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A Note on Weakly Compact Homomorphisms Between Uniform Algebras

by

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Summary. In this paper we present some results about weakly compact homomorphisms from a uniform algebra A into a uniform algebra B, which is an integral domain. The main result says that if $h : A \to B$ is a weakly compact homomorphism then $h^*(\Sigma_B)$ meets one, and only one, Gleason part of the spectrum Σ_A of A.

1. Introduction. In this paper we study weakly compact homomorphisms from a uniform algebra A (for the undefined terminology used in this section we refer the reader to the next section) into a uniform algebra B, which is an integral domain. That is, for a, b in B, the equality $ab = 0$ implies that $a = 0$ or $b = 0$. We recall that the disk algebra $A(D)$, the Hardy class $H^\infty(D)$, and in general the uniform algebras whose definitions involve analytic functions on a domain in the complex plane, as well as the subalgebras of these algebras, are integral domains. As a motivating result, let $\varphi \in A(D)$, $\|\varphi\| \leqslant 1$, be a given function, and consider the homomorphism $h : A(D) \to A(D)$, defined by $h(a) = a \circ \varphi$. Then, as proved by H. Kamowitz in [1], h is compact if, and only if, either φ is constant or $\varphi(\overline{D}) \subset D$. This result, rephrased in terms of the Gleason parts, states that the homomorphism h is compact if, and only if, $h^*(\Sigma)$ is contained in a single Gleason part of Σ, where Σ is the Gelfand spectrum of $A(D)$. The main result of the present paper is the following: Let A and B be two uniform algebras with B integral domain, and $h : A \to B$ be a weakly compact homomorphism. Then $h^*(\Sigma_B)$ meets only one Gleason part of Σ_A, where Σ_A and Σ_B are the Gelfand spectra of the algebras A and B, respectively. In particular,

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an arbitrary homomorphism $h : H^\infty(D) \to B$ is compact if, and only if, $h^*(\Sigma_B)$ meets exactly one Gleason part of $\Sigma_{H^\infty(D)}$. The main ingredient of the proofs are a result of [2] about Gleason parts of the uniform algebras and a result of Bourgain [3] stating that $H^\infty(D)$ has the Grothendieck property.

2. Notation and Preliminaries. Our notation and terminology are quite standard. For any Banach space X, we denote by X^* its continuous dual and by X_1 its closed unit ball. We always consider X as naturally embedded into its second dual X^{**}. For x in X and μ in X^* by $\langle x, \mu\rangle$ (and also by $\langle \mu, x\rangle$) we denote the natural duality between X and X^*.

A uniform algebra A on a compact Hausdorff space K is a closed subalgebra of the algebra of the continuous functions $C(K)$ from K into the field of the complex numbers, endowed with the supremum norm, that contains the unit element of $C(K)$ and separates the points of K. The spectrum of A is denoted by Σ_A. We denote the elements of A by small italic letters $a, b \ldots$; those of A^* by Greek letters $\lambda, \mu \ldots$; and those of A^{**} by m, n,... etc. The relation $\backsim$ defined on Σ_A by "$\lambda \backsim \mu$ if, and only if, $\|\lambda - \mu\| < 2$" is an equivalence relation. The corresponding equivalence classes are the "Gleason parts of Σ_A" [4, p. 67]. A one-point Gleason part of Σ_A is said to be a trivial part.

Uniform algebras being Arens regular, on the second dual A^{**} of any uniform algebra A there exists a unique separately weak*-weak* continuous Banach algebra multiplication extending that of A. This multiplication is known as the "Arens multiplication of A^{**}". We shall always consider A^{**} equipped with this multiplication, under which it is a commutative Banach algebra. Thus, for n, m in A^{**}, nm denotes the product of n and m in A^{**}. The Arens multiplication of A^{**} is constructed in three steps as follows: Let a, b be in A; λ in A^*; and, n, m be in A^{**}. We define the elements $a\lambda$, $m\lambda$ of A^* and nm of A^{**} by the identities.

$\langle \lambda.a, b\rangle = \langle ab, \lambda\rangle$
$\langle m\lambda, a\rangle = \langle m, \lambda.a\rangle$
$\langle nm, \lambda\rangle = \langle n, m\lambda\rangle$.

Ample information about Arens multiplications and Arens regularity notion can be found in the books [5] and [6]. Finally we remark that each $\lambda \in \Sigma_A$ is also a multiplicative functional on A^{**} so that for n, m in A,

$$\langle nm, \lambda\rangle = \langle n, \lambda\rangle\langle m, \lambda\rangle.$$

3. Main results. The letters A and B will denote two uniform algebras. The compact spaces on which they are realized are irrelevant for our purpose, thus we do not mention them.

Let P be a Gleason part of Σ_A. The main ingredient of the proofs is the following lemma, which, although not explicitly stated there, is contained in the paper [2, Theorem 2.2]. For the sake of completeness we include a proof of it. In this lemma by $I_P : \Sigma_A \to \{0,1\}$, $I_P(\lambda) = \begin{cases} 1 & \text{if } \lambda \in P, \\ 0 & \text{if } \lambda \notin P, \end{cases}$ we denote the characteristic function of P.

LEMMA 1. *There exists a linear functional $m_P : A^* \to C$ of norm one and extending I_P.*

Proof. The proof uses the Markov-Kakutani Fixed Point Theorem [7, p. 456]. We recall that two elements λ and μ in Σ_A are not equivalent if, and only if, there exists a sequence $(a_n)_{n\in N}$ in the unit ball of A such that $\langle \lambda, a_n \rangle \to 1$ and $\langle \mu, a_n \rangle \to 0$, see [4, p. 68]. Now fix a $\lambda \in P$. Then, for a given $\mu \in \Sigma_A \setminus P$, there exists a sequence $(a_n)_{n\in N}$ in A_1, such that $\langle a_n, \lambda \rangle \to 1$ and $\langle a_n, \mu \rangle \to 0$. Let m be any weak* cluster point of the sequence $(a_n)_{n\in N}$ in A^{**}. Then $\langle m, \lambda \rangle = 1$ and $\langle m, \mu \rangle = 0$. As $< m, \lambda >= 1$, the set $K = \{n \in A_1^{**} : \langle n, \lambda \rangle = 1\}$ is nonempty. Clearly this set is convex and weak* compact. For $q \in K$, let $\tau_q : K \to K$ be the affine mapping defined by $\tau_q(n) = qn$. Since A^{**} is commutative and τ_q is continuous on K in its relative weak* topology, by the Markov-Kakutani Fixed Point Theorem, there exists an element m_P in K such that, for all $n \in K$, $m_P.n = m_P$. Now, for any $\mu \in \Sigma_A \setminus P$ and any $n \in K$, $\langle \mu, n \rangle = 0$ since λ and μ are not equivalent, see [4, p. 68, Lemma 4]. It follows that $m_P = 0$ on $\Sigma_P \setminus P$. On the other hand, since $\langle m_P, \lambda \rangle = 1$, we must also have $\langle m_P, \mu \rangle = 1$ for any $\mu \in P$, see [4, p. 68, Lemma 4] again. Thus m_P extends I_P. As $m_P \in K$ and $\langle m_P, \lambda \rangle = 1$, the norm of m_P is one. □

Now let $h : A \to B$ be a (necessarily continuous) homomorphism. We remark that, since A is semisimple, h is identically zero if, and only if, $h^{**}(m_P) = 0$ for each Gleason part P of Σ_A. We also remark that, for any two distinct Gleason parts P and Q of Σ_A, $m_P m_Q = 0$ on Σ_A. The next result is the key result of the paper. In this and subsequent results, by a homomorphism we mean a non identically zero homomorphism.

THEOREM 2. *Assume that B is an integral domain, and let $h : A \to B$ be a weakly compact homomorphism. Then $h^*(\Sigma_B)$ meets one, and only one, Gleason part of Σ_A.*

Proof. We first observe that, since h is weakly compact, h^{**} maps A^{**} into B. We also remark that, since h is assumed to be non identically zero, for at least one Gleason part, say P of Σ_A, $h^{**}(m_P) \neq 0$. Thus, by the above remark, for any Gleason part Q of Σ_A distinct from P, $h^{**}(m_P m_Q) =$

$h^{**}(m_P)\ h^{**}(m_Q) = 0$ in B. Since B is an integral domain, we conclude that $h^{**}(m_Q) = 0$ for any Gleason part $Q \neq P$ of Σ_A. From this we get that, for any $\lambda \in \Sigma_B$ and any Gleason part $Q \neq P$ of Σ_A, $\langle h^{**}(m_Q), \lambda\rangle = \langle m_Q, h^*(\lambda)\rangle = 0$. It follows that $h^*(\Sigma_B) \setminus \{0\} \subseteq P$ so that $h^*(\Sigma_B)$ meets only the Gleason part P of Σ_A. □

Next we present some corollaries of this theorem.

COROLLARY 3. *Assume that B is an integral domain, and let $h : H^\infty(D) \to B$ be a homomorphism. Then the following assertions are equivalent.*

a) *h is compact.*

b) *h is weakly compact.*

c) *$h^*(\Sigma_B)$ meets only one Gleason part of $\Sigma_{H^\infty(D)}$.*

d) *$h^*(\Sigma_B)$ is norm separable.*

P r o o f . The implication a) $\Longrightarrow$ b) is obvious. The implication b) $\Longrightarrow$ c) follows from the preceding theorem. To prove the implication c) $\Rightarrow$ d), assume that $h^*(\Sigma_B) \setminus \{0\}$ is contained in a single Gleason part of $\Sigma_{H^\infty(D)}$. Then, by [2, Corollary 4.8], the set $h^*(\Sigma_B)$ is norm compact, and thus norm separable. To prove the implication d) $\Rightarrow$ a), suppose that $h^*(\Sigma_B)$ is norm separable. Then, under the weak* topology induced by $\sigma(H^\infty(D)^*, H^\infty(D))$, the space $h^*(\Sigma_B)$ is metrizable (see [2, Lemma 4.6]). Since, as proved in Bourgain [3], the Banach space $H^\infty(D)$ has the Grothendieck property (i.e. in $H^\infty(D)^*$ weak* convergent sequences are weakly convergent), $h^*(\Sigma_B)$ is weakly compact. But by [2, Corollary 4.8], weakly compact subsets of $\Sigma_{H^\infty(D)}$ are norm compact. Hence $h^*(\Sigma_B)$, so its absolutely convex closed hull $\overline{\operatorname{aco} h^*(\Sigma_B)}$, is norm compact. Then the weak* closed absolutely convex hull of $h^*(\Sigma_B)$, which is the same as $\overline{\operatorname{aco} h^*(\Sigma_B)}$, is norm compact. Since $h^*(B_1^*)$ is equal to the weak* closed absolutely convex hull of $h^*(\Sigma_B)$, we conclude that h^*, and thus h is compact. □

The next corollary is immediate since $H^\infty(D)$ is an integral domain.

COROLLARY 4. *Assume that the uniform algebra A is such that all the Gleason parts of Σ_A are trivial (e.g. $A = C(K)$), and $h : A \to H^\infty(D)$ be a homomorphism. Then h is weakly compact if, and only if, the range of h is one-dimensional.*

Let $\varphi : \Sigma_{H^\infty(D)} \to \Sigma_{H^\infty(D)}$ be a function such that, for each $a \in H^\infty(D)$, $\widehat{a} \circ \varphi$ is analytic on D. Let $h : H^\infty(D) \to H^\infty(D)$ be the homomorphism defined by $h(a) = \widehat{a} \circ \varphi$. The Gleason parts of $H^\infty(D)$ are described by K. Hoffman in [8]. As proved in this paper, if P is a non trivial Gleason part of $\Sigma_{H^\infty(D)}$, then there exists a continuous bijective mapping $L : D \to P$ such

that, for each $a \in H^\infty(D)$, $\widehat{a} \circ L$ is analytic on D. The next result, which is also immediate, is the exact analog of the Kamowitz's result mentioned above.

COROLLARY 5. *The homomorphism $h(a) = \widehat{a} \circ \varphi$ is (weakly) compact if, and only if, either φ is constant or $\widehat{\varphi}(\Sigma_{H^\infty(D)}) \subset L(D)$ for some Hoffman's analytic mapping L corresponding to a non trivial Gleason part P of $\Sigma_{H^\infty(D)}$.*

For closely related results we refer the reader to the paper [9]. Finally, we remark that, as one can see it easily, combining Theorem 2 above with Theorem 4.7 of [2], Corollary 3 above remains valid if we replace there the algebra $H^\infty(D)$ by any logmodular uniform algebra.

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