a new setting.

Lemma 4.1. *Let Γ be a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. If Σ_{H_1} and Σ_{H_2} are minimizing CMC hypersurfaces in $\mathbf{H}^{n+1}$ with $\partial_\infty \Sigma_{H_i} = \Gamma_i$ and $-1 < H_1 < H_2 < 1$, then Σ_{H_1} and Σ_{H_2} are disjoint.*

Proof: By Lemma 2.5, the regularity theorem for H-hypersurfaces near infinity [To], for sufficiently small $\rho > 0$, $\Sigma_H \cap \{x_{n+1} < \rho\}$ is a graph over $\Gamma \times (0, \rho)$ in upper half space model for $\mathbf{H}^{n+1}$, i.e. $\Sigma_H \cap \{x_{n+1} < \rho\} = u_H(\Gamma \times (0, \rho))$. Let $p \in \mathbf{H}^{n+1}$ be a point, and $R_0 > 0$ be sufficiently large that $\partial B_{R_0}(p) = S_{R_0}(p) \subset \{x_{n+1} < \rho\}$ (Note that $S_{R_0}(p)$ has mean curvature $\coth R_0 > 1 > H_2$). By p.599 in [To], the estimate $u_y = \frac{H}{\sqrt{1-H^2}}\sqrt{1 + |\nabla\varphi(x)|^2}$ implies that for sufficiently small $\epsilon > 0$, $u_{H_1}(x, \epsilon) < u_{H_2}(x, \epsilon)$ where $x \in \Gamma$ and $0 \le H_1 < H_2$. Hence, if $\gamma_i = S_{R_0}(p) \cap \Sigma_{H_i}$, then $\gamma_1 \cap \gamma_2 = \emptyset$.

Let $S_i = \Sigma_{H_1} \cap B_{R_0}(p)$ and $\gamma_i = \partial S_i$. Also, let Ω_i be the region which S_i separates in $B_{R_0}(p)$ where the mean curvature vector on S_i points outside of Ω_i (without loss of generality, we assume $0 < H_1 < H_2 < 1$. If both negative, just change the direction. If different signs, same argument works as area minimizing hypersurface Σ_0 would be a barrier between them [AR]). An alternative way to think about this picture is the Poincare ball model in $\mathbf{H}^{n+1}$ (See Figure 1). Hence, if we can show that Ω_2 is strictly included in Ω_1, we are done (S_1 cannot touch S_2



FIGURE 1. For $0 < H_1 < H_2 < 1$, S_1 is above S_2 near the boundary of the ball $B_{R_0}(p)$ by [To]

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because of maximum principle, Lemma 2.1). This is because this implies $S_1 \cap S_2 = \emptyset$, and we already know $\Sigma_{H_1} - S_1$ is disjoint from $\Sigma_{H_2} - S_2$ by [To].

Assume that $\Omega_2 - \Omega_1 \neq \emptyset$. Since Σ_{H_i} are minimizing H_i-hypersurfaces, then so is S_i, and hence $|S_i| + nH_i|\Omega_i|$ are minimizers for the functional $I_{H_i}(t) = A(t) + nH_iV(t)$ for the fixed boundary γ_i. Consider $\Omega_2' = \Omega_2 \cap \Omega_1$ and $S_2' = \partial\Omega_2' - S_{R_0}(p)$ which is another hypersurface in $B_{R_0}(p)$ with $\partial S_2' = \gamma_2$. Let $Q = \Omega_2 - \Omega_1$. Let $T_1 = \partial Q \cap S_1$, $T_2 = \partial Q \cap S_2$ with $\widehat{S}_2 = S_2 \cap \Omega_1$. Then, $S_2 = \widehat{S}_2 \cup T_2$, $S_2' = \widehat{S}_2 \cup T_1$, and $\Omega_2 = \Omega_2' \cup Q$. Since S_2 is a minimizing H_2-hypersurface, $|S_2| + nH_2|\Omega_2| \leq |S_2'| + nH_2|\Omega_2'|$. This implies $(|\widehat{S}_2| + |T_2|) + nH_2(|\Omega_2'| + |Q|) \leq (|\widehat{S}_2| + |T_1|) + nH_2|\Omega_2'|$. After cancelations, we get $|T_2| + nH_2|Q| \leq |T_1|$.

Similarly, let $\widehat{S}_1 = S_1 - T_1$ and $S_1' = \widehat{S}_1 \cup T_2$ and $\Omega_1' = \Omega_1 \cup Q$. Again, since S_1 is minimizing H_1-hypersurface, then $|S_1| + nH_1|\Omega_1| \leq |S_1'| + nH_1|\Omega_1'|$. This implies $(|\widehat{S}_1| + |T_1|) + nH_1(|\Omega_1|) \leq (|\widehat{S}_1| + |T_2|) + nH_1(|\Omega_1| + |Q|)$. After cancelations, we get $|T_1| \leq |T_2| + nH_1|Q|$. If we combine this with the previous inequality, we get $|T_2| + nH_2|Q| \leq |T_1| \leq |T_2| + nH_1|Q|$. This implies $H_2 \leq H_1$ which is a contradiction. □

In a sense, this lemma is a generalization of Lemma 3.1. Now, we generalize Theorem 3.3. The technique is basically same.

Theorem 4.2. *Let Γ be a $C^{2,\alpha}$ closed codimension-1 submanifold in $S^n_\infty(\mathbf{H}^{n+1})$. If for any $H \in (-1, 1)$, there exists a unique CMC hypersurface Σ_H with $\partial_\infty\Sigma_H =$*

Γ, *then, the collection of CMC hypersurfaces* $\{\Sigma_H\}$ *with* $H \in (-1,1)$ *foliates* $\mathbf{H}^{n+1}$.

Proof: By Lemma 4.1, $\Sigma_{H_1} \cap \Sigma_{H_2} = \emptyset$ for $H_1 \neq H_2$. By using Lemma 4.1 instead of Lemma 3.1, the proof of Step 1 of Theorem 3.3 adapts to this case easily, hence there cannot be any gap in the collection $\{\Sigma_H\}$. Similarly, the proof of Step 2 of Theorem 3 also goes through in this case, and this shows $\{\Sigma_H\}$ fill out the whole $\mathbf{H}^{n+1}$. The proof follows. □

5. Final Remarks

First we should note that Nelli-Spruck's uniqueness result does not automatically give a foliation when it is combined with our result.

Theorem 5.1. [NS] *Let* Ω *be a* $C^{2,\alpha}$ *mean convex domain in* $S^n_\infty(\mathbf{H}^{n+1})$ *and* $\Gamma = \partial\Omega$. *Then for each* $H \in (0,1)$ *there exists a complete embedded hypersurface* M *of* $\mathbf{H}^{n+1}$ *of constant mean curvature* H *with* $\partial_\infty M = \Gamma$. *Moreover,* M *can be represented as a graph* $x_{n+1} = u(x)$ *over* Ω *with* $u \in C^{2,\alpha}(\overline{\Omega})$ *and there is a unique such a graph.*

Unfortunately, this result for $C^{2,\alpha}$ mean convex domains does not give uniqueness in general. Other than the obvious reason $H \in (0,1)$, it only asserts the uniqueness of the graphs, where there still might be other CMC hypersurfaces which may not be represented as graph. So, for this case, Theorem 4.2 cannot be applied directly. However, for the question whether the collection of CMC hypersurfaces $\{\Sigma_H\}_{H\in(0,1)}$ in the theorem above is a foliation of graphs or not, our techniques might be used, if one can show that the pairwise disjointness of the graphs, and that the limit of the graphs is also a graph in that setting. Hence, with pairwise disjointness, the only possibility not to have a foliation would be to have a gap, but the existence of a gap in the foliation would contradict the uniqueness of graphs in the theorem above.

For the nonexistence question, by using the techniques of [Co2], it is possible to show the existence of a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that there is no foliation of $\mathbf{H}^3$ by CMC hypersurfaces $\{\Sigma_H\}$ with $\partial_\infty \Sigma_H = \Gamma$. In particular, such a foliation implies the uniqueness of not only minimizing CMC hypersurfaces, but also any type of CMC hypersurfaces with asymptotic boundary Γ. In dimension 3, by Hass's result in [Ha] and Anderson's result in [A2], there are examples of Jordan curves in $S^2_\infty(\mathbf{H}^3)$ bounding many minimal surfaces in $\mathbf{H}^3$. Hence, for those curves in $S^2_\infty(\mathbf{H}^3)$, there cannot be a foliation of $\mathbf{H}^3$ by CMC surfaces because of the maximum principle.

On the other hand, recently in [Wa], Wang showed that if a quasi-Fuchsian 3-manifold M contains a minimal surface whose principle curvature is less than 1, than M admits a foliation by CMC surfaces by using volume preserving mean curvature flow. If we lift this foliation to the universal cover, we get a foliation of $\mathbf{H}^3$ by CMC surfaces with same asymptotic boundary which is the limit set of M. However, the limit set of quasi-Fuchsian manifolds are far from being smooth, even they contain no rectifiable arcs ([Be]). Existence of one smooth point in the limit

set implies the group being Fuchsian, which means the limit set is a round circle in $S^2_\infty(\mathbf{H}^3)$. Even though the foliation constructed in [Wa] induces a foliation of $\mathbf{H}^3$ by CMC surfaces, since the limit sets in $S^2_\infty(\mathbf{H}^3)$ are nonrectifiable, Wang's results and the results in this paper should be considered in different contexts. Also in [Wa], Wang constructs a limit set Γ of a quasi-Fuchsian 3-manifold which is similar to the one in [Ha], where there cannot be a foliation of $\mathbf{H}^3$ by CMC surfaces with asymptotic boundary Γ. This also implies the existence of a quasi-Fuchsian 3-manifold which has no foliation by CMC surfaces.

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KOC UNIVERSITY, DEPARTMENT OF MATHEMATICS, SARIYER, ISTANBUL 34450 TURKEY
E-mail address: bcoskunuzer@ku.edu.tr