Check for updates

Received: 15 June 2022 | Revised: 5 April 2023 | Accepted: 14 April 2023

DOI: 10.1002/mana.202200256

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

On the local well-posedness of the 1D Green–Naghdi system on Sobolev spaces

Hasan İnci

Department of Mathematics, Koç University, Istanbul, Türkiye

Correspondence
Hasan İnci, Department of Mathematics, Koç University, Istanbul, Türkiye.
Email: hinci@ku.edu.tr

Abstract

In this paper, we consider the local well-posedness of the 1D Green–Naghdi system. This system describes the evolution of water waves over an uneven bottom in the shallow water regime in terms of the water depth h and the horizontal velocity u. Using a Lagrangian formulation of this system on a Sobolev-type diffeomorphism group, we prove local well-posedness for (h, u) in the Sobolev space $([1-\xi]+H^s(\mathbb{R}))\times H^{s+1}(\mathbb{R})$, $s>1/2$, where $\xi : \mathbb{R}\to\mathbb{R}$ is the parameterization of the bottom and where we assume that the water surface has an equilibrium at height 1. This improves the present local well-posedness range by one degree.

KEYWORDS

1D Green–Naghdi system, diffeomorphism group, local well-posedness

1 | INTRODUCTION

We consider for $t\geq 0$, $x\in\mathbb{R}$, the 1D Green–Naghdi system in the setting of a nonflat bottom as given in [9]

$$\begin{aligned} &h_t+(hu)_x=0,\\ &h(u_t+uu_x)+\left[\frac{1}{2}h^2g+\frac{1}{2}h^2\xi_x(u_t+uu_x)+\frac{1}{2}h^2u^2\xi_{xx}+\frac{1}{3}h^3(u_x^2-uu_{xx}-u_{tx})\right]_x \\ &=-h\xi_xg-h\xi_x^2(u_t+uu_x)-hu^2\xi_x\xi_{xx}-\frac{1}{2}h^2\xi_x(u_x^2-uu_{xx}-u_{tx}),\\ &h(t=0)=h_0,\ u(t=0)=u_0, \end{aligned} \tag{1}$$

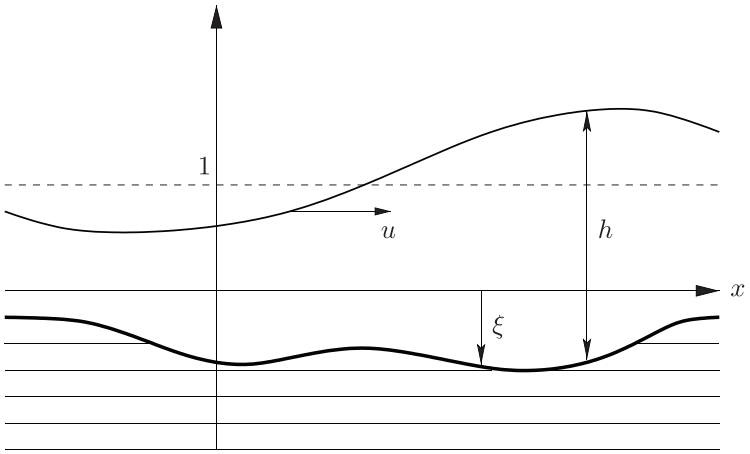
where $h(t,x)\in\mathbb{R}$ is the water depth, $u(t,x)\in\mathbb{R}$ is the horizontal velocity of the water wave, $\xi(x)\in\mathbb{R}$ is the shape of the bottom, and g is the gravitational constant (Figure 1). The Green–Naghdi system (1) models 1D wave propagation over an uneven bottom in the shallow water regime, that is, when the typical wavelength is much larger than the water depth. It is used as a model in coastal oceanography. The 2D version of (1) for a variable bottom is derived in a paper of Green and Naghdi [4]. The name "Green-Naghdi system" originates from this paper.

The case of a flat bottom, that is, when $\xi\equiv 0$, reads as

$$h_t+(hu)_x=0,\quad t\geq 0,\ x\in\mathbb{R},$$

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made.
© 2023 The Authors. *Mathematische Nachrichten* published by Wiley-VCH GmbH.

FIGURE 1 Water wave with speed u and water depth h.



$$h(u_t + uu_x) + ghh_x + \frac{1}{3}\left(h^3(u_x^2 - uu_{xx} - u_{tx})\right)_x = 0, \quad t \geq 0,\ x \in \mathbb{R}, \tag{2}$$

$$h(t=0) = h_0,\ u(t=0) = u_0.$$

Model (2) appeared in a paper by Serre [2] and later in [10]. People refer to (2) sometimes also as the "Serre-Green-Naghdi system."

Assuming that the water surface has an equilibrium at height 1, we denote by $\bar{h}$ the equilibrium water depth, that is,

$$\bar{h} = 1 - \xi.$$

For the equilibrium water depth, we will always assume

$$\inf_{x\in\mathbb{R}} \bar{h}(x) = \inf_{x\in\mathbb{R}} (1 - \xi(x)) > 0.$$

The functional spaces we consider for (1) are Sobolev spaces. Recall that for $s \geq 0$, the Sobolev space $H^s(\mathbb{R})$ is defined as

$$H^s(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid \|f\|_s < \infty\},$$

where the Sobolev norm $\|\cdot\|_s$ is given by

$$\|f\|_s = \left(\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s |\hat{f}(\xi)|^2\, d\xi\right)^{1/2}.$$

Here, we denote by $\hat{f}$ the Fourier transform of f. Sobolev spaces of negative order are defined as dual spaces

$$H^{-s}(\mathbb{R}) = (H^s(\mathbb{R}))',\ s \geq 0,$$

endowed with the dual norm, which we denote by $\|\cdot\|_{-s}$. The Sobolev Imbedding Theorem tells us that $H^s(\mathbb{R}) \hookrightarrow C_0^k(\mathbb{R})$ for $s > 1/2 + k$ and $k \geq 0$. Here, $C_0^k(\mathbb{R})$ is the space of C^k functions on $\mathbb{R}$, which vanish at infinity together with their derivatives up to order k. With this we can state our local well-posedness result for (1) as follows:

Theorem 1.1. *Let $s > 1/2$, $\xi \in H^{s+3}(\mathbb{R})$, and $\bar{h} := 1 - \xi$ satisfying $\inf_{x\in\mathbb{R}} \bar{h}(x) > 0$. Suppose that we are given $(h_0, u_0) \in (\bar{h} + H^s(\mathbb{R})) \times H^{s+1}(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} h_0(x) > 0$. Then, there is $T > 0$ and a unique solution (h, u) to (1) on $[0, T]$ of class*

$$(h, u) \in C([0,T]; (\bar{h} + H^s(\mathbb{R})) \times H^{s+1}(\mathbb{R})) \cap C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})) \times H^s(\mathbb{R})),$$

with $\inf_{x\in\mathbb{R}} h(t,x) > 0,\ 0 \leq t \leq T$. Moreover, the solution depends continuously on the data (h_0, u_0).

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Remark. In the statement of Theorem 1.1, one could lower the regularity requirements to $\xi \in L^\infty(\mathbb{R})$ and $\xi_x \in H^{s+2}(\mathbb{R})$. Another option for the bottom ξ would be to take $\xi \in C_b^{\lceil s\rceil+3}(\mathbb{R})$. Some statements of this paper require less regularity for ξ. But for the sake of uniformity, we will always assume $\xi \in H^{s+3}(\mathbb{R})$.

In [7], it was proved that (1) is locally well-posed for $(h,u) \in (\bar{h} + H^s(\mathbb{R})) \times H^{s+1}(\mathbb{R}),\ s > 3/2$. In [8], it was shown that the 1D Green–Naghdi system over a flat bottom (2) is locally well-posed for $(h,u) \in (1 + H^s(\mathbb{R})) \times H^{s+1}(\mathbb{R}),\ s > 3/2$. So Theorem 1.1 improves the local well-posedness range by one degree.

The first step in the proof of Theorem 1.1 is to rewrite (1) in nonlocal form. To do so, let us define for $h(t,x)$ and $\xi(x)$ the operator

$$A_{h,\xi} : u \mapsto h(1+\xi_x^2)u + \left[\frac{1}{2}h^2\xi_x u\right]_x - \frac{1}{2}h^2\xi_x u_x - \left[\frac{1}{3}h^3 u_x\right]_x.$$

With this, we have

$$\begin{aligned} A_{h,\xi}(u_t + uu_x) = h(1+\xi_x^2)(u_t+uu_x) + \left[\frac{1}{2}h^2\xi_x(u_t+uu_x)\right]_x \\ - \frac{1}{2}h^2\xi_x(u_{tx} + u_x^2 + uu_{xx}) - \left[\frac{1}{3}h^3\left(u_{tx} + uu_{xx} + u_x^2\right)\right]_x. \end{aligned}$$

Thus, using the second equation in (1), we can write

$$A_{h,\xi}(u_t + uu_x) = -hh_x g - \left[\frac{1}{2}h^2u^2\xi_{xx}\right]_x - \left[\frac{2}{3}h^3u_x^2\right]_x - h\xi_x g - hu^2\xi_x\xi_{xx} - h^2\xi_x u_x^2.$$

Supposing that $A_{h,\xi}$ is invertible, the second equation in (1) is equivalent to

$$u_t + uu_x = -A_{h,\xi}^{-1}\left(hh_x g + \left[\frac{1}{2}h^2u^2\xi_{xx}\right]_x + \left[\frac{2}{3}h^3u_x^2\right]_x + h\xi_x g + hu^2\xi_x\xi_{xx} + h^2\xi_x u_x^2\right) = -A_{h,\xi}^{-1}P(h,u,\xi). \tag{3}$$

We will see in the next section that $A_{h,\xi}$ is invertible under suitable assumptions on h.

The second step to prove Theorem 1.1 is to write (1) in Lagrangian variables, that is, in terms of the flow map φ of u. The flow map φ is defined by the ordinary differential equation (ODE)

$$\varphi_t(t,x) = u(t,\varphi(t,x)),\ t \geq 0,\ \varphi(0,x) = x,\ x \in \mathbb{R}.$$

Note that this defines for each t a diffeomorphism $\varphi(t) := \varphi(t,\cdot) : \mathbb{R} \to \mathbb{R}$. We can translate the first equation in (1) into Lagrangian coordinates by considering

$$\frac{d}{dt}(\varphi_x \cdot h\circ\varphi) = \varphi_x \cdot (u_x h)\circ\varphi + \varphi_x \cdot (h_t + uh_x)\circ\varphi = 0,$$

where we used $\frac{d}{dt}\varphi_x = u_x\circ\varphi \cdot \varphi_x$ and the last equality is a consequence of $h_t + (hu)_x = 0$. Thus, h expressed in Lagrangian variables is

$$h = \left(\frac{h_0}{\varphi_x}\right)\circ\varphi^{-1}. \tag{4}$$

To express the second equation of (1) in Lagrangian variables, we consider it in nonlocal form (3) and compute

$$\varphi_{tt} = \frac{d}{dt}u\circ\varphi = (u_t + uu_x)\circ\varphi = -\left(A^{-1}_{(h_0/\varphi_x)\circ\varphi^{-1},\xi}P\left((h_0/\varphi_x)\circ\varphi^{-1}, \varphi_t\circ\varphi^{-1}, \xi\right)\right)\circ\varphi = F(\varphi,\varphi_t,h_0,\xi), \tag{5}$$

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where we used the equation for $u_t + uu_x$ in (3), $h = (h_0/\varphi_x)\circ\varphi^{-1}$ and $u = \varphi_t\circ\varphi^{-1}$. Our strategy will be to prove that F is continuously differentiable on a suitable functional space and hence by solving the second-order ODE $\varphi_{tt} = F(\varphi, \varphi_t, h_0, \xi)$, we will get local well-posedness of (1).

2 | THE OPERATOR $A_{h,\xi}$

The goal of this section is to prove that $A_{h,\xi}$ is invertible on Sobolev spaces. But first we need some preliminary facts about multiplication of functions in Sobolev spaces. Let $s > 1/2$ and $0 \le \sigma \le s$. Then, multiplication

$$H^s(\mathbb{R}) \times H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ (f,g) \mapsto f\cdot g, \tag{6}$$

is continuous—see [6, Lemma 2.3]. For $1/2 < s < 1$, multiplication extends for $s - 1 \le \sigma < 0$ to a continuous bilinear map

$$H^s(\mathbb{R}) \times H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ (f,g) \mapsto f\cdot g. \tag{7}$$

This follows from the fact that there is a constant $C > 0$ s.t.

$$\left|\int_{\mathbb{R}} f\cdot g\cdot\phi\ dx\right| \le \|g\|_\sigma \|f\cdot\phi\|_{-\sigma} \le C\|f\|_s\|g\|_\sigma\|\phi\|_{-\sigma},$$

for all $\phi \in C_c^\infty(\mathbb{R})$, where we used $0 < -\sigma < s$.

Lemma 2.1. *Let $s > 1/2$, $\xi \in H^{s+3}(\mathbb{R})$, $\bar{h} = 1 - \xi$ with $\inf_{x\in\mathbb{R}} \bar{h}(x) > 0$, and $h \in \bar{h} + H^s(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} h(x) > 0$. Then,*

$$A_{h,\xi} : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R}),\ u \mapsto h(1+\xi_x^2)u + \left[\frac{1}{2}h^2\xi_x u\right]_x - \frac{1}{2}h^2\xi_x u_x - \left[\frac{1}{3}h^3 u_x\right]_x,$$

is invertible.

Proof. Consider the symmetric bilinear form on $H^1(\mathbb{R})$ given by

$$\langle u, v\rangle_{h,\xi} = \int_{\mathbb{R}} h(1+\xi_x^2)uv - \frac{1}{2}h^2\xi_x uv_x - \frac{1}{2}h^2\xi_x u_x v + \frac{1}{3}h^3 u_x v_x\ dx.$$

We want to show that $\langle\cdot,\cdot\rangle_{h,\xi}$ is equivalent to the H^1 inner product

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}} uv + u_x v_x\ dx,\ u, v \in H^1(\mathbb{R}).$$

By the Sobolev Imbedding Theorem, $H^s(\mathbb{R})$ imbeds into $C_0(\mathbb{R})$, the continuous functions vanishing at infinity. Thus,

$$|\langle u, u\rangle_{h,\xi}| \le C\langle u, u\rangle_{H^1}$$

for some $C > 0$. For the estimate from below, consider

$$\langle u, u\rangle_{h,\xi} = \int_{\mathbb{R}} h(1+\xi_x^2)u^2 - h^2\xi_x uu_x + \frac{1}{3}h^3u_x^2\ dx.$$

We have

$$|h^2\xi_x uu_x| = (|h|^{1/2}|\xi_x||u|)(|h|^{3/2}|u_x|) \le h\xi_x^2u^2 + \frac{1}{4}h^3u_x^2,$$

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where in the last step, we used $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ with $\varepsilon = 2$. Thus,

$$\langle u, u\rangle_{h,\xi} \geq \int_{\mathbb{R}} hu^2 + \frac{1}{12}h^3u_x^2\, dx.$$

Since $\inf_{x\in\mathbb{R}} h(x) > 0$, we get

$$\langle u, u\rangle_{h,\xi} \geq C\langle u, u\rangle_{H^1}$$

for some $C > 0$. Let $f \in H^{s-1}(\mathbb{R})$. Our goal is to show that

$$A_{h,\xi}(u) = f$$

has a unique solution $u \in H^{s+1}(\mathbb{R})$. We assume first $1/2 < s \leq 1$. Since $f \in H^{-1}(\mathbb{R})$, we get by the Riesz Representation Theorem a unique $u \in H^1(\mathbb{R})$ satisfying

$$\langle u, \phi\rangle_{h,\xi} = \langle f, \phi\rangle,\ \forall \phi \in C_c^\infty(\mathbb{R}),$$

where $\langle f, \phi\rangle$ is the duality pairing. In other words, we have

$$h(1+\xi_x^2)u + \left[\frac{1}{2}h^2\xi_x u\right]_x - \frac{1}{2}h^2\xi_x u_x - \left[\frac{1}{3}h^3u_x\right]_x = f$$

in $H^{-1}(\mathbb{R})$. Rewriting gives

$$\left[\frac{1}{3}h^3u_x\right]_x = h(1+\xi_x^2)u + \left[\frac{1}{2}h^2\xi_x u\right]_x - \frac{1}{2}h^2\xi_x u_x - f \in H^{s-1}(\mathbb{R}).$$

Thus, $h^3u_x \in H^s(\mathbb{R})$. From [6, Lemma B.4], we know that dividing by h^3 is a bounded linear map on $H^s(\mathbb{R})$. So $u_x \in H^s(\mathbb{R})$ and with that we conclude $u \in H^{s+1}(\mathbb{R})$ and that u satisfies $A_{h,\xi}(u) = f$ in $H^{s-1}(\mathbb{R})$. Suppose now $1 < s \leq 2$. From the previous step (case $s = 1$), we get $u \in H^2(\mathbb{R})$ satisfying $A_{h,\xi}(u) = f$ in L^2. Thus, we have

$$\left[\frac{1}{3}h^3u_x\right]_x = h\big(1+\xi_x^2\big)u + \left[\frac{1}{2}h^2\xi_x u\right]_x - \frac{1}{2}h^2\xi_x u_x - f \in H^{s-1}(\mathbb{R}).$$

Arguing as above we get $u \in H^{s+1}(\mathbb{R})$ satisfying $A_{h,\xi}(u) = f$ in $H^{s-1}(\mathbb{R})$. By continuing this bootstrap argument for $2 < s \leq 3,\ 3 < s \leq 4, \ldots$, we get that

$$A_{h,\xi} : H^{s+1}(\mathbb{R}) \to H^{s-1}(\mathbb{R})$$

is an isomorphism for $s > 1/2$. □

3 | LAGRANGIAN FORMULATION

In [6], the authors studied for $s > 1/2$ the functional space

$$\mathcal{D}^{s+1}(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi - \mathrm{id} \in H^{s+1}(\mathbb{R}),\ \varphi_x(x) > 0\ \forall x \in \mathbb{R}\},$$

where $\mathrm{id} : \mathbb{R} \to \mathbb{R},\ x \mapsto x$, is the identity map on $\mathbb{R}$. By the Sobolev Imbedding Theorem, $H^{s+1}(\mathbb{R}) \hookrightarrow C_0^1(\mathbb{R})$ the functional space $\mathcal{D}^{s+1}(\mathbb{R})$ consists of C^1 diffeomorphisms of $\mathbb{R}$ and $\mathcal{D}^{s+1}(\mathbb{R}) - \mathrm{id}$ is an open subset of $H^{s+1}(\mathbb{R})$. So as an open

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

subset of $H^{s+1}(\mathbb{R})$, it has naturally a differential structure. In [6, Lemma B.5], it was shown that the composition map

$$H^\sigma(\mathbb{R}) \times \mathcal{D}^{s+1}(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ (f,\varphi) \mapsto f\circ\varphi, \quad 0 \le \sigma \le s+1, \tag{8}$$

is continuous. Furthermore, we have that

$$H^{s+2}(\mathbb{R}) \times \mathcal{D}^{s+1}(\mathbb{R}) \to H^{s+1}(\mathbb{R}),\ (f,\varphi) \mapsto f\circ\varphi, \tag{9}$$

is a C^1 map—-see [6, Theorem B.1]. In [6, Lemma B.6], it was shown that

$$\mathcal{D}^{s+1}(\mathbb{R}) \to \mathcal{D}^{s+1}(\mathbb{R}),\ \varphi \mapsto \varphi^{-1}, \tag{10}$$

is continuous. In particular, we get that $(\mathcal{D}^{s+1}(\mathbb{R}), \circ)$ is a topological group.

From (6) and (7), we get that for $\varphi \in \mathcal{D}^{s+1}(\mathbb{R})$ multiplication with φ_x

$$M_{\varphi_x} : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ f \mapsto \varphi_x \cdot f, \quad \min\{s-1,0\} \le \sigma \le s,$$

is a continuous linear map. From [6, Lemma B.4], we know that division by φ_x

$$M_{\varphi_x}^{-1} : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ f \mapsto \frac{f}{\varphi_x}, \quad \min\{s-1,0\} \le \sigma \le s,$$

is continuous as well. The map

$$\mathcal{D}^{s+1}(\mathbb{R}) \to L(H^\sigma(\mathbb{R}); H^\sigma(\mathbb{R})),\ \varphi \mapsto M_{\varphi_x}, \quad \min\{s-1,0\} \le \sigma \le s,$$

is affine linear, hence it is analytic—see [5, Appendix A] for basic definitions and results on analyticity in Banach spaces. Here, $L(X;Y)$ is the space of continuous linear maps from X to Y. Using Neumann series, we see that inversion of isomorphisms is an analytic process. So

$$\mathcal{D}^{s+1}(\mathbb{R}) \to L(H^\sigma(\mathbb{R}); H^\sigma(\mathbb{R})),\ \varphi \mapsto M_{\varphi_x}^{-1}, \quad \min\{s-1,0\} \le \sigma \le s,$$

is analytic as well. In particular,

$$\mathcal{D}^{s+1}(\mathbb{R}) \times \left(\bar{h} + H^s(\mathbb{R})\right) \to \bar{h} + H^s(\mathbb{R}),\ (\varphi, f) \mapsto M_{\varphi_x}^{-1} f = \frac{f}{\varphi_x}, \tag{11}$$

is analytic.

If we denote the composition from the right with φ by R_φ, that is, $R_\varphi : f \mapsto f\circ\varphi$, we get from (8) that

$$R_\varphi : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ f \mapsto f\circ\varphi, \quad 0 \le \sigma \le s+1,$$

is a continuous linear map. If $1/2 < s < 1$, then this extends for $s-1 \le \sigma < 0$ to a continuous linear map

$$R_\varphi : H^\sigma(\mathbb{R}) \to H^\sigma(\mathbb{R}).$$

The reason is that there is a constant $C > 0$ such that we have for all test functions $\phi \in C_c^\infty(\mathbb{R})$,

$$\left|\int_{\mathbb{R}} f\circ\varphi \cdot \phi \, dx\right| = \left|\int_{\mathbb{R}} f \cdot \frac{\phi\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}} \, dx\right| \le C\|f\|_\sigma \|\phi\|_{-\sigma}. \tag{12}$$

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

This follows from $0 < -\sigma < s$ and the fact that division by φ_x and R_φ^{-1} are bounded linear maps $H^{-\sigma}(\mathbb{R}) \to H^{-\sigma}(\mathbb{R})$. Note that R_φ is an isomorphism with inverse $R_\varphi^{-1} = R_{\varphi^{-1}}$. Composition turns out to be still continuous in the specified negative range. More precisely, we have the following:

Lemma 3.1. *Let* $1/2 < s < 1$ *and* $s - 1 \le \sigma < 0$. *Then,*

$$H^\sigma(\mathbb{R}) \times \mathcal{D}^{s+1}(\mathbb{R}) \to H^\sigma(\mathbb{R}),\ (f, \varphi) \mapsto f \circ \varphi,$$

is continuous.

Proof. Let $(f_\bullet, \varphi_\bullet) \in H^\sigma(\mathbb{R}) \times \mathcal{D}^{s+1}(\mathbb{R})$. Note that

$$H^{-\sigma}(\mathbb{R}) \times \mathcal{D}^{s+1}(\mathbb{R}) \to H^{-\sigma}(\mathbb{R}),\ (\phi, \varphi) \mapsto \frac{\phi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}},$$

is continuous as $0 < -\sigma < s$. By the Uniform Boundedness Principle, there is a constant $C > 0$ and a neighborhood $V \subset \mathcal{D}^{s+1}(\mathbb{R})$ of $\varphi_\bullet$ s.t.

$$\left\| \frac{\phi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \right\|_{-\sigma} \le C \|\phi\|_{-\sigma},\ \varphi \in V.$$

Using this in (12), we get for some $C > 0$,

$$\|f \circ \varphi\|_\sigma \le C \|f\|_\sigma,\ f \in H^\sigma(\mathbb{R}), \varphi \in V.$$

Let $\varepsilon > 0$. Take $\tilde{f} \in C_c^\infty(\mathbb{R})$ with $\|\tilde{f} - f_\bullet\|_\sigma \le \varepsilon$. If we take $f \in H^\sigma(\mathbb{R})$ in an ε neighborhood of $f_\bullet$, we have

$$\|f \circ \varphi - f_\bullet \circ \varphi\|_\sigma \le \|(f - \tilde{f}) \circ \varphi\|_\sigma + \|\tilde{f} \circ \varphi - \tilde{f} \circ \varphi_\bullet\|_\sigma + \|(\tilde{f} - f_\bullet) \circ \varphi_\bullet\|_\sigma \le 3C\varepsilon + \|\tilde{f} \circ \varphi - \tilde{f} \circ \varphi_\bullet\|_\sigma.$$

From [6, Lemma 2.11], we know (in the mentioned reference the result is shown for s integer, but the proof directly applies to the noninteger case) that there is a neighborhood $W \subset V$ of $\varphi_\bullet$ s.t.

$$\|\tilde{f} \circ \varphi - \tilde{f} \circ \varphi\|_{L^2} \le \|\tilde{f}\|_1 \|\varphi - \varphi_\bullet\|_{s+1},\ \varphi \in W.$$

Since $\|\tilde{f} \circ \varphi - \tilde{f} \circ \varphi_\bullet\|_\sigma \le \|\tilde{f} \circ \varphi - \tilde{f} \circ \varphi_\bullet\|_{L^2}$, we see that there is a neighborhood $O \subset W$ of $\varphi_\bullet$ s.t.

$$\|f \circ \varphi - f_\bullet \circ \varphi_\bullet\|_\sigma \le (3C + 1)\varepsilon,\ \|f - f_\bullet\|_\sigma \le \varepsilon, \varphi \in O.$$

This shows that $(f, \varphi) \mapsto f \circ \varphi$ is continuous. □

Now consider the following relation due to the chain rule:

$$\left[\left(f \circ \varphi^{-1} \right)_x \right] \circ \varphi = \frac{f_x}{\varphi_x}.$$

This translates to $R_\varphi \partial_x R_\varphi^{-1} = M_{\varphi_x}^{-1} \partial_x$. Thus,

$$\mathcal{D}^{s+1}(\mathbb{R}) \to L(H^\sigma(\mathbb{R}); H^{\sigma-1}(\mathbb{R})),\ \varphi \mapsto R_\varphi \partial_x R_\varphi^{-1}, \quad \sigma = s, s+1, \tag{13}$$

is analytic.

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Remark. The composition map (8) has poor regularity. It is not more than continuous. The reason is that to take the derivative with respect to φ in $\varphi \mapsto f\circ\varphi$, we have to take the derivative of f, which leads to a loss of derivative. But the conjugation with R_φ^{-1} turns out to be smooth.

Suppose now that for some $T > 0$, we have $u \in C([0,T]; H^{s+1}(\mathbb{R}))$. In [5, Proposition 4.1], it was shown that there is a unique $\varphi \in C^1([0,T]; \mathcal{D}^{s+1}(\mathbb{R}))$ satisfying

$$\varphi_t(t) = u(t)\circ\varphi(t),\ 0 \le t \le T,\ \varphi(0) = \mathrm{id}.$$

Thus, $\mathcal{D}^{s+1}(\mathbb{R})$ is the right functional space for the Lagrangian variable φ.

Let $s > 1/2$, $\xi \in H^{s+3}(\mathbb{R})$, and $\bar{h} = 1 - \xi$ with $\inf_{x\in\mathbb{R}} \bar{h}(x) > 0$. We introduce

$$U_{\bar{h}}^s = \{h \in \bar{h} + H^s(\mathbb{R}) \mid \inf_{x\in\mathbb{R}} h(x) > 0\}.$$

Note that we can identify $U_{\bar{h}}^s$ with the open subset $U_{\bar{h}}^s - \bar{h} \subset H^s(\mathbb{R})$. With this we have the following:

Lemma 3.2. *Let $s > 1/2$, $\xi \in H^{s+3}(\mathbb{R})$, and $\bar{h} = 1 - \xi$ with $\inf_{x\in\mathbb{R}} \bar{h}(x) > 0$. Then,*

$$\mathcal{D}^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times U_{\bar{h}}^s \to H^{s+1}(\mathbb{R}),\ (\varphi, v, h_0) \mapsto F(\varphi, v, h_0, \xi),$$

is C^1. Here, F is the map from (5).

Proof. We write

$$F(\varphi, v, h_0, \xi) = -R_\varphi A^{-1}_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1} R_\varphi P((h_0/\varphi_x)\circ\varphi^{-1}, v\circ\varphi^{-1}, \xi),$$

with P from (3). We will first show that

$$\mathcal{D}^{s+1}(\mathbb{R}) \times U_{\bar{h}}^s \to L(H^{s-1}(\mathbb{R}); H^{s+1}(\mathbb{R})),\ (\varphi, h_0) \mapsto R_\varphi A^{-1}_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1}, \tag{14}$$

is C^1. Note that

$$R_\varphi A^{-1}_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1} = \left(R_\varphi A_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1}\right)^{-1}.$$

Let $f \in H^{s+1}(\mathbb{R})$. We then have

$$\begin{aligned} R_\varphi A_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1} f =& M_{\varphi_x}^{-1} h_0 \cdot \left(1 + (R_\varphi \xi_x)^2\right) \cdot f + R_\varphi \partial_x R_\varphi^{-1}\left[\frac{1}{2}\left(M_{\varphi_x}^{-1} h_0\right)^2 \cdot (R_\varphi \xi_x) \cdot f\right] \\ & - \frac{1}{2}\left(M_{\varphi_x}^{-1} h_0\right)^2 \cdot R_\varphi \xi_x \cdot R_\varphi \partial_x R_\varphi^{-1} f - R_\varphi \partial_x R_\varphi^{-1}\left[\frac{1}{3}\left(M_{\varphi_x}^{-1} h_0\right)^3 \cdot R_\varphi \partial_x R_\varphi^{-1} f\right]. \end{aligned}$$

From (11) resp. (13), we know that $M_{\varphi_x}^{-1} h_0$ resp. $R_\varphi \partial_x R_\varphi^{-1}$ depend analytically on φ and h_0. From (9), we know that $R_\varphi \xi_x$ depends in a C^1 fashion on φ. Thus,

$$\mathcal{D}^{s+1}(\mathbb{R}) \times U_{\bar{h}}^s \to L(H^{s+1}(\mathbb{R}); H^{s-1}(\mathbb{R})),\ (\varphi, h_0) \mapsto R_\varphi A_{(h_0/\varphi_x)\circ\varphi^{-1},\xi} R_\varphi^{-1},$$

is C^1. By using Neumann series, we know that inversion of isomorphisms is a smooth process. Hence, we get that the map in (14) is C^1. Next, consider

$$\mathcal{D}^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times U_{\bar{h}}^s \to H^{s-1}(\mathbb{R}),\ (\varphi, v, h_0) \mapsto R_\varphi P((h_0/\varphi_x)\circ\varphi^{-1}, v\circ\varphi^{-1}, \xi). \tag{15}$$

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

This is explicitly given by

$$R_\varphi P((h_0/\varphi_x)\circ\varphi^{-1}, v\circ\varphi^{-1}, \xi) = M_{\varphi_x}^{-1}h_0 \cdot R_\varphi\partial_x R_\varphi^{-1} M_{\varphi_x}^{-1}h_0 \cdot g + R_\varphi\partial_x R_\varphi^{-1}\left[\frac{1}{2}\left(M_{\varphi_x}^{-1}h_0\right)^2 \cdot v^2 \cdot R_\varphi\xi_{xx}\right]$$
$$+ R_\varphi\partial_x R_\varphi^{-1}\left[\frac{2}{3}\left(M_{\varphi_x}^{-1}h_0\right)^3\left(R_\varphi\partial_x R_\varphi^{-1}v\right)^2\right] + M_{\varphi_x}h_0 \cdot R_\varphi\xi_x \cdot g + M_{\varphi_x}^{-1}h_0 \cdot v^2 \cdot R_\varphi\xi_x \cdot R_\varphi\xi_{xx}$$
$$+ \left(M_{\varphi_x}^{-1}h_0\right)^2 \cdot R_\varphi\xi_x \cdot \left(R_\varphi\partial_x R_\varphi^{-1}v\right)^2.$$

As above, we have that $M_{\varphi_x}^{-1}h_0$ depends analytically on (φ, h_0), $R_\varphi\partial_x R_\varphi^{-1}$ depends analytically on φ, and $R_\varphi\xi_x$ depends in a C^1 way on φ. For $R_\varphi\xi_{xx}$, we write $R_\varphi\partial_x R_\varphi^{-1}R_\varphi\xi_x$ and use (9) combined with (13) to get that

$$\mathcal{D}^{s+1} \to H^s(\mathbb{R}),\ \varphi \mapsto R_\varphi\xi_{xx} = (R_\varphi\partial_x R_\varphi^{-1})\circ(R_\varphi\xi_x)$$

is C^1. We conclude that the map in (15) is C^1. By composing (14) and (15), we see that the map in the statement of Lemma 3.2 is C^1. This finishes the proof. □

Consider now for $(h_0, u_0) \in U_{\bar{h}}^s \times H^{s+1}(\mathbb{R})$ the second-order ODE on $\mathcal{D}^{s+1}(\mathbb{R})$

$$\varphi_{tt} = F(\varphi, \varphi_t, h_0, \xi),\ \varphi(0) = \mathrm{id},\ \varphi_t(0) = u_0. \tag{16}$$

By Lemma 3.2, this is a C^1 ODE. Hence, we can find solutions by applying the Picard–Lindelöf Theorem. We will use this in the next section to prove Theorem 1.1.

4 | LOCAL WELL-POSEDNESS OF THE GREEN–NAGHDI SYSTEM

For the proof of Theorem 1.1, we need the following technical lemma.

Lemma 4.1. *Let $s > 1/2$, $T > 0$, $\xi \in H^{s+3}(\mathbb{R})$, and $\bar{h} = 1 - \xi$ with $\inf_{x\in\mathbb{R}} \bar{h}(x) > 0$. Suppose $g \in C([0,T]; U_{\bar{h}}^s) \cap C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})))$ and $\varphi \in C^1([0,T]; \mathcal{D}^{s+1}(\mathbb{R}))$. Then, $g\circ\varphi^{-1} \in C([0,T]; U_{\bar{h}}^s) \cap C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})))$ with*

$$\frac{d}{dt}g(t)\circ\varphi^{-1}(t) = g_t(t)\circ\varphi(t)^{-1} - \left(\frac{g_x(t)\varphi_t(t)}{\varphi_x(t)}\right)\circ\varphi(t)^{-1},\ 0 \le t \le T.$$

Similarly we have $g\circ\varphi \in C([0,T]; U_{\bar{h}}^s) \cap C^1\left([0,T]; \left(\bar{h} + H^{s-1}(\mathbb{R})\right)\right)$ with

$$\frac{d}{dt}g(t)\circ\varphi(t) = g_t(t) + g_x(t)\circ\varphi(t) \cdot \varphi_t(t),\ 0 \le t \le T.$$

Proof. Take a sequence $(g^{(k)})_{k\ge1} \subset C^1([0,T]; U_{\bar{h}}^{s+1})$ s.t. $g^{(k)} \to g$ in $C([0,T]; U_{\bar{h}}^s) \cap C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})))$ as $k \to \infty$. By the Sobolev Imbedding Theorem $H^{s+1}(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, we can differentiate $g^{(k)}\circ\varphi^{-1}$ pointwise in t

$$\frac{d}{dt}g^{(k)}(t)\circ\varphi(t)^{-1} = g_t^{(k)}(t)\circ\varphi(t)^{-1} - \left(\frac{g_x^{(k)}(t)\varphi_t(t)}{\varphi_x(t)}\right)\circ\varphi(t)^{-1},\ 0 \le t \le T.$$

But this is an identity in $H^{s-1}(\mathbb{R})$ as well. Taking $k \to \infty$ and using the continuity properties of the composition (8) (Lemma 3.1 for negative indexes) and inversion (10) shows the claim for $g\circ\varphi^{-1}$. A similar argument applies to $g\circ\varphi$. We omit the proof. □

Now we can prove Theorem 1.1.

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof of Theorem 1.1. Suppose $(h_0, u_0) \in U^s_{\bar{h}} \times H^{s+1}(\mathbb{R})$. First we show the existence of a solution. By the Picard–Lindelöf Theorem, there is some $T > 0$ and a solution $\varphi \in C^2([0,T]; \mathcal{D}^{s+1}(\mathbb{R}))$ to (16). We define

$$h(t) := \left(\frac{h_0}{\varphi_x(t)}\right) \circ \varphi(t)^{-1}, \; u(t) = \varphi_t(t) \circ \varphi(t)^{-1}, \quad 0 \le t \le T.$$

By the continuity properties of the composition (8) and inversion (10), we get $(h,u) \in C([0,T]; U^s_{\bar{h}} \times H^{s+1}(\mathbb{R}))$. From Lemma 4.1, we conclude

$$(h,u) \in C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})) \times H^s(\mathbb{R})).$$

By the Sobolev imbedding, we get $u \in C^1([0,T] \times \mathbb{R})$. Taking pointwise the t derivative of $u \circ \varphi = \varphi_t$ gives

$$(u_t + uu_x) \circ \varphi = \frac{d}{dt} u \circ \varphi = \varphi_{tt} = -\left(A^{-1}_{h,\xi} P(h,u,\xi)\right) \circ \varphi.$$

Thus, pointwise

$$u_t + uu_x = -A^{-1}_{h,\xi} P(h,u,\xi).$$

But this is an identity in $H^s(\mathbb{R})$ as well since the left-hand side is in $H^s(\mathbb{R})$ and the right-hand side is in $H^{s+1}(\mathbb{R})$ as one can see from the proof of Lemma 3.2. So (h,u) solves the second equation in (1). To verify that (h,u) solves the first equation in (1), we take the t derivative of $\varphi_x \cdot h \circ \varphi = h_0$ by using Lemma 4.1 to get

$$\varphi_x \cdot (h_t + (hu)_x) \circ \varphi = 0.$$

This shows that we have $h_t + (hu)_x = 0$ in $H^{s-1}(\mathbb{R})$. Thus, the first equation in (1) holds. So the existence of solutions is established.

In Lagrangian variables, we get continuous dependence on (h_0, u_0) by ODE theory. The continuity properties of the composition (8) and inversion (10) now provide continuous dependence of (h,u) on (h_0, u_0). This proves continuous dependence on the data.

To prove uniqueness, we assume that we have two solutions

$$(h,u), (\tilde{h}, \tilde{u}) \in C([0,T]; U^s_{\bar{h}} \times H^{s+1}(\mathbb{R})) \cap C^1([0,T]; (\bar{h} + H^{s-1}(\mathbb{R})) \times H^s(\mathbb{R}))$$

satisfying (1). From [5, Proposition 4.1], we know that u resp. $\tilde{u}$ generate flow maps φ resp. $\tilde{\varphi}$ with

$$\varphi, \; \tilde{\varphi} \in C^1([0,T]; \mathcal{D}^{s+1}(\mathbb{R})).$$

As a consequence of $h_t + (hu)_x = 0$, we get

$$h = \left(\frac{h_0}{\varphi_x}\right) \circ \varphi^{-1}.$$

Taking pointwise the t derivative in the equation $\varphi_t = u \circ \varphi$, we get

$$\varphi_{tt} = (u_t + uu_x) \circ \varphi.$$

Thus, we get pointwise

$$\varphi_{tt} = -R_\varphi A^{-1}_{h,\xi} P(h,u,\xi) = F(\varphi, \varphi_t, h_0, \xi),$$

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where we used $h = (h_0/\varphi_x)\circ\varphi^{-1}$. This is an identity in $H^{s+1}(\mathbb{R})$ as well. So φ solves the ODE (16). Similarly, we get that $\tilde{\varphi}$ solves (16) with the same initial data. By the Uniqueness Theorem for ODEs, we get $\varphi \equiv \tilde{\varphi}$ on $[0,T]$ and with that $(h,u) \equiv (\tilde{h},\tilde{u})$ on $[0,T]$.

Putting existence, continuous dependence on the data and uniqueness together, we get the local well-posedness of (1). This finishes the proof. □

Remark. A natural question is whether one can adapt the methods of this paper to the 2D Green–Naghdi equations. We first have thought that this is probably not possible. The reason is that in the 2D case the operator corresponding to $A_{h,\xi}$ is not fully invertible. This leads to a "loss of derivative." To overcome this difficulty, Alvarez-Samaniego and Lannes [1] used a Nash–Moser scheme to construct solutions to the 2D Green–Naghdi equations. But the anonymous referee has called our attention to a work by Fujiwara and Iguchi [3] where the 2D Green–Naghdi equations are handled on suitable Sobolev spaces by using energy methods without using Nash–Moser iteration. This has changed our opinion. So a future project of ours will be trying to adapt the methods of this work to the 2D case, which is worth doing since there seems to be some room for improving the regularity of the solutions given in [3].

ACKNOWLEDGMENTS

The author thanks the anonymous referee for carefully reading the paper, giving valuable comments, and pointing out missing details. This has resulted in an improved version of the paper.

ORCID

Hasan İnci https://orcid.org/0000-0001-9626-5000

REFERENCES

[1] B. Alvarez-Samaniego and D. Lannes, *A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green-Naghdi equations*, Indiana Univ. Math. J. **57** (2008), no. 1, 97–131. https://doi.org/10.1512/iumj.2008.57.3200.

[2] S. François, *Contribution à l'étude des écoulements permanents et variables dans les canaux*, La Houille Blanche (1953), no. 6, 830–872. https://doi.org/10.1051/lhb/1953058.

[3] H. Fujiwara and T. Iguchi, *A shallow water approximation for water waves over a moving bottom*, Nonlinear dynamics in partial differential equations, Adv. Stud. Pure Math., vol. 64, Math. Soc. Japan, Tokyo, 2015, pp. 77–88. https://doi.org/10.2969/aspm/06410077.

[4] A. E. Green and P. M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78** (1976), no. 2, 237–246.

[5] H. Inci, *On a Lagrangian formulation of the incompressible Euler equation*, J. Partial Differ. Equ. **29** (2016), no. 4, 320–359. https://doi.org/10.4208/jpde.v29.n4.5.

[6] H. Inci, T. Kappeler, and P. Topalov, *On the regularity of the composition of diffeomorphisms*, Mem. Amer. Math. Soc. **226** (2013), no. 1062, vi+60. https://doi.org/10.1090/S0065-9266-2013-00676-4.

[7] S. Israwi, *Large time existence for 1D Green-Naghdi equations*, Nonlinear Anal. **74** (2011), no. 1, 81–93. https://doi.org/10.1016/j.na.2010.08.019.

[8] Y. A. Li, *A shallow-water approximation to the full water wave problem*, Comm. Pure Appl. Math. **59** (2006), no. 9, 1225–1285. https://doi.org/10.1002/cpa.20148.

[9] F. J. Seabra-Santos, D. P. Renouard, and A. M. Temperville, *Numerical and experimental study of the transformation of a solitary wave over a shelf or isolated obstacle*, J. Fluid Mech. **176** (1987), 117–134.

[10] C. H. Su and C. S. Gardner, *Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation*, J. Math. Phys. **10** (1969), 536–539. https://doi.org/10.1063/1.1664873.

How to cite this article: H. İnci, *On the local well-posedness of the 1D Green–Naghdi system on Sobolev spaces*, Math. Nachr. **297** (2024), 52–62. https://doi.org/10.1002/mana.202200256

15222616, 2024, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/mana.202200256 by Koc University, Wiley Online Library on [18/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License