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Area minimizing surfaces in mean convex 3-manifolds

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Abstract. In this paper, we give several results on area minimizing surfaces in strictly mean convex 3-manifolds. First, we study the genus of absolutely area minimizing surfaces in a compact, orientable, strictly mean convex 3-manifold M bounded by a simple closed curve in ∂M. Our main result is that for any $g \geq 0$, the space of simple closed curves in ∂M where all the absolutely area minimizing surfaces they bound in M has genus $\geq g$ is open and dense in the space $\mathcal{A}$ of nullhomologous simple closed curves in ∂M. For showing this we prove a bridge principle for absolutely area minimizing surfaces. Moreover, we show that for any $g \geq 0$, there exists a curve γ_g in $\mathcal{A}$ such that the minimum genus of the absolutely area minimizing surfaces γ_g bounds is exactly g.

As an application of these results, we further prove that the simple closed curves in ∂M bounding more than one minimal surface in M is an open and dense subset of $\mathcal{A}$. We also show that there are disjoint simple closed curves in ∂M bounding minimal surfaces in M which are not disjoint. This allows us to answer a question of Meeks, by showing that for any strictly mean convex 3-manifold M, there exists a simple closed curve Γ in ∂M which bounds a stable minimal surface which is not embedded. We also gave some applications of these results to the simple closed curves in $\mathbb{R}^3$.

1. Introduction

The Plateau problem concerns the existence of an area minimizing disk with boundary a given curve in an ambient manifold M. This problem was solved in the case when the ambient manifold is $\mathbb{R}^3$ by Douglas [6], and Rado [21] in the early 1930s. Later, it was generalized by Morrey [20] for Riemannian manifolds. In the 1980s, Meeks and Yau showed that if M is a mean convex 3-manifold, and Γ is a simple closed curve in ∂M, then any area minimizing disk with boundary Γ is embedded [17]. Later, White gave a generalization of this result to any genus [23].

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In the early 1960s, the same question was studied for absolutely area minimizing surfaces, i.e. for surfaces that minimize area among all orientable surfaces with a given boundary (without restriction on the genus). Using techniques from geometric measure theory, Federer and Fleming [9] were able to solve this problem by proving the existence of an absolutely area minimizing integral current (see also [5, 11] for the existence of a minimizing Cacciopoli set). In [1] Almgren, Schoen and Simon showed that this current is a smooth embedded surface away from its boundary and Hardt [13] showed that it is also smooth at the boundary, provided that the prescribed boundary is smooth and lies on a convex set. Later, Hardt and Simon [14], improved this boundary regularity result, by dropping the assumption that the prescribed boundary lies on a convex set.

It can be seen that there are two main versions of the Plateau problem; one of them is that of fixed genus (area minimizing in a fixed topological class), and the other one is with no restriction on the genus (absolutely area minimizing case). There have also been many important results on the *a priori* bounds on the genus of an absolutely area minimizing surface bounded by a given simple closed curve [14].

In this paper, we study the genus of absolutely area minimizing surfaces Σ in a compact, orientable, strictly mean convex 3-manifold M, with boundary $\partial\Sigma$ a simple closed curve lying in ∂M. We consider the stratification of the space of nullhomologous (bounding a surface in M) simple closed curves in ∂M with respect to the minimum genus of the absolutely area minimizing surfaces that they bound. In particular, we let $\mathcal{A}$ be the space of all nullhomologous simple closed curves in ∂M with the $\mathcal{C}^0$ topology (see Definition 3.1). By the previously mentioned results, any $\Gamma \in \mathcal{A}$ bounds an embedded absolutely area minimizing surface Σ in M with $\partial\Sigma = \Gamma$. Let $\mathcal{A}_g$ be the set of all the curves in $\mathcal{A}$ such that any embedded absolutely area minimizing surface in M that they bound has genus $\geq g$. Then, clearly

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_n \supset \cdots .$$

We show that for any $g \geq 0$, $\mathcal{A}_g$ is an open subset of $\mathcal{A}$ (Lemma 3.2).

Then, we define an operation, which we call *horn surgery*, on a given simple closed curve Γ in ∂M which adds a *thin handle* to an absolutely area minimizing surface Σ which Γ bounds in M. In other words, by modifying the boundary curve, the operation increases the genus of an absolutely area minimizing surface (see Figure 4 and Figure 5). Moreover, this new curve can be made as close as we want to the original curve. Hence by using the horn surgery operation, it is easy to show that $\mathcal{A}_g$ is not only open, but also dense in $\mathcal{A}$ (Theorem 3.22). Furthermore, to define the horn surgery operation, we prove *a bridge principle for absolutely area minimizing surfaces* (Theorem 3.8).

Next, we study the space $\mathcal{B}_g$ of all the curves in $\mathcal{A}$, such that the minimum genus of the embedded absolutely area minimizing surfaces in M that they bound is exactly g, i.e. $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$. Then, clearly

$$\mathcal{A} = \bigcup_{g=0}^{\infty} \mathcal{B}_g$$

and

$$\mathcal{B}_g \cap \mathcal{B}_{g'} = \emptyset \quad \text{for any } g \neq g'.$$

Again by using the horn surgery operation, we show that for any $g \geq 0$, $\mathcal{B}_g$ is not empty (Theorem 3.20). In other words, we show that for any $g \geq 0$, there is a simple closed curve

Γ_g in ∂M that bounds an absolutely area minimizing surface Σ with genus g. Also, we prove that for any g, $\mathcal{B}_g$ is nowhere dense in $\mathcal{A}$ (Corollary 3.23). Furthermore, we show that there exist simple closed curves Γ in ∂M which bound two absolutely area minimizing surfaces of different genus (Theorem 3.25).

In Section 4, we further derive several interesting results about minimal surfaces in strictly mean convex 3-manifolds by using the previously mentioned results. We show that some important properties for properly embedded area minimizing surfaces in such manifolds are not valid for properly embedded minimal surfaces. First, we show that *curves that bound more than one minimal surface* are generic for a strictly mean convex 3-manifold M (Theorem 4.1). In [3], and [4], it is proven that a generic nullhomotopic simple closed curve in ∂M bounds a unique area minimizing disk, and similarly, a generic nullhomologous simple closed curve in ∂M bounds a unique absolutely area minimizing surface in M. However, here we show that when we relax the condition of being area minimizing to just minimal, the situation is completely opposite.

Then, we generalize Peter Hall's results [12], which answers Meeks' questions, to any strictly mean convex 3-manifold M. As previously mentioned, in [17], Meeks and Yau proved that any area minimizing disk in a mean convex 3-manifold bounded by a simple closed curve in ∂M must be embedded. After establishing this result, Meeks posed the question of whether or not the same holds for stable minimal surfaces. Then, Hall constructed an example of a simple closed curve Γ in $S^2 = \partial\mathbb{B}^3$ where $\mathbb{B}^3$ is the unit 3-ball in $\mathbb{R}^3$, such that Γ bounds a stable minimal disk M in B^3 which is not embedded. This shows that if we relax the area minimizing condition to just being minimal again, the embeddedness result of Meeks and Yau is no longer valid. We generalize Hall's example, and show that this is true for any strictly mean convex 3-manifold (Theorem 4.8).

To construct the nonembedded minimal examples, we first show that for any strictly mean convex 3-manifold M, there are disjoint simple closed curves Γ_1 and Γ_2 in ∂M that bound minimal surfaces, M_1 and M_2 respectively, in M, such that $M_1 \cap M_2 \neq \emptyset$ (Theorem 4.4). This is very interesting since if M is a mean convex 3-manifold with trivial second homology, and Γ_1, Γ_2 are two disjoint simple closed curves in ∂M, then any two absolutely area minimizing surfaces Σ_1, Σ_2 with $\partial\Sigma_1 = \Gamma_1$, $\partial\Sigma_2 = \Gamma_2$ are also disjoint [3]. The same holds for area minimizing disks, too. Hence, again the case of minimal surfaces, and that of area minimizing are very different. Then, by connecting these intersecting minimal surfaces, which have disjoint boundaries, with a *bridge*, we construct nonembedded examples of minimal surfaces in the strictly mean convex 3-manifold M. Thus, we show that for any such M, there exists a simple closed curve Γ in ∂M which bounds a stable minimal surface, and in particular a stable minimal disk, in M, which is not embedded (Theorem 4.8).

In Section 5, we give some applications of these results to the simple closed curves in $\mathbb{R}^3$. First we generalize Theorem 3.22 ($\mathcal{A}_g$ is open dense in $\mathcal{A}$) for the simple closed curves in $\mathbb{R}^3$ (Theorem 5.1). Then, we show that the simple closed curves in $\mathbb{R}^3$ bounding *more than one minimal surface* are dense in the space of simple closed curves in $\mathbb{R}^3$ (Theorem 5.3). Finally, we prove that the simple closed curves bounding a nonembedded stable minimal surface in $\mathbb{R}^3$ are abundant in $\mathbb{R}^3$ (Theorem 5.5).

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2. Preliminaries

In this section we give some standard definitions as well as an overview of the basic known results which we use in the following sections.

Definition 2.1. An *area minimizing disk* is a disk which has the smallest area among all disks with the same boundary. An *area minimizing surface* is a surface which has the smallest area among all surfaces with the same genus and the same boundary. An *absolutely area minimizing surface* is a surface which has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

Definition 2.2. Let M be a compact Riemannian 3-manifold with boundary. Then M is called *mean convex* (or sufficiently convex) if the following conditions hold:

(i) ∂M is piecewise smooth.

(ii) Each smooth subsurface of ∂M has nonnegative mean curvature with respect to the inward normal.

(iii) There exists a Riemannian manifold N such that M is isometric to a submanifold of N and each smooth subsurface S of ∂M extends to a smooth embedded surface S' in N such that $S' \cap M = S$.

If in (ii) of the definition we require that each smooth subsurface has strictly positive mean curvature, then M is called *strictly mean convex*.

Definition 2.3. A simple closed curve is called an *extreme curve* if it is on the boundary of its convex hull. A simple closed curve is called an *H-extreme curve* if it is a curve in the boundary of a mean convex manifold M.

Definition 2.4. An embedded compact surface S in a 3-manifold M is called *properly embedded* if $S \cap \partial M = \partial S$.

We now state the main facts which we will be using in the following sections.

Theorem 2.5 ([18,19]). *Let M be a compact, mean convex 3-manifold, and $\Gamma \subset \partial M$ be a nullhomotopic simple closed curve. Then, there exists an area minimizing disk $D \subset M$ with $\partial D = \Gamma$. Moreover, all such disks are properly embedded in M and they are pairwise disjoint. Furthermore, if $\Gamma_1, \Gamma_2 \subset \partial M$ are two disjoint simple closed curves, then two area minimizing disks D_1 and D_2 spanning Γ_1 and Γ_2 respectively are also disjoint.*

There is an analogous fact for area minimizing surfaces, too.

Theorem 2.6 ([1, 9, 13]). *Let M be a compact, strictly mean convex 3-manifold, and $\Gamma \subset \partial M$ be a nullhomologous simple closed curve. Then there exists an absolutely area minimizing surface $\Sigma \subset M$ with $\partial \Sigma = \Gamma$ and each such Σ is smooth away from its boundary and it is smooth around points of the boundary where Γ is smooth.*

Remark 2.7 (on the proof of Theorem 2.6). The regularity results in [1, 13] extend from the ambient space being $\mathbb{R}^n$ to M, a manifold as in the Theorem 2.6, as explained in [23, Theorem 6.3].

Theorem 2.8 ([23]). *Let M be a compact, orientable, strictly mean convex 3-manifold, and Γ be a simple closed curve in ∂M. For each $g \geq 0$, define*

$$a(g) = \inf |\Sigma|$$

where the infimum is taken over all piecewise smooth embedded surfaces Σ in M, with genus g and boundary Γ and where $|\Sigma|$ denotes the area of a surface Σ. Then, if $a(g) < a(g-1)$, the infimum $a(g)$ is attained.

Remark 2.9. In particular, this theorem implies that if Γ_0 is a simple closed curve in ∂M where g_0 is the minimum genus of a surface bounded by Γ_0, then there exists an area minimizing surface Σ_{g_0} of genus g_0. This is because by the proof of [23, Theorem 3.4], the condition $a(g+1) < a(g)$ is to make sure that there is no collapsing (genus dropping) in the limit surface achieving the infimum. Hence, if there is no such surface S of genus g in M with $\partial S = \Gamma$, such a collapsing is not possible. Hence, one can take $a(g) = \infty$ in the theorem if there is no such surface S of genus g in M with $\partial S = \Gamma$. This would imply $a(g_0) < a(g_0 - 1)$ and the existence of an area minimizing surface Σ_{g_0}

Theorem 2.10 ([23]). *Let M be a compact, orientable, strictly mean convex 3-manifold, Γ be a simple closed curve in ∂M and Σ be a stable surface in M with $\partial \Sigma = \Gamma$. Assume that for a geodesic ball $\mathcal{B}(x, R)$ in M we have*

$$\sup_{\mathcal{B}(y,r) \subset \mathcal{B}(x,R)} \frac{|\Sigma \cap \mathcal{B}(y,r)|}{\pi r^2} = C < \infty.$$

Then the following hold:

- *If $\mathcal{B}(x, R) \cap \Gamma = \emptyset$, then the principal curvatures of $M \cap \mathcal{B}(x, R/2)$ are bounded by a constant that depends only on R, C.*
- *If $\mathcal{B}(x, R) \cap \Gamma$ is $C^{2,\alpha}$, then the principal curvatures of $M \cap \mathcal{B}(x, R/2)$ are bounded by a constant that depends only on R, C and Γ.*

Finally, we state two results on the generic uniqueness of area minimizing disks, and absolutely area minimizing surfaces for H-extreme curves.

Theorem 2.11 ([3]). *Let M be a compact, orientable, mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. Then the following hold:*

(i) *For a generic nullhomotopic (in M) simple closed curve Γ in ∂M, there exists a unique area minimizing disk D in M with $\partial D = \Gamma$.*

(ii) *For a generic nullhomologous (in M) simple closed curve Γ in ∂M, there exists a unique absolutely area minimizing surface Σ in M with $\partial \Sigma = \Gamma$.*

Convention. Throughout the paper, all the manifolds will be assumed to be compact, orientable, and strictly mean convex unless otherwise stated. We will also assume that all the surfaces are orientable.

3. Stratification of the space of simple closed curves in ∂M

In this section, we study the space of nullhomologous simple closed curves in ∂M where M is a compact, orientable, strictly mean convex 3-manifold. We stratify this space with respect to the minimum genus of the absolutely area minimizing surfaces that its curves bound.

Let $\mathcal{A}$ denote the space of nullhomologous simple closed curves in ∂M, equipped with the $\mathcal{C}^0$ topology. We give here the precise definition of the neighborhoods defining this topology.

Definition 3.1. Let $\Gamma \in \mathcal{A}$. Then there exists a continuous function

$$f : S^1 \to \partial M$$

such that $f(S^1) = \Gamma$ where $S^1 = \partial \mathbb{B}^2$ and $\mathbb{B}^2$ is the unit 2-ball in $\mathbb{R}^2$.

An ϵ-neighborhood of Γ, $\mathcal{U}_\Gamma(\epsilon)$, in the $\mathcal{C}^0$ topology consists of all $\widetilde{\Gamma} \in \mathcal{A}$ for which there exists a continuous function

$$g : S^1 \to \partial M$$

such that

$$g(S^1) = \widetilde{\Gamma} \quad \text{and} \quad \sup_{x \in S^1} \operatorname{dist}_{\partial M}(f(x), g(x)) < \epsilon,$$

where $\operatorname{dist}_{\partial M}$ denotes the Riemannian distance on ∂M.

Note that, for ϵ small enough, so that the ϵ-neighborhood of Γ in ∂M, defined by

$$N_\epsilon(\Gamma) = \{x \in \partial M \mid \operatorname{dist}_{\partial M}(x, \Gamma) < \epsilon\},$$

is topologically an annulus, we have that $\widetilde{\Gamma} \in \mathcal{U}_\Gamma(\epsilon)$ is equivalent to $\widetilde{\Gamma} \subset N_\epsilon(\Gamma)$ and $\widetilde{\Gamma} \sim \Gamma$, i.e. $\widetilde{\Gamma}$ is homotopic equivalent to Γ.

We define a relation $\varphi : \mathcal{A} \to \mathbb{N}$, by

$$\varphi(\Gamma) = \min_{\Sigma} \operatorname{genus}(\Sigma)$$

where the minimum is taken over all absolutely area minimizing surfaces Σ in M with $\partial\Sigma = \Gamma$ and naturally $\operatorname{genus}(\Sigma)$ denotes the genus of a surface Σ.

Now, define

$$\mathcal{A}_g = \{\Gamma \in \mathcal{A} \mid \varphi(\Gamma) \geq g\} \quad \text{and} \quad \mathcal{B}_g = \varphi^{-1}(g) = \mathcal{A}_g \setminus \mathcal{A}_{g+1}.$$

In particular, $\mathcal{A}_g$ denotes the space of simple closed curves in ∂M such that the minimum genus of the absolutely area minimizing surfaces that they bound is greater than or equal to g; $\mathcal{B}_g$ denotes the space of simple closed curves in ∂M such that the minimum genus of the absolutely area minimizing surfaces that they bound is exactly g. Clearly,

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_n \supset \cdots.$$

Lemma 3.2. *For any $g \geq 0$, $\mathcal{A}_g$ is open in $\mathcal{A}$.*

Proof. Let $\Gamma_0 \in \mathcal{A}_g$ be a simple closed curve in ∂M. Since $\Gamma_0 \in \mathcal{A}_g$, we have

$$\varphi(\Gamma_0) \geq g,$$

i.e. if Σ is an absolutely area minimizing surface in M with $\partial\Sigma = \Gamma_0$, then genus$(\Sigma) \geq g$.

Assume now that the lemma is not true. Then for any $\epsilon > 0$, there exists $\Gamma_\epsilon \in \mathcal{U}_{\Gamma_0}(\epsilon)$, such that $\Gamma_\epsilon \notin \mathcal{A}_g$ where $\mathcal{U}_{\Gamma_0}(\epsilon)$ is the ϵ-neighborhood of Γ_0 in the $\mathcal{C}^0$ topology, as in Definition 3.1. Note that for ϵ small enough, $\Gamma_\epsilon \sim \Gamma$ (cf. Definition 3.1). Hence, for ϵ small enough, there exists a region A_ϵ in ∂M between Γ_ϵ and Γ, i.e. $\partial A_\epsilon = \Gamma_\epsilon \cup \Gamma$, for which we have

$$|A_\epsilon| \xrightarrow{\epsilon \to 0} 0$$

where $|\cdot|$ denotes the area of a surface. This implies that for any absolutely area minimizing surface Σ_ϵ, with $\partial\Sigma_\epsilon = \Gamma_\epsilon$, we have $|\Sigma_\epsilon| \leq |\Sigma| + |A_\epsilon|$ where Σ is any absolutely area minimizing surface with boundary equal to Γ_0. Therefore there exists a sequence $\{\Gamma_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$, such that

$$\Gamma_i \xrightarrow{i \to \infty} \Gamma_0$$

in the $\mathcal{C}^0$ topology and such that for each i, there exists an absolutely area minimizing surface Σ_i, with $\partial\Sigma_i = \Gamma_i$, genus$(\Sigma_i) \leq g - 1$ and such that their areas satisfy

$$|\Sigma_i| \xrightarrow{i \to \infty} |\Sigma|.$$

Let now $S_i = [\![\Sigma_i]\!]$, be the corresponding integral currents. Then, since the areas $|\Sigma_i|$ are uniformly bounded, by the Federer–Fleming compactness theorem for integral currents [9], after passing to a subsequence $S_i \to S_0$, in the sense of currents, where S_0 is an integral current, with

$$\underline{\underline{M}}(S_0) \leq \lim_{i\to\infty} \underline{\underline{M}}(S_i) = |\Sigma|;$$

here $\underline{\underline{M}}$ denotes the mass of the current (see [22]). Hence $\underline{\underline{M}}(S_0) = |\Sigma_0|$ and thus (using Theorem 2.6) $S_0 = [\![\Sigma_0]\!]$ where Σ_0 is an absolutely area minimizing surface with $\partial\Sigma_0 = \Gamma_0$ and since $\Gamma_0 \in \mathcal{A}_g$, we have that genus$(\Sigma_0) \geq g$.

Note now that, since S_i are absolutely area minimizing, in the above convergence, the corresponding radon measures converge $\mu_{S_i} \to \mu_S$ (for a proof of this see [22, Theorem 34.5]). Furthermore, since each Σ_i is a minimal surface, the monotonicity formula holds [22], i.e. for each i, each $x \in \overline{M} \setminus \Gamma_i$ and for r small enough (so that $\mathcal{B}(x, r) \cap \Gamma_i = \emptyset$), $r^{-2}|\Sigma_i|$ is an increasing function of r. Therefore we can apply the principal curvature bound of White, as given in Theorem 2.10, to conclude that the convergence $\Sigma_i \to \Sigma_0$ is actually smooth in compact sets of $\overline{M} \setminus \Gamma_0$. Let now K be a compact subset of M, such that $\Sigma_0 \cap K$ has genus $\geq g$. Then the previous convergence, along with the Gauss–Bonnet Theorem, implies that the same should be true for the surfaces Σ_i, for i big enough, i.e. that genus$(\Sigma_i \cap K) \geq g$. But this contradicts the fact that genus$(\Sigma_i) \leq g - 1$. □

3.1. A bridge principle for absolutely area minimizing surfaces. In this section we define a surgery operation on a simple closed curve Γ in ∂M, by gluing a "bridge" on Γ. Let α be a simple path in ∂M such that α intersects Γ transversely and $\Gamma \cap \alpha = \{x, y\}$ are the end-

points of α. Let $\epsilon_0 > 0$ be sufficiently small so that for almost every $0 < \epsilon < \epsilon_0$, $\partial N_\epsilon(\alpha) \cap \Gamma$ consists of exactly four points, namely $\{x_\epsilon^+, x_\epsilon^-, y_\epsilon^+, y_\epsilon^-\}$, where $N_\epsilon(\alpha)$ is the ϵ neighborhood of α in ∂M, i.e.

$$N_\epsilon(\alpha) = \{x \in \partial M \mid \text{dist}_{\partial M}(x, \alpha) < \epsilon\}.$$

Then, Γ divides $N_\epsilon(\alpha)$ into three components, which we call C_x, C_y and S_α^ϵ, where S_α^ϵ is the component containing α and C_x, C_y are the remaining caps of $N_\epsilon(\alpha)$ near x and y respectively (see Figure 1).

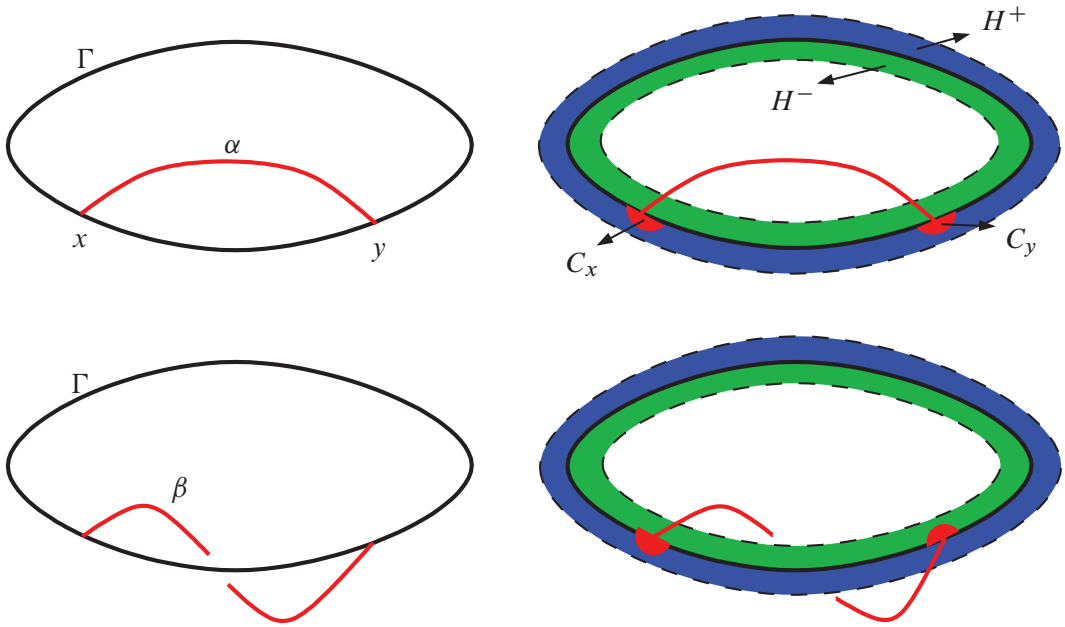


Figure 1. The path α is Γ-admissible as both C_x and C_y belong to H^+. The path β is not Γ-admissible as C_x belongs to H^+ while C_y belongs to H^-.

For orientation issues, that are explained in Remark 3.11, we further assume that the path α is such that C_x and C_y are in the same side of Γ. In particular we will be considering paths α satisfying the following definition.

Definition 3.3. Let $\Gamma \in \mathcal{A}$ and let α be a simple path in ∂M such that α intersects Γ transversely and $\Gamma \cap \alpha = \{x, y\}$ are the endpoints of α. We say that α is a Γ-*admissible path* if the following holds:

If $N_{\epsilon'}(\Gamma)$ is a small annular neighborhood of Γ in ∂M, and $N_{\epsilon'}(\Gamma) \setminus \Gamma = H^+ \cup H^-$ where H^+ is an annulus, say the "positive side" of Γ, and H^- is the other annulus, say the "negative side" of Γ, then either both C_x and C_y intersect H^+ or they both intersect H^- where C_x, C_y are as above (see Figure 1).

The component S_α^ϵ as constructed above is a tiny strip ("bridge") of width ϵ, around α. For its boundary we have $\partial S_\alpha^\epsilon = (\alpha_\epsilon^+ \cup \alpha_\epsilon^-) \cup (\beta_\epsilon^x \cup \beta_\epsilon^y)$ where β_ϵ^x is the arc segment of Γ between x_ϵ^+ and x_ϵ^- containing x, and similarly β_ϵ^y is the arc segment of Γ between y_ϵ^+ and y_ϵ^- containing y. Also, α_ϵ^+ is a simple path in ∂M connecting x_ϵ^+ to y_ϵ^+, and α_ϵ^- is a simple path in ∂M connecting x_ϵ^- to y_ϵ^- in ∂M (see Figure 2). "Gluing" this strip S_α^ϵ on Γ results to the following definition.

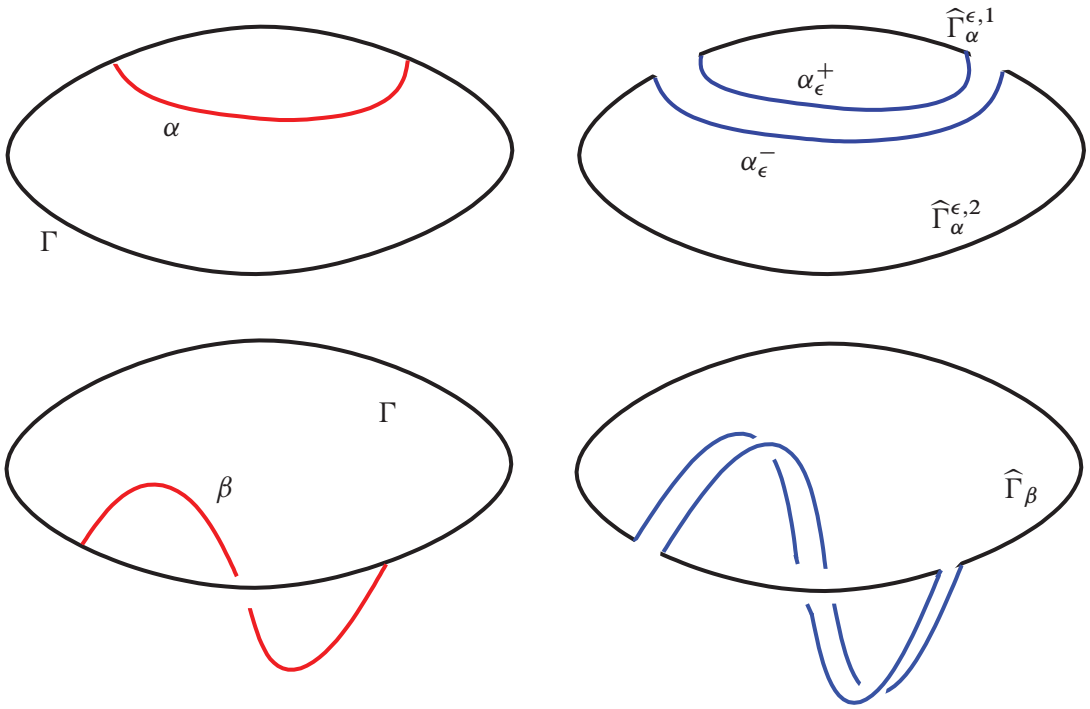


Figure 2. If α is Γ-admissible, after the surgery along S^{ϵ}_{α}, the new curve $\widehat{\Gamma}^{\epsilon}_{\alpha}$ has two components, $\widehat{\Gamma}^{\epsilon,1}_{\alpha}$ and $\widehat{\Gamma}^{\epsilon,2}_{\alpha}$. When β is not Γ-admissible, then the resulting curve $\widehat{\Gamma}^{\epsilon}_{\beta}$ has only one component.

Definition 3.4. Let $\Gamma \in \mathcal{A}$ and let α be a Γ-admissible path. Then we define a new curve

$$\widehat{\Gamma}^{\epsilon}_{\alpha} = \Gamma \sharp S^{\epsilon}_{\alpha},$$

obtained by gluing on Γ a bridge S^{ϵ}_{α} of width ϵ around α. With the above notation $\Gamma \sharp S^{\epsilon}_{\alpha}$ is the curve

$$\widehat{\Gamma}^{\epsilon}_{\alpha} = \Gamma \sharp S^{\epsilon}_{\alpha} = (\Gamma \setminus (\beta^{x}_{\epsilon} \cup \beta^{y}_{\epsilon})) \cup (\alpha^{+}_{\epsilon} \cup \alpha^{-}_{\epsilon}).$$

Since α is Γ-admissible, $\widehat{\Gamma}^{\epsilon}_{\alpha}$ is the union of two simple closed curves in ∂M, which we will denote by $\widehat{\Gamma}^{\epsilon,1}_{\alpha}$ and $\widehat{\Gamma}^{\epsilon,2}_{\alpha}$ (see Figure 2).

The above gluing construction is also possible when instead of a single simple closed curve, we have two disjoint curves and α is a path joining them. In particular, let $\Gamma_1, \Gamma_2 \in \mathcal{A}$ be such that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then the above gluing operation goes through in exactly the same way as described, with Γ replaced by $\Gamma_1 \cup \Gamma_2$. In this case we have no orientation issues, but we want α to be a path "connecting" the two curves. In particular we will be considering paths α satisfying the following definition.

Definition 3.5. Let $\Gamma_1, \Gamma_2 \in \mathcal{A}$ be such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and let α be a simple path in ∂M such that α intersects Γ_1 and Γ_2 transversely and

$$(\Gamma_1 \cup \Gamma_2) \cap \alpha = \{x, y\}$$

are the endpoints of α. We say that α is a path *connecting* Γ_1 and Γ_2 if $x \in \Gamma_1$ and $y \in \Gamma_2$.

As in the case of a simple closed curve we can consider S^ϵ_α, a tiny strip ("bridge") of width ϵ around α with $\partial S^\epsilon_\alpha = (\alpha^+_\epsilon \cup \alpha^-_\epsilon) \cup (\beta^x_\epsilon \cup \beta^y_\epsilon)$ where $S^\epsilon_\alpha, \alpha^+_\epsilon, \alpha^-_\epsilon, \beta^x_\epsilon$ and β^y_ϵ are defined exactly as before. Connecting Γ_1 and Γ_2 via this strip S^ϵ_α results to the following definition.

Definition 3.6. Let $\Gamma_1, \Gamma_2 \in \mathcal{A}$ be such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and let α be a path connecting them. Then we define a new curve

$$\widehat{\Gamma}^\epsilon_\alpha = \Gamma \sharp S^\epsilon_\alpha,$$

obtained by gluing on Γ a bridge S^ϵ_α of width ϵ around α, where $\Gamma = \Gamma_1 \cup \Gamma_2$. With the previous notation $\Gamma \sharp S^\epsilon_\alpha$ is the curve

$$\widehat{\Gamma}^\epsilon_\alpha = \Gamma \sharp S^\epsilon_\alpha = (\Gamma \setminus (\beta^x_\epsilon \cup \beta^y_\epsilon)) \cup (\alpha^+_\epsilon \cup \alpha^-_\epsilon).$$

Since α is a path connecting Γ_1 and Γ_2, $\widehat{\Gamma}^\epsilon_\alpha$ is a simple closed curve.

Remark 3.7. We remark here that if the curve Γ is smooth around $\Gamma \cap \alpha$ where α is a Γ-admissible path, then we can consider gluing the bridge S^ϵ_α smoothly along Γ, so that the resulting curve $\widehat{\Gamma}^\epsilon_\alpha = \widehat{\Gamma}^{\epsilon,1}_\alpha \cup \widehat{\Gamma}^{\epsilon,2}_\alpha$, as in Definition 3.4, is smooth around $x^\pm, y^\pm$.

Similarly, if Γ_1 and Γ_2 are smooth around $(\Gamma_1 \cup \Gamma_2) \cap \alpha$ where α is a path connecting Γ_1 and Γ_2, then we consider gluing the bridge S^ϵ_α smoothly along Γ_1 and Γ_2, so that the resulting curve $\widehat{\Gamma}^\epsilon_\alpha$, as in Definition 3.6, is smooth around $x^\pm, y^\pm$.

In the rest of the paper and as long as these smoothness assumptions are satisfied we will always consider constructing the curve $\widehat{\Gamma}^\epsilon_\alpha$ to be smooth around α.

Theorem 3.8 (A bridge principle for absolutely area minimizing surfaces). *Assume that either*

(a) $\Gamma \in \mathcal{A}$ *and* α *is a* Γ*-admissible simple path in* ∂M *(as in Definition* 3.3*)*

or

(b) $\Gamma = \Gamma_1 \cup \Gamma_2$ *where* $\Gamma_1, \Gamma_2 \in \mathcal{A}$ *are such that* $\Gamma_1 \cap \Gamma_2 = \emptyset$ *and* α *is a simple path in* ∂M *connecting them (as in Definition* 3.5*).*

Let $\widehat{\Gamma}^\epsilon_\alpha = \Gamma \sharp S^\epsilon_\alpha$ *be the curve obtained by attaching on* Γ *a strip of width* ϵ *around* α *(see Definitions* 3.4 *and* 3.6 *for cases* (a) *and* (b) *respectively). Then, for any* $\eta > 0$ *there exists an* $\epsilon_\alpha > 0$ *such that for almost every* $0 < \epsilon < \epsilon_\alpha$*, any absolutely area minimizing surface in* M*,* Σ^ϵ_α*, with* $\partial \Sigma^\epsilon_\alpha = \widehat{\Gamma}^\epsilon_\alpha$ *is* η*-close in Hausdorff distance and in area, to the surface obtained by attaching an* ϵ*-strip around* α *to one of the absolutely area minimizing surface bounded by* Γ*, i.e.*

$$\operatorname{dist}(\Sigma^\epsilon_\alpha, \Sigma \cup S^\epsilon_\alpha) = \max\Big\{ \sup_{x \in \Sigma^\epsilon_\alpha} \operatorname{dist}(x, \Sigma \cup S^\epsilon_\alpha), \sup_{x \in \Sigma \cup S^\epsilon_\alpha} \operatorname{dist}(x, \Sigma^\epsilon_\alpha) \Big\} < \eta$$

and

$$||\Sigma^\epsilon_\alpha| - |\Sigma|| < \eta$$

for some absolutely area minimizing surface Σ*, with* $\partial \Sigma = \Gamma$.

Proof. We will prove this by contradiction. Assume that the theorem is not true, then there exist Γ, α satisfying either the assumption (a) or the assumption (b) of the theorem, and some $\eta > 0$, such that the following holds: For any $\epsilon_\alpha > 0$, there exists an $\epsilon < \epsilon_\alpha$ and an

absolutely area minimizing surface Σ^ϵ_α in M, with $\partial\Sigma^\epsilon_\alpha = \widehat{\Gamma}^\epsilon_\alpha$, such that for any absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$ either

$$\operatorname{dist}(\Sigma^\epsilon_\alpha, \Sigma \cup S^\epsilon_\alpha) \geq \eta \tag{3.1}$$

or

$$||\Sigma^\epsilon_\alpha| - |\Sigma|| \geq \eta. \tag{3.2}$$

Hence there exists a sequence $\epsilon_i \downarrow 0$, and surfaces $\Sigma^{\epsilon_i}_\alpha$ as above such that for any absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$, either (3.1) or (3.2) holds, with ϵ replaced by ϵ_i.

Let $T_i = [\![\Sigma^{\epsilon_i}_\alpha]\!]$ be the corresponding currents. Note that for any absolutely area minimizing current T_0, with $\partial T_0 = [\![\Gamma]\!]$, we have that

$$\partial T_i = \partial(T_0 + [\![S^{\epsilon_i}_\alpha]\!])$$

and thus

$$\underline{\underline{M}}(T_i) \leq \underline{\underline{M}}(T_0 + [\![S^{\epsilon_i}_\alpha]\!]) \leq \underline{\underline{M}}(T_0) + C\epsilon_i,$$

where $\underline{\underline{M}}(T)$ denotes the mass of a current T and C is a constant that depends only on α (in particular it is independent of i). Hence, the currents T_i have uniformly bounded masses, and thus by the Federer–Fleming Compactness Theorem [9], after passing to a subsequence, $T_i \to T$, in the sense of currents, where T is an integral current. This current convergence also gives that $\partial T_i \to \partial T$ in the sense of currents and thus $\partial T = \Gamma$. Furthermore, since $\{T_i\}$ is a sequence of absolutely area minimizing currents, we get that T is also absolutely area minimizing, and moreover the corresponding Radon measures also converge $\mu_{T_i} \to \mu_T$ (cf. [22, Theorem 34.5]). By Theorem 2.6 now, we know how such an area minimizing current looks like, in particular we have that $T = [\![\Sigma_0]\!]$ where Σ_0 is an absolutely area minimizing surface which is smooth and embedded away from Γ. The measure convergence implies that

$$||\Sigma^{\epsilon_i}_\alpha| - |\Sigma_0|| \xrightarrow{i\to\infty} 0$$

and so (3.2) cannot hold. Furthermore we claim that for i big enough

$$\sup_{x\in\Sigma_0\cup S^{\epsilon_i}_\alpha} \operatorname{dist}(x, \Sigma^{\epsilon_i}_\alpha) \leq \eta.$$

To see this note first that for $\epsilon_i < \eta$ and $x \in S^{\epsilon_i}_\alpha$ we have

$$\operatorname{dist}(x, \Sigma^{\epsilon_i}_\alpha) \leq \operatorname{dist}(x, \partial S^{\epsilon_i}_\alpha) < \epsilon_i < \eta.$$

Hence if the claim is not true, then for a sequence $i \to \infty$ there is a sequence of $x_i \in \Sigma_0$ such that

$$\operatorname{dist}(x_i, \Sigma^{\epsilon_i}_\alpha) > \eta.$$

After passing to a further subsequence if necessary we have $x_i \to x_0$ for some $x_0 \in \overline{\Sigma}_0$. Since $x_0 \in \overline{\Sigma}_0$ we have that $\mu_T(B_{\eta/2}(x_0)) \neq 0$. By the measure convergence $\mu_{T_i} \to \mu_T$ now we get that

$$\lim_{i\to\infty} \mu_{T_i}(B_{\eta/2}(x_0)) = \mu_T(B_{\eta/2}(x_0)) \neq 0,$$

which implies that for i big enough there exists a $y_i \in \operatorname{spt} T_i \cap B_{\eta/2}(x_0) = \Sigma^{\epsilon_i}_\alpha \cap B_{\eta/2}(x_0)$ and thus

$$\eta < \operatorname{dist}(x_i, \Sigma^{\epsilon_i}_\alpha) \leq \operatorname{dist}(x_i, y_i) < \frac{\eta}{2},$$

which is a contradiction.

Therefore we can assume that for all i big enough we have

$$\sup_{x\in\Sigma_\alpha^{\epsilon_i}} \operatorname{dist}(x, \Sigma\cup S_\alpha^{\epsilon_i}) \geq \eta$$

for any absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$. In particular, for each i, there exists $x_i \in \Sigma_\alpha^{\epsilon_i}$ such that

$$\operatorname{dist}(x_i, \Sigma\cup S_\alpha^{\epsilon_i}) \geq \eta$$

for any absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$.

By passing to a further subsequence if necessary we also have that

$$x_i \to x_0$$

for some $x_0 \in \overline{M}$. Note that by the assumption on x_i, we have that $\operatorname{dist}(x_i, \Sigma_0\cup\alpha) \geq \eta$ and hence we also have that $\operatorname{dist}(x_0, \Sigma_0\cup\alpha) \geq \eta$.

By the above convergence, there exists an i_0, such that for all $i \geq i_0$ we have

$$\mathcal{B}\left(x_i, \frac{\eta}{4}\right) \subset \mathcal{B}\left(x_0, \frac{\eta}{2}\right) \subset \mathcal{B}(x_i, \eta)$$

where $\mathcal{B}(x,\rho)$ denotes a geodesic ball of radius ρ and centered at x, in M. Hence, using the monotonicity formula (cf. [22]) and the fact that $x_i \in \operatorname{spt} T_i \setminus \operatorname{spt}\partial T_i$

$$\mu_{T_i}\left(\mathcal{B}\left(x_0, \frac{\eta}{2}\right)\right) \geq \mu_{T_i}\left(\mathcal{B}\left(x_i, \frac{\eta}{4}\right)\right) \geq \pi\left(\frac{\eta}{4}\right)^2.$$

Using now the measure convergence, we have that

$$\mu_T\left(\mathcal{B}\left(x_0, \frac{\eta}{2}\right)\right) = \lim_i \mu_{T_i}\left(\mathcal{B}\left(x_0, \frac{\eta}{2}\right)\right) \geq \pi\left(\frac{\eta}{4}\right)^2$$

and therefore

$$\mu_T(\mathcal{B}(x_i, \eta)) \geq \mu_T\left(\mathcal{B}\left(x_0, \frac{\eta}{2}\right)\right) \geq \pi\left(\frac{\eta}{4}\right)^2$$

which implies that

$$\operatorname{spt} T \cap \mathcal{B}(x_i, \eta) \neq \emptyset \Rightarrow \operatorname{dist}(x_i, \Sigma_0\cup\alpha) < \eta.$$

This contradicts the choice of x_i and thus neither

$$\operatorname{dist}(x_i, \Sigma\cup S_\alpha^{\epsilon_i}) \geq \eta$$

can hold for every i. This finishes the proof of the theorem. □

Remark 3.9. Let $\{\epsilon_i\}_{i\in\mathbb{N}}$ be a sequence with $\epsilon_i \downarrow 0$ and for each i let $\Sigma_\alpha^{\epsilon_i}$ be an area minimizing surface with $\partial\Sigma_\alpha^{\epsilon_i} = \widehat{\Gamma}_\alpha^{\epsilon_i}$ as in Theorem 3.8. Then the proof of Theorem 3.8 shows that after passing to a subsequence, $\Sigma_\alpha^{\epsilon_i} \to \Sigma$ where Σ is an absolutely area minimizing surface with $\partial\Sigma = \Gamma$ and the convergence is with respect to Hausdorff distance and with respect to measure. Therefore we can argue as in the proof of Lemma 3.2, using White's curvature bound, Theorem 2.10 (which is applicable because of the measure convergence and the monotonicity formula [22]), to conclude that this convergence is smooth in compact sets of $\overline{M}\setminus(\Gamma\cup\alpha)$ and in fact it is also smooth around points of $\Gamma\setminus\alpha$ where Γ is smooth. In particular, if Γ is smooth, then it is a smooth convergence in compact sets of $\overline{M}\setminus\alpha$.

Remark 3.10. Note that the result of Theorem 3.8 is not an extension of the classical bridge principle to absolutely area minimizing surfaces. In the classical bridge principle, one starts with a fixed stable minimal surface T with boundary Γ, and after adding a thin bridge S^ϵ_α to Γ, one gets a new surface $T \sharp S^\epsilon_\alpha$ with boundary $\widehat{\Gamma}^\epsilon_\alpha$ which is close to the original surface T, cf. [19, 24].

In our case, if the curve Γ bounds more than one absolutely area minimizing surface, say $\{\Sigma_1, \Sigma_2, \dots, \Sigma_k\}$, then the absolutely area minimizing surface Σ^ϵ_α bounded by $\widehat{\Gamma}^\epsilon_\alpha$ is close to $\Sigma_{i_0} \cup S^\epsilon_\alpha$ for some $1 \leq i_0 \leq k$. Therefore, one may not get a modified absolutely area minimizing surface Σ^ϵ_α which is close to Σ_i for each i. The absolutely area minimizing surface Σ_{i_0} which is close to Σ^ϵ_α depends on the choice of the simple path α and then Σ_{i_0} is the limit of the sequence of absolutely area minimizing surfaces $\{\Sigma^{\epsilon_i}_\alpha\}$ in M as it appears in the proof of Theorem 3.8. At this point, we conjecture that Σ_{i_0} must be one of the two canonical absolutely area minimizing surfaces Σ^+, Σ^- bounding Γ as in [3, Lemma 4.2], since they are the extremal absolutely area minimizing surfaces with boundary Γ.

On the other hand, if one relaxes the condition of area minimizing to just minimal for the new surface, then the classical bridge principle for extreme curves is valid by using the techniques in Section 4. There, we show how one can apply [24] (see Theorem 3.12 below) to get a *minimal surface* S with $\partial S = \widehat{\Gamma}^\epsilon_\alpha$, which is close to $\Sigma_{i_0} \cup S^\epsilon_\alpha$ for a specified area minimizing surface Σ_{i_0} with $\partial \Sigma_{i_0} = \Gamma$.

Remark 3.11 (Orientation issues). If Σ is an oriented surface whose boundary is a simple closed curve Γ, gluing a strip $S \sim I \times I$ "trivially" along the boundary Γ will give another oriented surface whose boundary consists of two simple closed curves. However, if the strip S glued to Σ along Γ has a "twist", then the new surface will not be oriented anymore, and the boundary will be a simple closed curve again. The Γ-admissibility condition on α expresses this difference. Therefore, the surgery along a Γ-admissible path α gives an oriented surface with 2 boundary components, whereas the surgery along a non-Γ-admissible path β would be gluing a twisted strip along Γ to Σ, and give a nonorientable surface whose boundary is a simple closed curve again (see Figure 3).

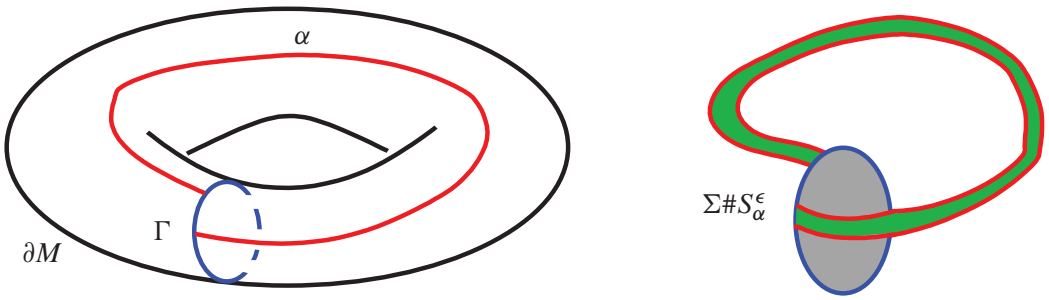


Figure 3. If α is not Γ-admissible, when we attach the strip S^ϵ_α to the oriented surface Σ in M with boundary Γ, we get a nonorientable surface. In this picture, $\Sigma \# S^\epsilon_\alpha$ is a Möbius band.

Next we show in Corollary 3.15 how one can relate the genus of the surfaces Σ^ϵ_α to that of Σ where $\Sigma^\epsilon_\alpha, \Sigma$ are as in Theorem 3.8. For the proof, we need White's bridge principle for stable minimal surfaces [24]. For the convenience of the reader, we state it here in the form that we will apply it, using our notation so far.

Theorem 3.12 ([24]). *Let Γ, α satisfy assumption* (a) *or* (b) *of Theorem* 3.8. *We further assume that both Γ and α are smooth. Let Σ be a smooth and strictly stable surface with $\partial\Sigma = \Gamma$. Let also Σ_n be a sequence of minimal surfaces converging smoothly to Σ and such that for $\Gamma_n := \partial\Sigma_n$, we have that $W \cap \Gamma_n = W \cap \Gamma$, for an open set W containing α. Then:*

(i) *There is an open set U containing $\Sigma \cup \alpha$ such that Σ is the unique area minimizing surface in $\overline{U}$.*

(ii) *If for a sequence $\epsilon_n \downarrow 0$, T_n minimizes area among all surfaces in $\overline{U}$ with boundary $\widehat{\Gamma}^{\epsilon_n}_{n\alpha}$ where $\widehat{\Gamma}^{\epsilon_n}_{n\alpha}$ are the smooth curves given in Definition* 3.4 *or* 3.6 *(see also Remark* 3.7*), using now Γ_n instead of Γ, then:*

- $|T_n| \xrightarrow{n\to\infty} |\Sigma|$.
- *T_n converges smoothly to Σ on compact subsets of $\overline{M} \setminus \alpha$.*
- *For sufficiently large n, T_n and $\Sigma_n \cup S^{\epsilon}_{\alpha}$ are diffeomorphic.*
- *For sufficiently large n, T_n is unique and strictly stable.*

Remark 3.13. In [24], part (i) of the above theorem is actually proven without any assumptions on the regularity of Γ and furthermore the theorem holds, without necessarily Γ lying in the boundary of M. Also, in [24] it is actually proven (and we will need it later) that Theorem 3.12 still holds if Σ is an immersed surface. Note also that although Γ might not lie in ∂M, as long as $\Gamma \cap W \subset \partial M$ and thus also $\Gamma_n \cap W \subset \partial M$, the construction of the curves $\widehat{\Gamma}^{\epsilon_n}_{n\alpha}$ given in Definitions 3.4 and 3.6 still makes sense.

Lemma 3.14. *Let Γ, α satisfy assumption* (a) *or* (b) *of Theorem* 3.8. *We further assume that α is smooth, Γ is smooth in an open set W containing α and for any absolutely area minimizing surface Σ with $\partial\Sigma = \Gamma$, $\mathrm{genus}(\Sigma) < \infty$. Then there exists an $\epsilon_\alpha > 0$ such that for almost every $0 < \epsilon < \epsilon_\alpha$ and for any absolutely area minimizing surface $\Sigma^{\epsilon}_{\alpha}$, with $\partial\Sigma^{\epsilon}_{\alpha} = \widehat{\Gamma}^{\epsilon}_{\alpha}$ where $\widehat{\Gamma}^{\epsilon}_{\alpha}$ is as in Definition* 3.4 *or* 3.6 *in case assumption* (a) *or* (b) *is satisfied respectively, the following hold:*

(i) *There exists a surface $\Sigma^{\epsilon}_{0\alpha} \subset \Sigma^{\epsilon}_{\alpha}$ such that*

$$\chi(\Sigma^{\epsilon}_{0\alpha}) = \chi(\Sigma \cup \alpha),$$

for some absolutely area minimizing surface Σ with $\partial\Sigma = \Gamma$ and where χ denotes the Euler characteristic, and furthermore

- *under assumption* (a), *$\Sigma^{\epsilon}_{0\alpha}$ has two boundary components,*
- *under assumption* (b), *$\Sigma^{\epsilon}_{0\alpha}$ has a unique boundary component.*

(ii) *In particular, if Γ is smooth, then the surfaces $\Sigma^{\epsilon}_{0\alpha}$ in* (i) *can be taken to be equal to $\Sigma^{\epsilon}_{\alpha}$ and thus we have that*

$$\chi(\Sigma^{\epsilon}_{\alpha}) = \chi(\Sigma \cup \alpha).$$

Proof. We prove this lemma by contradiction. Assume the lemma is not true, then for any ϵ_α, there exists an $0 < \epsilon < \epsilon_\alpha$ such that the conclusion (i) of the lemma (and in particular conclusion (ii), in case Γ is smooth) is not true. Hence there exists a sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$, with $\epsilon_i \downarrow 0$ such that for all i, the lemma with $\epsilon = \epsilon_i$ fails.

Let $\Sigma_\alpha^{\epsilon_i}$ be a sequence of absolutely area minimizing surfaces with $\partial\Sigma_\alpha^{\epsilon_i} = \widehat{\Gamma}_\alpha^{\epsilon_i}$. Then by Theorem 3.8 and in particular Remark 3.9, there exists an absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$, such that after passing to a subsequence, $\Sigma_\alpha^{\epsilon_i} \to \Sigma$, with the convergence being with respect to the Hausdorff distance, the measure and also it is a smooth convergence in compact sets of $\overline{M} \setminus (\Gamma \cup \alpha)$ and it is also smooth around points of $\Gamma \setminus \alpha$ where Γ is smooth. Finally we let $g_0 = \text{genus}(\Sigma) < \infty$

Let now $\Gamma_0 \subset \Sigma$ be a smooth curve on Σ, such that $\Gamma_0 = \Gamma$ in $W \supset \alpha$, $\Gamma_0 \sim \Gamma$ and such that the part of Σ bounded by Γ_0, which we call Σ_0, has genus g_0. Note that Σ_0 is absolutely area minimizing and it is smooth (by Theorem 2.6 and [14]). We furthermore take Γ_0, so that it is not identical to Γ, so that Σ_0 is strictly contained in Σ and therefore it is strictly stable (because Σ is absolutely area minimizing), cf. [10].

Now, because of the fact that the convergence $\Sigma_\alpha^{\epsilon_i} \to \Sigma$ is smooth in compact sets of $\overline{M} \setminus (\Gamma \cup \alpha)$ and also around points of $\Gamma \setminus \alpha$ where Γ is smooth, we can find curves $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} \subset \Sigma_\alpha^{\epsilon_i}$ such that

- under hypothesis (a), i.e. when Γ is a single simple closed curve and α is a Γ-admissible path, $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} \subset \Sigma_\alpha^{\epsilon_i}$ consists of two connected components, and under hypothesis (b), i.e. when $\Gamma = \Gamma_1 \cup \Gamma_2$ and α is a path connecting them, $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} \subset \Sigma_\alpha^{\epsilon_i}$ consists of a single connected component,
- $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} = \widehat{\Gamma}_\alpha^{\epsilon_i}$ in $W \supset \widehat{\Gamma}_\alpha^{\epsilon_i} \cap S_\alpha^{\epsilon_i}$,
- $\widehat{\Gamma}_{0\alpha}^{\epsilon_i}$ is smooth away from $S_\alpha^{\epsilon_i}$ (hence $\widehat{\Gamma}_{0\alpha}^{\epsilon_i}$ is smooth),
- $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} \to \Gamma_0$ smoothly away from α.

Let $\Sigma_{0\alpha}^{\epsilon_i}$ be the part of $\Sigma_\alpha^{\epsilon_i}$ bounded by $\widehat{\Gamma}_{0\alpha}^{\epsilon_i}$. Then, because of the convergences $\Sigma_\alpha^{\epsilon_i} \to \Sigma$ and $\widehat{\Gamma}_{0\alpha}^{\epsilon_i} \to \Gamma_0$, as described above, we have that $\Sigma_{0\alpha}^{\epsilon_i}$ converges to Σ_0 and the convergence $\Sigma_{0\alpha}^{\epsilon_i} \to \Sigma_0$ is smooth in compact sets of $\overline{M} \setminus \alpha$.

Furthermore, letting $\Gamma_i = \partial(\Sigma_{0\alpha}^{\epsilon_i} \cup S_\alpha^{\epsilon_i})$, we have that Γ_i is smooth, $\widehat{\Gamma}_{0\alpha}^{\epsilon_i}$ is the curve that we obtain by attaching a bridge around α on the curve Γ_i as described in Definitions 3.4 and 3.6 and $\Gamma_i \to \Gamma_0$ smoothly.

Note, that since Σ_0 is strictly stable, by White's bridge principle, and in particular by (i) of Theorem 3.12 (see also Remark 3.13), there exists an open set U containing $\Sigma_0 \cup \alpha$, such that Σ_0 is the unique area minimizing surface with boundary Γ_0 in $\overline{U}$. Since the convergence $\Sigma_{0\alpha}^{\epsilon_i} \to \Sigma_0$ is also with respect to the Hausdorff distance, for any $\eta > 0$, there exists an i_0 such that for any $i \geq i_0$, $\Sigma_{0\alpha}^{\epsilon_i}$ is in an η-neighborhood of $\Sigma_0 \cup \alpha$. Taking η small enough (so that the η-neighborhood of $\Sigma_0 \cup \alpha$ is contained in U) we have that there exists an i_0 such that for any $i \geq i_0$, $\Sigma_{0\alpha}^{\epsilon_i} \subset U$. Since $\Sigma_{0\alpha}^{\epsilon_i}$ are absolutely area minimizing, they also minimize area among all surfaces with the same boundary in $\overline{U}$.

Let Σ_i now be an area minimizing surface in U, such that $\partial\Sigma_i = \Gamma_i$. Then

$$\begin{aligned}\text{Area}(\Sigma_i) &\leq \text{Area}(\Sigma_{0\alpha}^{\epsilon_i}) + \text{Area}(S_\alpha^{\epsilon_i}) \\ &\leq \text{Area}(\Sigma_\alpha^{\epsilon_i}) + \text{Area}(S_\alpha^{\epsilon_i}),\end{aligned}$$

so after passing to a subsequence, Σ_i converge to an area minimizing surface in U with boundary equal to Γ_0, and since Σ_0 is the unique such surface, we have that $\Sigma_i \to \Sigma_0$. Arguing as in the proof of Theorem 3.8 and in particular as in Remark 3.9, we can apply White's curvature estimate, Theorem 2.10, to conclude that the convergence is smooth.

Note now that $\Sigma^{\epsilon_i}_{0\alpha}$ is absolutely area minimizing and thus by White's bridge principle, in particular by (ii) of Theorem 3.12, for i large enough $\Sigma^{\epsilon_i}_{0\alpha}$ and $\Sigma_i \cup S^{\epsilon_i}_{\alpha}$ are diffeomorphic. Since $\Sigma_i \to \Sigma_0$ smoothly and $(\Sigma_0 \cup S^{\epsilon_i}_{\alpha}) \sim (\Sigma_0 \cup \alpha)$, we have that $\Sigma^{\epsilon_i}_{0\alpha}$ is homotopic equivalent to $\Sigma_0 \cup \alpha$, which by construction is homotopic equivalent to $\Sigma \cup \alpha$, and thus $\chi(\Sigma^{\epsilon_i}_{0\alpha}) = \chi(\Sigma \cup \alpha)$. This leads to a contradiction, since we have assumed that (i) of the lemma with $\epsilon = \epsilon_i$ fails for any i and thus finishes the proof of part (i) of the lemma.

If Γ is smooth, then we can pick Γ_0, so that $\Sigma \setminus \Sigma_0$ is topologically an annulus. Furthermore, since $\Gamma_0 = \Gamma$ in $W \supset \alpha$, there exists an open set $\mathcal{O} \subset M \setminus \alpha$, such that $\mathcal{O} \supset \Sigma \setminus \Sigma_0$, $\mathcal{O} \supset \Sigma^{\epsilon_i}_{\alpha} \setminus \Sigma^{\epsilon_i}_{0\alpha}$ for i large enough and the convergence $\Sigma^{\epsilon_i}_{\alpha} \to \Sigma$ is smooth in $\mathcal{O}$. This, along with the convergence $\Sigma^{\epsilon_i}_{0\alpha} \to \Sigma_0$ and the fact that $\Sigma \setminus \Sigma_0$ is topologically an annulus, implies that for large i, $\Sigma^{\epsilon_i}_{\alpha} \setminus \Sigma^{\epsilon_i}_{0\alpha}$ is also topologically an annulus and therefore

$$\chi(\Sigma^{\epsilon_i}_{\alpha}) = \chi(\Sigma^{\epsilon_i}_{0\alpha}) = \chi(\Sigma \cup \alpha).$$

This leads also to a contradiction, since we have assumed that if Γ is smooth, then (ii) of the lemma with $\epsilon = \epsilon_i$ fails for any i and thus finishes the proof of the lemma. □

Corollary 3.15. *Let* Γ, α *satisfy assumption* (a) *or* (b) *of Theorem* 3.8. *We further assume that* α *is smooth,* Γ *is smooth in an open set* W *containing* α *and for any absolutely area minimizing surface* Σ, *with* $\partial\Sigma = \Gamma$, Σ *is connected and* $\mathrm{genus}(\Sigma) < \infty$. *Then, for almost every* $0 < \epsilon < \epsilon_\alpha$ *where* ϵ_α *is as in Lemma* 3.14, *and for any absolutely area minimizing surface* $\Sigma^{\epsilon}_{\alpha}$, *with* $\partial\Sigma^{\epsilon}_{\alpha} = \widehat{\Gamma}^{\epsilon}_{\alpha}$ *where* $\widehat{\Gamma}^{\epsilon}_{\alpha}$ *is as in Definition* 3.4 *or* 3.6 *in case assumption* (a) *or* (b) *is satisfied respectively, the following holds:*

(i) *Under assumption* (a)

$$\mathrm{genus}(\Sigma^{\epsilon}_{\alpha}) \geq \mathrm{genus}(\Sigma),$$

and under assumption (b)

$$\mathrm{genus}(\Sigma^{\epsilon}_{\alpha}) \geq \mathrm{genus}(\Sigma) + 1$$

for some absolutely area minimizing surface Σ *with* $\partial\Sigma = \Gamma$.

(ii) *If in particular* Γ *is smooth, then under assumption* (a)

$$\mathrm{genus}(\Sigma^{\epsilon}_{\alpha}) = \mathrm{genus}(\Sigma),$$

and under assumption (b)

$$\mathrm{genus}(\Sigma^{\epsilon}_{\alpha}) = \mathrm{genus}(\Sigma) + 1.$$

Proof. Let S be an orientable surface with k boundary components. Then by the classification of orientable surfaces, S is a genus g surface if and only if $\chi(S) = (2-k) - 2g$.

Let $\Sigma^{\epsilon}_{0\alpha} \subset \Sigma^{\epsilon}_{0}$ be as in (i) of Lemma 3.14, so that $\chi(\Sigma^{\epsilon}_{0\alpha}) = \chi(\Sigma \cup \alpha)$ for some absolutely area minimizing surface Σ with $\partial\Sigma = \Gamma$ and so that $\Sigma^{\epsilon}_{0\alpha}$ has two boundary components in case assumption (a) is satisfied and a unique boundary component in case assumption (b) is satisfied. Let also $g_0 = \mathrm{genus}(\Sigma) < \infty$.

By realizing the arc α as an edge in the triangulation, we have $\chi(\Sigma \cup \alpha) = \chi(\Sigma) - 1$.

Now under hypothesis (a), Σ is a genus g_0 surface with one boundary component, therefore we have $\chi(\Sigma \cup \alpha) = \chi(\Sigma) - 1 = -2g_0$. Since, $\Sigma^{\epsilon}_{0\alpha}$ has two boundary components and $\chi(\Sigma^{\epsilon}_{0\alpha}) = \chi(\Sigma \cup \alpha) = -2g_0$, we have that $\Sigma^{\epsilon}_{0\alpha}$ is a genus g_0 surface with two boundary components.

On the other hand, under hypothesis (b), as Σ is a connected surface with two boundary components, $\chi(\Sigma \cup \alpha) = \chi(\Sigma) - 1 = -2g_0 - 1$ where g_0 is the genus of Σ. Since, $\Sigma^\epsilon_{0\alpha}$ has a unique boundary component and $\chi(\Sigma^\epsilon_{0\alpha}) = \chi(\Sigma \cup \alpha) = -2g_0 - 1$, we have that $\Sigma^\epsilon_{0\alpha}$ is a genus $g_0 + 1$ surface with one boundary component.

Note now that since $\Sigma^\epsilon_{0\alpha} \subset \Sigma^\epsilon_\alpha$, $\text{genus}(\Sigma^\epsilon_\alpha) \geq \text{genus}(\Sigma^\epsilon_{0\alpha})$. This finishes the proof of (i) of the lemma.

Part (ii) is immediate from (ii) of Lemma 3.14, since if Γ is smooth, then $\Sigma^\epsilon_{0\alpha} = \Sigma^\epsilon_\alpha$. □

3.2. Horn surgery and thin handles. In the previous part, by adding a bridge near a Γ-admissible path α to the simple closed curve Γ, we get

$$\widehat{\Gamma}^\epsilon_\alpha = \Gamma \sharp S^\epsilon_\alpha$$

(as in Definition 3.4) which is the union of two simple closed curves $\widehat{\Gamma}^{\epsilon,1}_\alpha$ and $\widehat{\Gamma}^{\epsilon,2}_\alpha$ in ∂M. Now, we repeat this process one more time, by connecting the two simple closed curves $\widehat{\Gamma}^{\epsilon,1}_\alpha$ and $\widehat{\Gamma}^{\epsilon,2}_\alpha$ via a bridge around a new path τ, connecting the two curves. Thus, we get a simple closed curve $\widehat{\Gamma}^{\epsilon,\delta}_{\alpha,\tau} = \widehat{\Gamma}^\epsilon_\alpha \sharp S^\delta_\tau$ (as in Definition 3.5 and where δ here denotes the width of the new bridge) which bounds an absolutely area minimizing surface whose genus, we will show, is strictly greater than the genus of an absolutely area minimizing surface Σ in M with $\partial\Sigma = \Gamma$. In other words, by adding two bridges $S_\epsilon(\alpha)$ and $S_\delta(\tau)$ successively to a simple closed curve Γ, we modify an absolutely area minimizing surface it bounds by adding a thin handle. We will refer to this operation on Γ as *the horn surgery* (see Figure 4), and to the resulting additional handle in the new absolutely area minimizing surface as *the thin handle* (see Figure 5). Now, let us give the formal construction and definitions.

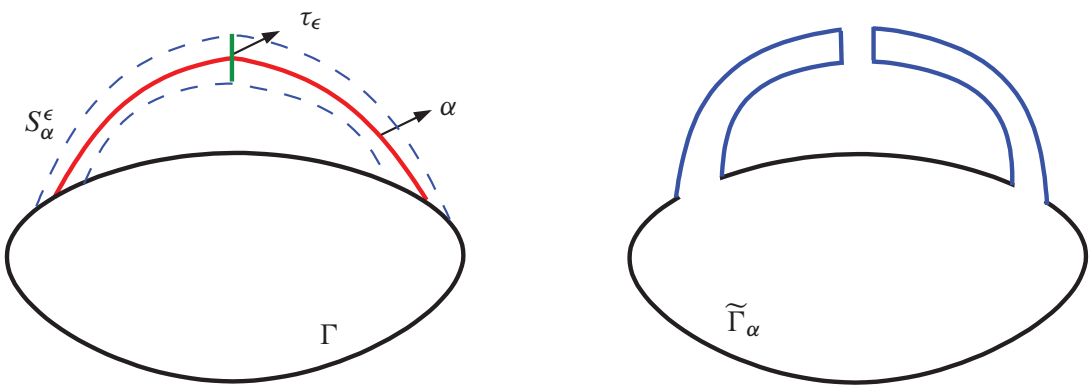


Figure 4. The horn surgery on Γ along α.

We will use the notation introduced in Section 3.1. Let Γ, α and $\widehat{\Gamma}^\epsilon_\alpha$ be as in Definition 3.4. Let z be the midpoint of α. Let, $z^+_\epsilon, z^-_\epsilon$ be the midpoints of $\alpha^+_\epsilon, \alpha^-_\epsilon$ respectively. Let τ_ϵ be a small path from z^+_ϵ to z^-_ϵ through z in ∂M. For $0 < \delta \ll \epsilon$, let

$$S^\delta_\tau = N_\delta(\tau_\epsilon) \cap S^\epsilon_\alpha$$

be a tiny strip ("bridge") of width δ around τ_ϵ connecting the simple closed curves $\widehat{\Gamma}^{\epsilon,1}_\alpha$ and $\widehat{\Gamma}^{\epsilon,2}_\alpha$ where $N_\delta(\tau_\epsilon)$ is the δ neighborhood of τ_ϵ in ∂M. Here, we assume δ is sufficiently small so that for any $0 < \delta' < \delta$, $\partial S^{\delta'}_\tau \cap \text{int}(S^\epsilon_\alpha)$ consists of exactly two arc segments, which we call τ^+_ϵ and τ^-_ϵ. Then $\partial S^\delta_\tau = (\tau^+_\epsilon \cup \tau^-_\epsilon) \cup (\gamma^1_\delta \cup \gamma^2_\delta)$ where $\gamma^i_\delta \subset \widehat{\Gamma}^{\epsilon,i}_\alpha$, $i = 1, 2$.

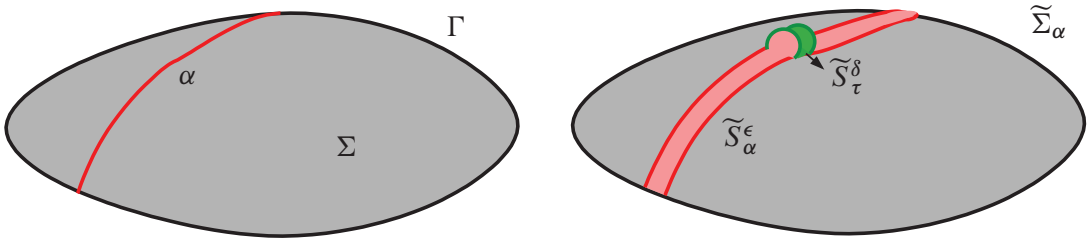


Figure 5. Adding a *thin gandle* to Σ along α. Here, $\widetilde{S}^\epsilon_\alpha$ and $\widetilde{S}^\delta_\tau$ represent the parts of $\widetilde{\Sigma}_\alpha$, which are close to the strips S^ϵ_α and S^δ_τ in ∂M.

Definition 3.16. We define *the horn surgery* on $\Gamma \in \mathcal{A}$ along a Γ-admissible path α to be the curve

$$\widetilde{\Gamma}_\alpha = \Gamma \sharp S^\epsilon_\alpha \sharp S^\delta_\tau$$

with τ being a path passing through the midpoint of α as described above (recall Definitions 3.4 and 3.6). With the above notation this curve is given

$$\widetilde{\Gamma}_\alpha = (\widehat{\Gamma}^\epsilon_\alpha \setminus (\gamma^1_\delta \cup \gamma^2_\delta)) \cup (\tau^+_\epsilon \cup \tau^-_\epsilon).$$

Hence, $\widetilde{\Gamma}_\alpha$ is a simple closed curve in ∂M (see Figure 4).

Remark 3.17. By Remark 3.7, we note here that if α is smooth and the curve Γ is smooth around α, then $\widehat{\Gamma}^\epsilon_\alpha$ is smooth around $\widehat{\Gamma}^\epsilon_\alpha \cap S^\epsilon_\alpha$ and in particular around $\widehat{\Gamma}^\epsilon_\alpha \cap \tau$. Then we can take the path τ to be smooth and we can consider gluing the bridge S^δ_τ smoothly along $\widehat{\Gamma}^\epsilon_\alpha$, so that the resulting curve $\widetilde{\Gamma}_\alpha$ is smooth around $\widehat{\Gamma}^\epsilon_\alpha \cap S^\delta_\tau$ and hence it is also smooth around $\widehat{\Gamma}^\epsilon_\alpha \cap S^\epsilon_\alpha$.

In the rest of the paper and as long as these smoothness assumptions are satisfied we will always consider constructing the curve $\widetilde{\Gamma}_\alpha$ to be smooth around α.

Let $\Gamma \in \mathcal{A}_g$, and α be a Γ-admissible path. Now, we claim that after the horn surgery on Γ along α, $\widetilde{\Gamma}_\alpha$ is in $\mathcal{A}_{g+1}$. In other words, the horn surgery on the boundary curve Γ will add a thin handle to an area minimizing surface Σ with boundary Γ and thus increase its genus (see Figure 5).

Lemma 3.18 (Thin handle). *Let $\Gamma \in \mathcal{A}_g$, and α be a Γ-admissible smooth path, such that α is smooth, Γ is smooth in an open set W containing α and for any absolutely area minimizing surface Σ with $\partial\Sigma = \Gamma$, $\operatorname{genus}(\Sigma) < \infty$. Then there exists an ϵ_α, such that for almost every $0 < \epsilon < \epsilon_\alpha$ the following holds:*

(i) *There is a sufficiently small $\delta_\tau > 0$, such that for any $0 < \delta < \delta_\tau$, after the horn surgery on Γ along α (see Definition 3.16), $\widetilde{\Gamma}_\alpha \in \mathcal{A}_{g+1}$. In particular, for any absolutely area minimizing surface $\widetilde{\Sigma}_\alpha$ with $\partial\widetilde{\Sigma}_\alpha = \widetilde{\Gamma}_\alpha$, $\operatorname{genus}(\widetilde{\Sigma}_\alpha) \geq \operatorname{genus}(\Sigma) + 1$ where Σ is an absolutely area minimizing surface with $\partial\Sigma = \Gamma$.*

(ii) *If Γ is smooth, then $\widetilde{\Gamma}_\alpha$ is a smooth curve in $\mathcal{A}_{g+1}$ and in particular, for any absolutely area minimizing surface $\widetilde{\Sigma}_\alpha$ with $\partial\widetilde{\Sigma}_\alpha = \widetilde{\Gamma}_\alpha$, $\operatorname{genus}(\widetilde{\Sigma}_\alpha) = \operatorname{genus}(\Sigma) + 1$ where Σ is an absolutely area minimizing surface with $\partial\Sigma = \Gamma$.*

Proof. Recall (Remarks 3.7, 3.17) that $\widehat{\Gamma}^{\epsilon}_{\alpha}$ and $\widetilde{\Gamma}_{\alpha}$ are both smooth around S^{ϵ}_{α}. Note first that for any $\epsilon < \epsilon_{\alpha}$, with ϵ_{α} as in Corollary 3.15, and for any absolutely area minimizing surface $\Sigma^{\epsilon}_{\alpha}$ with $\partial\Sigma^{\epsilon}_{\alpha} = \widehat{\Gamma}^{\epsilon}_{\alpha}$

$$\text{genus}(\Sigma^{\epsilon}_{\alpha}) \geq \text{genus}(\Sigma)$$

for an absolutely area minimizing surface Σ with $\partial\Sigma = \Gamma$ and the above inequality is actually an equality if Γ is smooth.

We apply now again Corollary 3.15 with Γ, α, $\Sigma^{\epsilon}_{\alpha}$ and Σ replaced by $\widehat{\Gamma}^{\epsilon}_{\alpha}$, τ, $\widetilde{\Sigma}_{\alpha}$ and $\Sigma^{\epsilon}_{\alpha}$ (note that here $\Sigma^{\epsilon}_{\alpha}$, $\widetilde{\Sigma}_{\alpha}$ denote *any* absolutely area minimizing surfaces with boundary equal to $\widehat{\Gamma}^{\epsilon}_{\alpha}$ and $\widetilde{\Gamma}_{\alpha}$ respectively). Since now we use the hypothesis (b) of Corollary 3.15, we get that there exists a $\delta_{\tau} > 0$ such that for any $0 < \delta < \delta_{\tau}$ and for any absolutely area minimizing surface $\widetilde{\Sigma}_{\alpha}$ with $\partial\widetilde{\Sigma}_{\alpha} = \widetilde{\Gamma}_{\alpha}$

$$\text{genus}(\widetilde{\Sigma}_{\alpha}) \geq \text{genus}(\Sigma^{\epsilon}_{\alpha}) + 1 \geq \text{genus}(\Sigma) + 1$$

for an absolutely area minimizing surface $\Sigma^{\epsilon}_{\alpha}$ with boundary equal to $\widehat{\Gamma}^{\epsilon}_{\alpha}$ and an absolutely area minimizing surface Σ with boundary equal to Γ. In the case when Γ is smooth (cf. Remark 3.17), the above inequalities are actually equalities and thus

$$\text{genus}(\widetilde{\Sigma}_{\alpha}) = \text{genus}(\Sigma^{\epsilon}_{\alpha}) + 1 = \text{genus}(\Sigma) + 1. \qquad \square$$

Remark 3.19. In the horn surgery construction, the Γ-admissible path α determines the $\epsilon_{\alpha} > 0$, and the path τ determines the $\delta_{\tau} > 0$. Since τ depends on the $0 < \epsilon < \epsilon_{\alpha}$, δ_{τ} also depends on ϵ_{α}. From now on, we will assume the choices for the horn surgery on Γ along α are canonical. In other words, we always consider $0 < \epsilon < \epsilon_{\alpha}$, τ and $0 < \delta < \delta_{\tau}$, so that the horn surgery is well defined. Furthermore from now on we assume that ϵ_{α} and δ_{τ} are small enough so that Lemma 3.18 holds, i.e. if Γ is smooth around α, then for $\widetilde{\Gamma}_{\alpha} = \Gamma \sharp S^{\epsilon}_{\alpha} \sharp S^{\delta}_{\tau}$ we can apply Lemma 3.18 and in particular if Γ is smooth, then we can apply (ii) of Lemma 3.18.

Now, let us consider the space $\mathcal{B}_g$. We show that $\mathcal{B}_g$ is nonempty for any $g \geq 0$. As noted before, $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$. In Lemma 3.2 we proved that for any g, $\mathcal{A}_g$ is open, but this does not imply that $\mathcal{B}_g$ is nonempty. In order to show this, we need to rule out the case $\mathcal{A}_g = \mathcal{A}_{g+1}$. In other words, we will show that for any $g \geq 0$, there is a simple closed curve $\Gamma_g \in \mathcal{A}$ such that the minimum genus among all absolutely area minimizing surfaces with boundary Γ_g is exactly g.

Theorem 3.20. *For any $g \geq 0$, $\mathcal{B}_g$ is nonempty.*

Proof. Since $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$, it suffices to show the existence of a simple closed curve $\Gamma_g \in \mathcal{A}_g$ such that Γ_g bounds an absolutely area minimizing surface Σ of genus g.

Clearly, $\mathcal{B}_0$ is not empty, as we can take a sufficiently small smooth simple closed curve in ∂M, so that the absolutely area minimizing surface bounded by this curve is a smooth disk in M, say D_0, and since M is strictly mean convex, D_0 is also properly embedded, i.e. $D_0 \cap \partial M = \partial D$.

Now, let Γ_0 be a smooth curve in $\mathcal{B}_0$ and take a Γ_0-admissible and smooth path α_1. Then, by (ii) of Lemma 3.18, the horn surgery on Γ_0 along α_1 will give us a simple closed and smooth curve $\widetilde{\Gamma_{0\alpha_1}}$, say Γ_1, such that $\Gamma_1 \in \mathcal{B}_1$.

We can argue now by induction, that for any k there exists a smooth curve in $\mathcal{B}_k$ as follows: Assume that there exists a smooth curve $\Gamma_{k-1} \in \mathcal{B}_{k-1}$ and take a Γ_{k-1}-admissible and smooth path α_k. The horn surgery on Γ_{k-1} along α_k (Lemma 3.18) gives a new smooth curve Γ_k with $\Gamma_k \in \mathcal{B}_k$ and thus $\mathcal{B}_k \neq \emptyset$. □

Remark 3.21. Intuitively, this gives a construction of adding a handle to an absolutely area minimizing surface by modifying the boundary curve. Note that, since the handles can be as small as we want, the area of the new surfaces can be as close as we want to the area of the original surface (cf. Theorem 3.8). Furthermore, by the proof of Theorem 3.20, we actually conclude that not only can we always find a simple closed curve in $\mathcal{B}_g$, but also we can find a smooth simple closed curve in $\mathcal{B}_g$.

In Lemma 3.2, we proved that for any $g > 0$, $\mathcal{A}_g$ is open in $\mathcal{A}$, and in Theorem 3.20, we showed that $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$ is nonempty. Now, by using the handle construction in Lemma 3.18, we will prove that $\mathcal{A}_g$ is not only open in $\mathcal{A}$, but also dense in $\mathcal{A}$.

Theorem 3.22. *For any $g \geq 0$, $\mathcal{A}_g$ is dense in $\mathcal{A}$, i.e. $\overline{\mathcal{A}}_g = \mathcal{A}$.*

Proof. Fix $g > 0$. It suffices to show that for any $\Gamma_0 \in \mathcal{A}$, and for any $\mu > 0$, there exists a simple closed curve $\Gamma \in \mathcal{A}_g$ with $d(\Gamma, \Gamma_0) < \mu$ where by $d(\Gamma, \Gamma_0) < \mu$ we mean that Γ is in a μ-neighborhood of Γ_0 in the $\mathcal{C}^0$ topology (cf. Definition 3.1).

Let $\Gamma_0 \in \mathcal{A}$. If $\Gamma_0 \in \mathcal{A}_g$, then we are done as we can take $\Gamma = \Gamma_0$. Hence we can assume that $\Gamma_0 \notin \mathcal{A}_g$, which implies that for some g_0, with $0 \leq g_0 \leq g-1$, Γ_0 belongs to $\mathcal{B}_{g_0} \subset \mathcal{A}_{g_0}$. Since $\mathcal{A}_{g_0}$ is open, we can take $\Gamma_1 \in \mathcal{A}_{g_0}$, so that Γ_1 is piecewise smooth and

$$d(\Gamma_0, \Gamma_1) < \frac{\mu}{2}.$$

If $\Gamma_1 \in \mathcal{A}_g$, then we can take $\Gamma = \Gamma_1$ and we are done. Therefore, we assume that $\Gamma_1 \in \mathcal{B}_{g_1}$, with $g > g_1 \geq g_0$ and we will construct Γ by $h = g - g_1$ horn surgeries on Γ_1.

Recall the construction in Lemma 3.18. Since Γ_1 is piecewise smooth, we can find a Γ_1-admissible and smooth path α_1 satisfying the hypothesis of Lemma 3.18, i.e. so that Γ_1 is smooth in an open neighborhood of α_1. This implies that since $\Gamma_1 \in \mathcal{A}_{g_1}$, $\widetilde{\Gamma_{1\alpha}} \in \mathcal{A}_{g_1+1}$. Let $\Gamma_2 = \widetilde{\Gamma_{1\alpha}}$, so that Γ_2 is obtained by horn surgery on Γ_1 along the Γ_1-admissible path α_1. In general we let Γ_k to be the curve obtained by horn surgery on Γ_{k-1} along a Γ_{k-1}-admissible smooth path α_{k-1}, satisfying the hypothesis of Lemma 3.18. Then

$$\Gamma_k \in \mathcal{A}_{g_1+k-1}, \quad \Gamma_k \sim \Gamma_{k-1}$$

and since we are free to choose the admissible paths α_k as short as we want, we can take the curves Γ_k so that

$$d(\Gamma_{k-1}, \Gamma_k) < \frac{1}{2h}\mu,$$

for any given $\mu > 0$. Let $\Gamma = \Gamma_{h+1}$. Then, by construction $\Gamma_h \in \mathcal{A}_{g_1+h} = \mathcal{A}_g$, and

$$d(\Gamma_0, \Gamma) < \frac{\mu}{2} + h\frac{\mu}{2h} = \mu.$$

The proof follows. □

Corollary 3.23. *For any $g \geq 0$, $\mathcal{B}_g$ is nowhere dense in $\mathcal{A}$.*

Proof. We have $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$ and thus

$$\overline{\mathcal{B}}_g = \overline{\mathcal{A}}_g \setminus \text{int}(A_{g+1}) = \mathcal{A} \setminus A_{g+1},$$

since $\mathcal{A}_g$ is dense in $\mathcal{A}$ by Theorem 3.22 and $\mathcal{A}_{g+1}$ is open in $\mathcal{A}$, by Lemma 3.2. Since $\mathcal{A}_{g+1}$ is also dense in $\mathcal{A}$, $\text{int}(\overline{\mathcal{B}_g}) = \emptyset$. □

Remark 3.24. Note that $\mathcal{B}_\infty := \bigcap_g \mathcal{A}_g \neq \emptyset$. To see this, even though $\mathcal{B}_\infty = \bigcap_g \mathcal{A}_g$ where all the $\mathcal{A}_g$ are open and dense in $\mathcal{A}$, the classical Baire Category Theorem argument ($\bigcap_g \mathcal{A}_g$ is dense in $\mathcal{A}$) would not work since $\mathcal{A}$ is not a complete metric space. However, it is possible to construct a simple rectifiable curve Γ_∞ in ∂M which bounds an absolutely area minimizing surface Σ with infinite genus. To see this, take a simple closed curve Γ in ∂M, and do the horn surgery infinitely many times by choosing sufficiently small admissible paths $\{\alpha_1, \alpha_2, \dots\}$ successively (see Figure 6).

Note also that since the above construction can be done on any piecewise smooth $\Gamma \in \mathcal{A}$, we actually have that $\mathcal{B}_\infty$ is dense in $\mathcal{A}$.

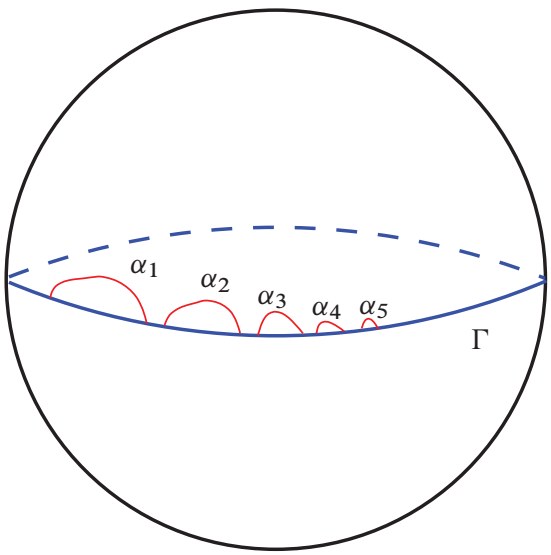


Figure 6. Construction of a rectifiable curve in ∂M bounding an absolutely area minimizing surface of infinite genus in M by doing infinitely many horn surgeries.

To summarize, so far in this section, we have shown that

$$\mathcal{A} = \mathcal{A}_0 \supsetneq \mathcal{A}_1 \supsetneq \mathcal{A}_2 \supsetneq \cdots \supsetneq A_g \supsetneq \cdots$$

where $\mathcal{A}_g$ is an open dense subset of $\mathcal{A}$ for any $g \geq 0$. Moreover, we have shown that the set $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$ is not empty, and nowhere dense in $\mathcal{A}$ for any $g \geq 0$.

We finish this section with a result which states that there are simple closed curves bounding absolutely area minimizing surfaces with different genus. In other words, we show that the relation $\mathfrak{g} : \mathcal{A} \to \mathbb{N}$ such that $\mathfrak{g}(\Gamma)$ is defined to be the genus of an absolutely area minimizing surface Σ_Γ in M with $\partial \Sigma_\Gamma = \Gamma$, is not a function.

Theorem 3.25. *Let M be a compact, orientable, strictly mean convex 3-manifold. Then, for any $g \geq 0$, there exists a simple closed curve Γ in ∂M such that Γ bounds two different absolutely area minimizing surfaces Σ_1 and Σ_2 in M with $g = \text{genus}(\Sigma_1) \neq \text{genus}(\Sigma_2)$.*

Proof. Let Γ be a smooth curve in $\mathcal{B}_g$ (cf. Theorem 3.20, Remark 3.21). Let α be a smooth and Γ-admissible path in ∂M. Then, by Theorem 3.8, for any $\eta_0 > 0$, there is an $\epsilon_0 > 0$ such that any absolutely area minimizing surface Σ_α with boundary $\widehat{\Gamma}_\alpha^{\epsilon_0}$ is η_0-close to the surface $\Sigma \cup S_\alpha^{\epsilon_0}$, for some absolutely area minimizing surface Σ, with $\partial\Sigma = \Gamma$ (where $\widehat{\Gamma}_\alpha^{\epsilon_0}, S_\alpha^{\epsilon_0}$ are as in Definition 3.4). Recall that since Γ is smooth, we consider this bridge construction so that $\widehat{\Gamma}_\alpha^{\epsilon_0}$ is smooth (cf. Remark 3.7). Now foliate the strip $S_\alpha^{\epsilon_0}$ with arcs $\{\tau_\delta\}_{|\delta|\leq c}$ where τ_δ is parametrized by the arclength of α, with $|\alpha| = 2c$, such that $\tau_0 = \tau$ and $\tau_{\pm c} \subset \Gamma$. We also construct the arcs τ_δ, so that they are smooth and hence after the horn surgery, the curves

$$\widetilde{\Gamma}_\alpha^\delta = \Gamma \sharp S_\alpha^{\epsilon_0} \sharp S_\tau^\delta = (\widehat{\Gamma}_\alpha^{\epsilon_0} \setminus (\gamma_\delta^1 \cup \gamma_\delta^2)) \cup (\tau_\delta \cup \tau_{-\delta})$$

are smooth where S_τ^δ represents the strip in the surface $S_\alpha^{\epsilon_0}$ separated by the arcs τ_δ and $\tau_{-\delta}$ and $\partial S_\tau^\delta = \gamma_\delta^1 \cup \gamma_\delta^2 \cup \tau_\delta \cup \tau_{-\delta}$ (cf. Definition 3.16, Remark 3.17). Then

$$\widetilde{\Gamma}_\alpha^c = \Gamma \quad \text{and} \quad \widetilde{\Gamma}_\alpha^0 = \widehat{\Gamma}_\alpha^{\epsilon_o}.$$

Now we note the following:

(i) Let $\delta \in (0, c]$, $\{\delta_i\}_{i\in\mathbb{N}}$ be a sequence in $[0, c]$ such that $\delta_i \to \delta$ and let $\widetilde{\Sigma}_\alpha^{\delta_i}$ be a sequence of absolutely area minimizing surfaces with $\partial\widetilde{\Sigma}_\alpha^{\delta_i} = \widetilde{\Gamma}_\alpha^{\delta_i}$. Then, arguing as in Theorem 3.8 (see in particular Remark 3.9), with the use of the compactness theorem in [9] and White's curvature estimate, Theorem 2.10, we have that, after passing to a subsequence, $\widetilde{\Sigma}_\alpha^{\delta_i}$ converges smoothly to $\widetilde{\Sigma}_\alpha^\delta$ where $\widetilde{\Sigma}_\alpha^\delta$ is an absolutely area minimizing surface with $\partial\widetilde{\Sigma}_\alpha^\delta = \widetilde{\Gamma}_\alpha^\delta$. This, along with the Gauss–Bonnet Theorem implies that

$$\lim_i \text{genus}(\widetilde{\Sigma}_\alpha^{\delta_i}) = \text{genus}(\widetilde{\Sigma}_\alpha^\delta).$$

(ii) By Lemma 3.18, there is a $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, $\widetilde{\Gamma}_\alpha^\delta$ is a smooth curve in $\mathcal{B}_{g+1}$ and on the other hand we know that

$$\widetilde{\Gamma}_\alpha^c = \Gamma \in \mathcal{B}_g.$$

Moreover, by (i) we have that $\delta_0 < c$.

Let δ_1 be the infimum of all $\delta \in [0, c]$, such that $\widetilde{\Gamma}_\alpha^\delta \in \mathcal{B}_g$. By (i) and (ii) above we know that $\delta_1 \in [\delta_0, c) \subset (0, c)$. Let now $\Gamma_0 = \widetilde{\Gamma}_\alpha^{\delta_1}$. Then Γ_0 is a smooth curve and $\Gamma_0 \neq \Gamma$. Applying condition (i) with a sequence $\delta_i \downarrow \delta_1$ and a sequence $\widetilde{\Sigma}_\alpha^{\delta_i}$ of absolutely area minimizing surfaces with genus equal to g (which we can do since we can pick the δ_i, so that for each i, $\widetilde{\Gamma}_\alpha^{\delta_i} \in \mathcal{B}_g$), we conclude that there exists an absolutely area minimizing surface Σ_1 with $\partial\Sigma_1 = \Gamma_0$ and $\text{genus}(\Sigma_1) = g$. Applying now (i) with a sequence $\delta_i \uparrow \delta_1$ and a sequence $\widetilde{\Sigma}_\alpha^{\delta_i}$ of absolutely area minimizing surfaces with genus not equal to g (which we can do since for each i, $\widetilde{\Gamma}_\alpha^{\delta_i} \notin \mathcal{B}_g$), we conclude that there exists an absolutely area minimizing surface Σ_2 with $\partial\Sigma_2 = \Gamma_0$ and $\text{genus}(\Sigma_2) \neq g$. Hence, this curve Γ_0, satisfies the hypothesis of the theorem. □

4. Results on minimal surfaces in mean convex domains

In this section, we will give interesting results on uniqueness, embeddedness and the genus of minimal surfaces which are properly embedded in a mean convex 3-manifold. In particular, we generalize the examples of Peter Hall [12] in the unit 3-ball $\mathbb{B}^3$, addressing a question by Bill Meeks, to any strictly mean convex 3-manifold.

First, by using the results of the previous section, we show that the space $\mathfrak{C}$ of simple closed curves in ∂M bounding more than one minimal surface in M is generic in the set of simple nullhomologous closed curves in ∂M where M is a strictly mean convex 3-manifold.

Before giving the rigorous proof of this theorem, we give an outline of the main argument used. If ∂M is homeomorphic to S^2, then the theorem follows directly from the fact that $\mathcal{A}_1$ is open dense in $\mathcal{A}$, and $\mathcal{A}_1 \subset \mathfrak{C}$. This is because any simple closed curve Γ would be nullhomotopic in M as $\partial M \simeq S^2$, and by the result of Meeks and Yau (see Theorem 2.5), Γ bounds an area minimizing disk D in M. Hence, any simple closed curve Γ in $\mathcal{A}_1$ automatically bounds two different minimal surfaces where one of them is the absolutely area minimizing surface Σ with genus ≥ 1, and the other one is the area minimizing disk D. The same argument works in general with a slight modification. In particular, we will show that for an open dense subset $\mathfrak{C}$, any simple closed curve Γ in this subset bounds an absolutely area minimizing surface Σ in M where Σ is not the smallest genus surface which Γ bounds in M. By a theorem of White, which we have stated in Theorem 2.8, there is an area minimizing surface S of smallest genus. This implies Γ bounds at least two different minimal surfaces, where one of them is the absolutely area minimizing surface Σ, and the other one is the area minimizing representative S of the smallest genus surfaces which Γ bounds in M.

Theorem 4.1 (Curves bounding more than one minimal surface are generic). *Let M be a strictly mean convex 3-manifold. Let $\mathcal{A}$ be the space of nullhomologous simple closed curves in ∂M equipped with the $\mathcal{C}^0$ topology. Let $\mathfrak{C} \subset \mathcal{A}$ represent the simple closed curves in ∂M bounding more than one embedded stable minimal surface in M. Then, $\mathfrak{C}$ is generic in $\mathcal{A}$ in the sense that $\mathfrak{C}$ contains an open dense subset of $\mathcal{A}$.*

Remark 4.2. In particular we show the following: Let $\mathrm{AAM}(\Gamma)$, $\mathrm{AM}(\Gamma)$ be the set of *absolutely area minimizing* and that of *area minimizing* surfaces respectively bounded by Γ. Then the set

$$\{\Gamma \in \mathcal{A} \mid \exists \Sigma_1 \in \mathrm{AAM}(\Gamma), \Sigma_2 \in \mathrm{AM}(\Gamma) : \Sigma_1 \neq \Sigma_2\}$$

is open and dense in $\mathcal{A}$.

Proof. We first show that $\mathfrak{C}$ is dense in $\mathcal{A}$. Let Γ_0 be any nullhomologous simple closed curve in ∂M. It suffices to show that for any $\epsilon > 0$, there exists a $\Gamma \in \mathfrak{C}$, such that $d(\Gamma, \Gamma_0) < \epsilon$ where by $d(\Gamma, \Gamma_0) < \epsilon$ we mean that Γ is in a ϵ-neighborhood of Γ_0 in the $\mathcal{C}^0$ topology (cf. Definition 3.1).

Assume that g_0 is the minimum genus of a surface bounded by Γ_0. By Theorem 2.8 and Remark 2.9, there exists an area minimizing surface Σ_{g_0} of genus g_0. If $\Gamma_0 \notin \mathcal{B}_{g_0}$, then Γ_0 bounds an absolutely area minimizing surface Σ which has to be different from Σ_{g_0}. In this case Γ_0 bounds at least two different stable minimal surfaces, namely Σ_{g_0} and Σ, so we can take $\Gamma = \Gamma_0$. Hence we can assume that $\Gamma_0 \in \mathcal{B}_{g_0}$.

Given $g > g_0$, by Theorem 3.22, there exists a $\Gamma \in \mathcal{A}_g$, so that $d(\Gamma_0, \Gamma) < \epsilon$ and for ϵ small enough Γ is homotopic to Γ_0. We denote the annulus in ∂M between Γ and Γ_0 by A_ϵ, as we will need it later.

We claim that Γ bounds two different stable minimal surfaces. Note first that since $\Gamma \in \mathcal{A}_g$ and $g > g_0$, it bounds a stable minimal surface Σ_1 of genus bigger than g_0 (for example take an absolutely area minimizing surface that has minimum genus). To show the existence of the second surface, assume first that Γ is nullhomotopic. Then, by the theorem of Meeks and Yau, Theorem 2.5, Γ bounds a stable minimal disk Σ_2. Assume now that Γ is not nullhomotopic, i.e. it bounds no disk. Let $\widetilde{g}_\Gamma$ be the minimum genus of a surface bounded by Γ, i.e.

$$\widetilde{g}_\Gamma = \min\{g \mid \Gamma \text{ bounds a surface of genus } g\}.$$

We have that $0 < \widetilde{g}_\Gamma \leq g_0$; the second inequality being true because $\partial(\Sigma \cup A_\epsilon) = \Gamma$ where Σ is an area minimizing surface of genus g_0 and with $\partial\Sigma = \Gamma_0$. This implies that the minimum genus of a surface bounded by Γ is less than or equal to g_0. By Theorem 2.8, there exists an area minimizing surface Σ_2 of genus $\widetilde{g}_\Gamma$, which is therefore a stable minimal surface .

We have actually showed that the set

$$\mathcal{C}' = \bigcup_{g>0} \{\Gamma \in \mathcal{A}_g \mid \Gamma \text{ bounds an area minimizing surface of genus} \leq g-1\} \subset \mathcal{C}$$

is dense in $\mathcal{A}$. Next we show that $\mathcal{C}'$ is also open.

Let $\Gamma_0 \in \mathcal{C}'$. Then for some $g > 0$, $\Gamma_0 \in \mathcal{A}_g$ and it bounds an area minimizing surface of genus $\leq g-1$. Since $\mathcal{A}_g$ is open, there is an $\epsilon > 0$ such that for any $\Gamma \in \mathcal{A}$ with $d(\Gamma_0, \Gamma) < \epsilon$, Γ is homotopic to Γ_0 and belongs in $\mathcal{A}_g$.

Now, we can argue as before to show that any Γ with $d(\Gamma, \Gamma_0) < \epsilon$ bounds an area minimizing surface of genus $\leq g-1$. In particular we have the following: since $\Gamma \in \mathcal{A}_g$, it bounds an area minimizing surface Σ_1 of genus $\geq g$. If Γ is nullhomotopic in M, then, as before, by Theorem 2.5, it bounds an area minimizing disk Σ_2. If Γ is not nullhomotopic, then we let

$$\widetilde{g}_\Gamma = \min\{g \mid \Gamma \text{ bounds a surface of genus g}\}.$$

We have that $0 < \widetilde{g}_\Gamma \leq g-1$ as before, and by Theorem 2.8, there exists an area minimizing surface Σ_2 of genus $\widetilde{g}_\Gamma$.

This shows $\mathcal{C}'$ is open and dense in $\mathcal{A}$. Since $\mathcal{C}' \subset \mathcal{C}$, the proof follows. □

Remark 4.3. Note that by [3] and [4], a generic simple closed curve in ∂M bounds a unique area minimizing disk, and similarly, a generic simple closed curve in ∂M bounds a unique absolutely area minimizing surface in M. The result above shows that when we relax the condition being area minimizing to just minimal, the situation is opposite. In other words, a generic simple closed curve bounds a unique absolutely area minimizing surface, while a generic simple closed curve bounds more than one minimal surface. Hence the situations for minimal and absolutely area minimizing surfaces are quite different.

Now, we generalize Peter Hall's examples to a more general setting. In [12], Peter Hall gave examples of nonembedded stable minimal surfaces in the unit ball $\mathbb{B}^3$ of $\mathbb{R}^3$. To generalize this result to any strictly mean convex 3-manifold M, we first show that there are disjoint simple closed curves Γ_1 and Γ_2 in ∂M and stable minimal surfaces S_1 and S_2 in M with $\partial S_i = \Gamma_i$, $i = 1, 2$, that are not disjoint, i.e. $S_1 \cap S_2 \neq \emptyset$.

Theorem 4.4 (Disjoint extreme curves may not bound disjoint minimal surfaces). *Let M be a strictly mean convex 3-manifold. Then, there are stable minimal surfaces S_1 and S_2 in M with $S_1 \cap S_2 \neq \emptyset$ and $\partial S_1 \cap \partial S_2 = \emptyset$. Moreover, these surfaces can be chosen to be disks.*

Proof. We argue by contradiction using the technique in [3, Theorem 3.2], along with Theorem 4.1

Assume that the theorem is not true. By Theorem 4.1, there exist $\Gamma_0 \in \mathcal{A}$ and $\epsilon > 0$, such that any $\Gamma \in \mathcal{A}$ in an ϵ-neighborhood of Γ_0 is homotopic to Γ_0 and bounds two distinct stable minimal surfaces. Taking ϵ small enough, $A_{\Gamma_0} = \{x \in \partial M \mid \text{dist}_{\partial M}(x, \Gamma_0) < \epsilon\}$ is an annulus and we foliate this annulus by curves $\Gamma_t \in \mathcal{A}$, $t \in [-\epsilon, \epsilon]$. Each Γ_t, by the choice of ϵ, bounds two stable minimal surfaces Σ_t^+, Σ_t^-. If $H_2(M, \mathbb{Z}) = 0$, then both Σ_t^+ and Σ_t^- are separating in M and so there is a region *between* them. We let N_t be that region, i.e. N_t is such that $\partial N_t = \Sigma_t^+ \cup \Sigma_t^-$ as in [3]. Then for $t_1 < t_2$, $N_{t_1} \cap N_{t_2} = \emptyset$. This is because either $N_{t_1} \subset N_{t_2}$, which is impossible since $\Gamma_{t_1} \cap \Gamma_{t_2} = \emptyset$, or else $\partial N_{t_1} \cap \partial N_{t_2} \neq \emptyset$, which cannot hold because of our assumption that the theorem is false. Let $N = \bigcup_t N_t$. Then, for any $t \in [-\epsilon, \epsilon]$, $|N_t| > 0$ where $|N_t|$ now represents the volume of N_t, and

$$\sum_t |N_t| = |N| \leq |M| < \infty.$$

This implies that $|N_t| \neq 0$ for at most countably many t. But when $|N_t| = 0$ and since $\Sigma_t^\pm$ are smooth, this implies $\Sigma_t^+ = \Sigma_t^-$. This is a contradiction as we have assumed that for any t, Γ_t bounds two distinct stable minimal surfaces Σ_t^+ and Σ_t^-. Hence for some $t_1 < t_2$, we must have $\partial N_{t_1} \cap \partial N_{t_2} \neq \emptyset$, i.e. $(\Sigma_{t_1}^+ \cup \Sigma_{t_1}^-) \cap (\Sigma_{t_2}^+ \cup \Sigma_{t_2}^-) \neq \emptyset$ even though $\Gamma_{t_1} \cap \Gamma_{t_2} = \emptyset$. The proof follows.

For the case $H_2(M, \mathbb{Z}) \neq 0$, a similar argument from [4] works with a slight modification. If $H_2(M, \mathbb{Z}) \neq 0$, then the surfaces Σ_t^+ and Σ_t^- may not be separating in M, and we cannot talk about the region N_t *between* them. However, in this case, we can restrict ourselves to a small neighborhood of Γ_0 in M, say T_{Γ_0}, which is a very thin solid torus with $T_{\Gamma_0} \cap \partial M = A_{\Gamma_0}$. Now, the argument in the previous paragraph works, if we replace our ambient manifold M with T_{Γ_0} and the surfaces $\Sigma_t^\pm$ with $S_t^\pm = \Sigma_t^\pm \cap T_{\Gamma_0}$. Then for any t, S_t^+ and S_t^- are separating in T_{Γ_0}, and if we assume that the theorem is false, then $(S_t^+ \cup S_t^-) \cap (S_s^+ \cup S_s^-) = \emptyset$ for any $s \neq t$, since $S_t^\pm \subset \Sigma_t^\pm$. Hence, by defining N_t as the region between S_t^+ and S_t^- in T_{Γ_0}, the whole summation argument goes through, and we get a contradiction in this case, too. This proves the existence of intersecting stable minimal surfaces with disjoint boundaries.

Finally, we need to show that these surfaces can be chosen to be disks. For this we work as follows. We want to prove that there are stable minimal disks D_1 and D_2 in M with $D_1 \cap D_2 \neq \emptyset$ and $\partial D_1 \cap \partial D_2 = \emptyset$. Let p be a point in the smooth part of ∂M. Let $\epsilon > 0$ be sufficiently small so that $\overline{B}_\epsilon(p)$ is strictly mean convex (i.e. $\epsilon < \rho$ where ρ is the convexity radius at p). Let $\gamma \in \mathcal{B}_0$ with $\gamma \subset \partial M \cap B_\epsilon(p)$ and let E be the absolutely area minimizing surface, which is a disk, with $\partial E = \gamma$. By construction, E separates M into two parts and we let Ω be the "small" part containing p. Then, Ω is topologically a 3-ball, and any properly embedded surface would be separating in Ω. Let Γ_0 be in $\mathcal{A}_1$ and such that it has an annular neighborhood A_{Γ_0}, for which $A_{\Gamma_0} \subset \Omega \cap \partial M$. Since $\mathcal{A}_1$ is open by Theorem 3.2 (and by taking the annular neighborhood small enough), we can foliate A_{Γ_0} with a collection $\{\Gamma_t\} \in \mathcal{A}_1$ with $t \in [-\epsilon, \epsilon]$. Then for any $t \in [-\epsilon, \epsilon]$, the absolutely area minimizing sur-

face Σ_t with boundary Γ_t lies in Ω, because E is absolutely area minimizing surface and by [3], $E \cap \Sigma_t = \emptyset$. The surface Σ_t separates Ω into two parts, say Ω_t^+ (the part containing p) and Ω_t^-. By construction, both Ω_t^+ and Ω_t^- are mean convex. Moreover, Γ_t is null-homotopic in both Ω_t^+ and Ω_t^- as $\partial\Omega$ is a sphere. Hence, by the theorem of Meeks and Yau, Theorem 2.5, there are area minimizing disks D_t^+ and D_t^- in Ω_t^+ and Ω_t^- respectively. Since Σ_t has genus greater than or equal to one, $D_t^\pm \neq \Sigma_t$, and hence $D_t^+ \neq D_t^-$. This shows that for any $t \in [-\epsilon, \epsilon]$, Γ_t bounds two stable minimal disks D_t^+ and D_t^- in M.

Assume that $D_t^\pm \cap D_s^\pm = \emptyset$ for any $t \neq s$. Then, we define N_t as the region between D_t^+ and D_t^- as in the first paragraph. Notice that by assumption, $N_t \cap N_s = \emptyset$ and by construction, $|N_t| > 0$ for any t. Then, the argument in the first paragraph gives a contradiction as before and thus the proof follows. □

Remark 4.5. The result of Theorem 4.4 is interesting even for the case when M is a strictly mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. For such an M, any two properly embedded area minimizing surfaces Σ_1 and Σ_2 are disjoint, provided that their boundaries Γ_1 and Γ_2 in ∂M are disjoint by the Meeks–Yau exchange roundoff trick [3, Lemma 4.1]. However, this theorem shows that if one relaxes the condition being area minimizing to just being minimal, this is no longer true.

Remark 4.6. Note that the proof of Theorem 4.4 also shows that for any simple closed curve Γ in ∂M, it is possible to find *nearby* curves Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ while $S_1 \cap S_2 \neq \emptyset$ where S_i, $i = 1, 2$, is a stable minimal surface with boundary Γ_i. In particular, we can find such curves Γ_1 and Γ_2 in an ϵ-neighborhood of Γ, for any ϵ, such that the ϵ-neighborhood of Γ in ∂M is an annulus and moreover if Γ is smooth, then we can find such curves Γ_1, Γ_2 so that they are also smooth. Furthermore, by the proof of Theorem 4.1, it is easy to see that we can take S_2 to be absolutely area minimizing and S_1 to be area minimizing of genus g where g is less than or equal to the minimum genus of a surface bounded by Γ.

Now, we need a version of a bridge principle to construct a nonembedded stable minimal surface in a strictly mean convex 3-manifold whose boundary is a simple closed curve.

Lemma 4.7 (Bridge principle for extreme curves). *Let M be a strictly mean convex 3-manifold, and Γ_1 and Γ_2 be two disjoint smooth simple closed curves bounding two stable minimal surfaces S_1 and S_2 respectively. Let α be a smooth path in ∂M connecting Γ_1 and Γ_2, as in* Definition 3.5. *Then, for any $\epsilon > 0$, there is a stable minimal surface $S_1 \sharp_\alpha S_2$ which is ϵ-close to $S_1 \cup S_2 \cup \alpha$.*

Proof. This lemma is a direct application of the Meeks and Yau Bridge principle [19, Theorem 7], noting only that in their proof one can pick the bridge so that its boundary lies in ∂M, since M is strictly convex. □

Finally, by using the previous result, and by the bridge principle above, we generalize Peter Hall's examples of nonembedded stable minimal surfaces in $\mathbb{B}^3$ to any strictly mean convex 3-manifold M. In particular, we show that for any strictly mean convex 3-manifold M, there exists a simple closed curve Γ in ∂M which bounds a nonembedded stable minimal surface S in M.

Theorem 4.8 (Nonembedded stable minimal surfaces). *Let M be a strictly mean convex 3-manifold. Then, there is a simple closed curve Γ in ∂M such that Γ bounds a nonembedded stable minimal surface S in M. Furthermore there is a simple closed curve Γ in ∂M such that Γ bounds a nonembedded stable minimal disk.*

Proof. Take S_1, S_2 as in Theorem 4.4, so that $\partial S_1, \partial S_2$ are smooth (cf. Remark 4.6). Then S_1, S_2 intersect transversally, (using a local graphical representation and the maximum principle one sees that they cannot be tangent). Now, apply Lemma 4.7 to construct a minimal surface that is ϵ-close to $S_1 \cup S_2 \cup \alpha$ where α is a curve connecting ∂S_1 and ∂S_2. Since S_1, S_2 intersect transversally, for ϵ small enough the new surface has at least one self intersection. For the disk case, take the surfaces as disks as in Theorem 4.4. The proof follows. □

Remark 4.9. Note that using Remark 4.6, we actually have many curves satisfying Theorem 4.8. In particular, given a smooth curve $\Gamma_0 \in \mathcal{A}$ and ϵ, such that the ϵ-neighborhood of Γ_0 in ∂M is an annulus, there exists Γ in the ϵ-neighborhood of Γ_0, satisfying Theorem 4.8.

Remark 4.10. Again, when we compare this result with the area minimizing case, we see that this cannot happen in the area minimizing case. By the regularity results of [1], any area minimizing surface Σ in M bounded by a simple closed curve Γ in ∂M must be smooth in the interior. Hence, if we relax the condition being area minimizing to being just minimal, we see that this is no longer true.

5. Applications to curves in $\mathbb{R}^3$

In this section we apply our results to simple closed curves in $\mathbb{R}^3$. So far, we discussed the simple closed curves in the boundary of a strictly mean convex 3-manifold, and proved several results about the area minimizing surfaces in the manifold that these curves bound. Now, we change our focus to the simple closed curves in $\mathbb{R}^3$ and give extensions of these results to this case.

We use the notation from the previous sections. In particular, $\mathcal{A}$ denotes the space of simple closed curves in $\mathbb{R}^3$, and $\mathcal{A}_g \subset \mathcal{A}$ denotes the space of simple closed curves in $\mathbb{R}^3$ such that the minimum genus of the absolutely area minimizing surfaces that they bound is greater than or equal to g, while $\mathcal{B}_g$ denotes the space of simple closed curves in ∂M such that the minimum genus of the absolutely area minimizing surfaces that they bound is exactly g, i.e. $\mathcal{B}_g = \mathcal{A}_g - \mathcal{A}_{g+1}$. Note that $\mathcal{A}_0 = \mathcal{A}$ because of the existence of absolutely area minimizing surfaces with boundary any given simple closed curve in $\mathbb{R}^3$, cf. [8]. Furthermore we let $\mathcal{A}^k$ denote the $\mathcal{C}^k$ smooth simple closed curves in $\mathbb{R}^3$ and we let $\mathcal{A}^k_g$ be $\mathcal{A}^k \cap \mathcal{A}_g$.

First, we extend Lemma 3.2 and Theorem 3.22 to the case of curves in $\mathbb{R}^3$.

Theorem 5.1. *For any $g \geq 0$, $\mathcal{A}_g$ is open and dense in $\mathcal{A}$ equipped with the $\mathcal{C}^0$ topology. Moreover, for any $k \geq 0$, $\mathcal{A}^k_g$ is open and dense in $\mathcal{A}^k$ equipped with $\mathcal{C}^0$ topology.*

Proof. First, we show that $\mathcal{A}_g$ is open in $\mathcal{A}$. The proof of this fact is the same as the proof of Lemma 3.2, with the only difference that here $\mathcal{U}_{\Gamma_0}(\epsilon)$ is an ϵ-neighborhood of Γ_0 in $\mathbb{R}^3$ with respect to the $\mathcal{C}^0$ topology; whereas in the proof of Lemma 3.2 we restricted this

neighborhood in ∂M; the boundary of the mean convex manifold M. Restricting ourselves now to $\mathcal{C}^k$ curves shows that $\mathcal{A}_g^k$ is open in $\mathcal{A}^k$ with respect to the $\mathcal{C}^0$ topology.

Now, we show that $\mathcal{A}_g$ is dense in $\mathcal{A}$. Let Γ be a simple closed curve in $\mathbb{R}^3$. Let Σ be the absolutely area minimizing surface in $\mathbb{R}^3$ with $\partial\Sigma = \Gamma$. Then, by [19, p. 159, Corollary 1] , for a sequence Σ_i of proper subsurfaces ($\Sigma_i \subsetneq \Sigma$) with $\Sigma_i \to \Sigma$, there are strictly mean convex neighborhoods N_i of Σ_i in $\mathbb{R}^3$ such that $\Gamma_i := \partial\Sigma_i \subset \partial N_i$.

For a set X, let $\mathcal{A}(X)$ denote the space of simple closed curves in ∂X and let $\mathcal{A}_g(X)$ denote the space of simple closed curves in ∂X such that the minimum genus of the absolutely area minimizing surfaces that they bound in X is greater than or equal to g. With this notation, $\mathcal{A}_g = \mathcal{A}_g(\mathbb{R}^3)$. Since N_i is strictly mean convex, by Theorem 3.22, $\mathcal{A}_g(N_i)$ is dense in $\mathcal{A}(N_i)$, the space of simple closed curves in ∂N_i. Hence, for each i, there is a sequence $\{\alpha_i^j\}_{j\in\mathbb{N}}$ of simple closed curves in $\mathcal{A}_g(N_i)$ such that

$$\alpha_i^j \xrightarrow{j\to\infty} \Gamma_i,$$

with respect to the $\mathcal{C}^0$ topology, where recall that $\Gamma_i = \partial\Sigma_i \subset \partial N_i$.

Let S_i^j be an absolutely area minimizing surface in N_i with boundary $\partial S_i^j = \alpha_i^j$ and with genus$(S_i^j) \geq g$. Notice that S_i^j is absolutely area minimizing in N_i, and this does not necessarily imply that it is also absolutely area minimizing in $\mathbb{R}^3$. We claim however that for any $i > 0$, there is a $j_0 > 0$, such that for any $j \geq j_0$, S_i^j is also absolutely area minimizing in $\mathbb{R}^3$. Hence, when we prove the claim, the density of $\mathcal{A}_g$ in $\mathcal{A}$ follows.

First, note that Σ_i is the unique absolutely area minimizing surface in $\mathbb{R}^3$ with $\partial\Sigma_i = \Gamma_i$. This because if there was another absolutely area minimizing surface Σ_i' in $\mathbb{R}^3$ with $\partial\Sigma_i' = \Gamma_i$, then $\Sigma' = (\Sigma \setminus \Sigma_i) \cup \Sigma_i'$ would be an absolutely area minimizing surface with $\partial\Sigma' = \Gamma$, since $|\Sigma_i| = |\Sigma_i'|$. However, Σ' then would have a singularity along Γ_i, and this contradicts the regularity theorem for absolutely area minimizing surfaces [8].

Now, fix $i_0 > 0$. Assume that the above claim is not true and hence there is a sequence $\{j_k\}_{k\in\mathbb{N}}$ with

$$j_k \xrightarrow{k\to\infty} \infty,$$

such that $S_{i_0}^{j_k}$ is not an absolutely area minimizing surface in $\mathbb{R}^3$ for any $k \in \mathbb{N}$. Let $T_{i_0}^{j_k}$ be the absolutely area minimizing surface in $\mathbb{R}^3$ with $\partial T_{i_0}^{j_k} = \alpha_{i_0}^{j_k}$. Since

$$\alpha_{i_0}^{j_k} \xrightarrow{k\to\infty} \Gamma_{i_0},$$

and Σ_{i_0} is the unique absolutely area minimizing surface in $\mathbb{R}^3$ with $\partial\Sigma_{i_0} = \Gamma_{i_0}$, then after passing to a subsequence if necessary,

$$T_{i_0}^{j_k} \xrightarrow{k\to\infty} \Sigma_{i_0}$$

where the convergence is with respect to measure and to Hausdorff distance. Since N_{i_0} is strictly mean convex, there is a neighborhood N^δ of N_{i_0} in $\mathbb{R}^3$ such that the nearest point projection map $\pi : N^\delta \to N_{i_0}$ does not increase the area, i.e. $|\pi(S)| \leq |S|$ for a surface S in N^δ. Since

$$T_{i_0}^{j_k} \xrightarrow{k\to\infty} \Sigma_{i_0}$$

in Hausdorff distance, there exists a $k_0 > 0$ such that for any $k \geq k_0$, $T_{i_0}^{j_k} \subset N^\delta$. Hence,

$$|\pi(T_{i_0}^{j_k})| \leq |T_{i_0}^{j_k}|,$$

and since $\pi(T_{i_0}^{j_k}) \subset N_{i_0}$ with $\partial\pi(T_{i_0}^{j_k}) = \alpha_{i_0}^{j_k}$,

$$|S_{i_0}^{j_k}| \leq |\pi(T_{i_0}^{j_k})| \leq |T_{i_0}^{j_k}|.$$

This shows that $S_{i_0}^{j_k}$ is also absolutely area minimizing in $\mathbb{R}^3$ for $k \geq k_0$, and the claim follows with $j_0 = j_{k_0}$.

This shows that for any $i > 0$, there is a $j_0 > 0$ such that $\alpha_i^j \in \mathcal{A}_g(\mathbb{R}^3)$ for all $j \geq j_0$. Hence, $\mathcal{A}_g$ is not only open, but also dense in $\mathcal{A}$ with respect to the $\mathcal{C}^0$ topology. Notice that if the initial curve Γ is smooth ($\mathcal{C}^k$ smooth), then we can also take the curves α_i^j to be as smooth as Γ by Lemma 3.18. This implies that $\mathcal{A}_g^k$ is dense in $\mathcal{A}^k$ in the $\mathcal{C}^0$ topology. □

Remark 5.2. Note that from the proof of Theorem 5.1 we can conclude that $\mathcal{A}_g^k$ is open in $\mathcal{A}^k$ with respect to the $\mathcal{C}^k$ topology. However, the statement that $\mathcal{A}_g^k$ is dense in $\mathcal{A}^k$ with respect to the $\mathcal{C}^k$ topology is far from being true. To see that, let γ be a $\mathcal{C}^2$ simple closed curve in $\mathbb{R}^3$, that is furthermore contained in a plain. Then, by [16], there is a $\mathcal{C}^2$ neighborhood $N(\gamma)$ of γ such that for any $\gamma' \in N(\gamma)$, there exists a unique minimal surface S in $\mathbb{R}^3$ with $\partial S = \gamma'$, and S is a graph over the plane (hence a disk). This shows that $\gamma \in \text{int}(\mathcal{B}_0)$, and thus $\mathcal{A}_1$ (or any $\mathcal{A}_g$ for $g \geq 1$) is not dense in $\mathcal{A}$ in the $\mathcal{C}^2$ topology. The main reason why the techniques of this paper do not generalize to the smooth topology is that the horn surgery produces curves that are "close" to the original curve in the $\mathcal{C}^0$ topology but not in any $\mathcal{C}^k$ topology for $k \geq 2$.

Next, we apply the generic non-uniqueness result of Section 4 to curves in $\mathbb{R}^3$. Let $\mathfrak{C}$ be the space of simple closed curves in $\mathbb{R}^3$ bounding more than one embedded stable minimal surface. We show that $\mathfrak{C}$ is dense in $\mathcal{A}$ with respect to the $\mathcal{C}^0$ topology.

Theorem 5.3 (Curves bounding more than one minimal surface are dense). *Let $\mathcal{A}$ be the space of simple closed curves in $\mathbb{R}^3$ equipped with the $\mathcal{C}^0$ topology. Let $\mathfrak{C} \subset \mathcal{A}$ represent the simple closed curves in ∂M bounding more than one embedded stable minimal surface in $\mathbb{R}^3$. Then, $\mathfrak{C}$ is dense in $\mathcal{A}$ with respect to the $\mathcal{C}^0$ topology. Moreover, for any $k \geq 0$, $\mathfrak{C} \cap \mathcal{A}^k$ is dense in $\mathcal{A}^k$ with respect to the $\mathcal{C}^0$ topology.*

Proof. Let $\widehat{\mathfrak{C}} = \{\Gamma \in \mathcal{A} \mid \exists \Sigma_1 \in \text{AAM}(\Gamma), \Sigma_2 \in \text{AM}(\Gamma) : \Sigma_1 \neq \Sigma_2\}$ where similarly to Theorem 4.1, we let $\text{AAM}(\Gamma)$ and $\text{AM}(\Gamma)$ be the set of absolutely area minimizing and area minimizing surfaces respectively bounded by Γ. Clearly $\widehat{\mathfrak{C}} \subset \mathfrak{C}$. Hence, if we show that $\widehat{\mathfrak{C}}$ is dense in $\mathcal{A}$, we are done.

Let Γ, Σ, Σ_i and N_i be as in the proof of Theorem 5.1, i.e. $\Gamma \in \mathcal{A}$, $\Sigma \in \text{AAM}(\Gamma)$, $\Sigma_i \subsetneq \Sigma$, with $\Sigma_i \to \Sigma$ and N_i are strictly mean convex neighborhoods of the surfaces Σ_i such that $\Gamma_i := \partial\Sigma_i \subset \partial N_i$. Analogously with $\mathcal{A}(X)$, as defined in the proof of Theorem 5.1, for a set X, we let $\mathfrak{C}(X)$ represent the curves in $\mathcal{A}(X)$ that bound more than one stable minimal surface in X. Since N_i is strictly mean convex, Theorem 4.1 implies that $\mathfrak{C}(N_i)$ is open and dense in $\mathcal{A}(N_i)$. The embedded minimal surfaces in $N(\Sigma_i)$ are also minimal in $\mathbb{R}^3$. The smooth case is similar. The proof follows. □

Remark 5.4. Notice that Theorem 4.1 says not only density, but also genericity (containing open dense subset) of $\mathfrak{C}$ in $\mathcal{A}$, when the ambient manifold M is strictly mean convex and the curves are in ∂M. Here, genericity of $\mathfrak{C}$ for the curves in $\mathbb{R}^3$ cannot be proved with

the techniques of Theorem 4.1 as White's result in [23] (cf. Theorem 2.8) is for strictly mean convex manifolds; this theorem gives us an embedded minimal surface of smallest genus when the manifold is strictly mean convex.

On the other hand, instead of Theorem 2.8, one might want to consider [15, Corollary 2.1] to get a minimal representative of the smallest genus surface, and generalize these arguments as it applies to $\mathbb{R}^n$. However, Jost's result gives us an *immersed* (not necessarily embedded) least area representative of the smallest genus surface the curve bounds (which is always a disk in our case). Hence, we cannot use it here, as we are looking for embedded minimal surfaces.

Finally, we will generalize Theorem 4.8 to simple closed curves in $\mathbb{R}^3$.

Theorem 5.5. *For any simple closed curve Γ in $\mathbb{R}^3$ and for any $\epsilon > 0$, there is a smooth simple closed curve Γ' in $N_\epsilon(\Gamma)$; an ϵ-neighborhood of Γ, such that Γ' bounds a nonembedded stable minimal surface in $\mathbb{R}^3$.*

Proof. Let Γ, Γ_i, Σ, Σ_i and N_i be as in the proof of Theorems 5.1 and 5.3, i.e. $\Gamma \in \mathcal{A}$, $\Sigma \in \mathrm{AAM}(\Gamma)$, $\Sigma_i \subsetneq \Sigma$, with $\Sigma_i \to \Sigma$ and N_i are strictly mean convex neighborhoods of Σ_i such that $\Gamma_i := \partial\Sigma_i \subset \partial N_i$. Since N_i is strictly mean convex, Theorem 4.8 (see also Remark 4.9) implies that for any $\epsilon > 0$, there is a simple closed curve $\Gamma_i' \subset N_\epsilon(\Gamma_i) \cap \partial N_i$ where $N_\epsilon(\Gamma_i)$ is an ϵ-neighborhood of Γ_i in the $\mathcal{C}^0$ topology, such that Γ_i' bounds a nonembedded stable minimal surface in N_i. Since any minimal surface in N_i is also minimal in $\mathbb{R}^3$ and since, by construction, the curves can be taken smooth, the proof follows. □

Remark 5.6. The result above shows that the simple close curves in $\mathbb{R}^3$ bounding nonembedded minimal surfaces, say $\mathfrak{D}$, are very abundant. Note that our result does not imply the density of $\mathfrak{D}$ in $\mathcal{A}$. Even though the curve Γ' is in ϵ neighborhood of Γ, it is not $\mathcal{C}^0$-close to Γ (see proof of Theorem 4.8).

On the other hand, even though we cannot say whether $\mathfrak{D}$ is dense in $\mathcal{A}$ in the $\mathcal{C}^0$ topology or not, it is easy to see that $\mathfrak{D} \cap \mathcal{A}^2$ is not dense in $\mathcal{A}^2$ in the $\mathcal{C}^2$ topology. By [7], any minimal surface Σ bounded by a simple closed curve Γ in $\mathbb{R}^3$ with total curvature $k(\Gamma) \leq 4\pi$ must be embedded. Hence, a simple closed curve Γ satisfying $k(\Gamma) < 4\pi$ has a $\mathcal{C}^2$-neighborhood $N(\Gamma)$ such that for any $\Gamma' \in N(\Gamma)$ the following holds: any minimal surface Σ' in $\mathbb{R}^3$ bounded by Γ' is embedded. Similarly, the result by Meeks [16] mentioned in the previous Remark 5.2 gives such a neighborhood of a curve Γ. This shows that $\mathfrak{D}$ is not dense in $\mathcal{A}^2$ in the $\mathcal{C}^2$ topology.

6. Final remarks

In the last section, for curves in $\mathbb{R}^3$, we considered the questions of openness and density not only for simple closed curves but also for smooth curves. The same thing can be done for curves on the boundary of a mean convex manifold M and similar questions can be considered in the smooth category. Recall that $\mathcal{A}_g$ denotes the space of simple closed curves in ∂M such that the minimum genus of the absolutely area minimizing surfaces that they bound is greater than or equal to g, and $\mathcal{B}_g$ denotes the space of simple closed curves in ∂M such that the minimum genus of the absolutely area minimizing surfaces that they bound is exactly g. Also, $\mathcal{A}$

denotes the space of nullhomologous simple closed curves in ∂M equipped with the $\mathcal{C}^0$ topology. A natural extension is to consider the results of this paper in the smooth category. In other words, let $\mathcal{A}^k$ be equal to $\mathcal{A} \cap C^k$, the space of nullhomologous curves that are $\mathcal{C}^k$ smooth, considered again with the $\mathcal{C}^0$ topology.

Let also $\mathcal{A}_g^k = \mathcal{A}_g \cap C^k$ and $\mathcal{B}_g^k = \mathcal{B}_g \cap C^k$ be the $\mathcal{C}^k$ smooth curves in $\mathcal{A}_g$ and in $\mathcal{B}_g$ respectively. Then, it is easy to show that $\mathcal{A}_g^k$ is open in $\mathcal{A}^k$ by using the techniques of this paper (Lemma 3.2). Furthermore, if ∂M is smooth, then the fact that $\mathcal{A}_g$ is open and dense in $\mathcal{A}$ (Lemma 3.22), and $\mathcal{A}^k$ is dense in $\mathcal{A}$ implies that $\mathcal{A}_g^k$ is dense in $\mathcal{A}$. If ∂M is not smooth, we only get that $\mathcal{A}_g^k$ is dense in $\mathcal{A}^k$.

Another question would be whether the similar results are true in the mean convex setting instead of strictly mean convex manifolds. Unfortunately, most of the results in Section 3, like $\mathcal{A}_g$ is open dense in $\mathcal{A}$, are not true for mean convex 3-manifolds. The main obstacle is the following. Let M be a mean convex 3-manifold and S be a subsurface in ∂M which is a minimal surface. Take a sufficiently small disk D in S such that D is the absolutely area minimizing surface for its boundary. Then, for any simple closed curve in the interior of D, the absolutely area minimizing surface would be the disk it bounds in D. Hence, any simple closed curve Γ in $\text{int}(D) \subset \partial M$ would automatically be in $\mathcal{B}_0$ and therefore there is no way to approximate this Γ with the curves in $\mathcal{A}_g$, for $g \geq 1$, since there is an open neighborhood of Γ in A which lies entirely in $\mathcal{B}_0$. This example shows that Theorem 3.22 ($\mathcal{A}_g$ is dense in $\mathcal{A}$) is not true for mean convex 3-manifolds whose boundary contains a minimal subsurface. For an extreme example, if M is a mean convex manifold whose boundary is an absolutely area minimizing sphere, then clearly $\mathcal{A} = \mathcal{B}_0$. The reason for this is that the Horn Surgery Lemma fails when the absolutely area minimizing surface it bounds is in the boundary of the manifold. What in particular fails, is that we cannot apply White's bridge principle, Theorem 3.12. In [24], there is no restriction on M being strictly mean convex, however it is required that the arc α is not tangential at its endpoints to the tangent halfspace of the area minimizing surface. Of course this is not the case when both α and the absolutely area minimizing surface lie in ∂M. Hence, the relevant results depending on Theorem 3.12, would not be valid for mean convex 3-manifolds.

In Section 4, we showed that the case of absolutely area minimizing surfaces and that of minimal surfaces are quite different when bounding extreme curves. For a strictly mean convex 3-manifold M, while generically a simple closed curve in ∂M bounds a unique absolutely area minimizing surface in M (see [3, 4]), the situation is opposite for minimal surfaces, as generically a simple closed curve in ∂M bounds more than one minimal surface in M (Theorem 4.1). Also, for $H_2(M, Z) = 0$, while absolutely area minimizing surfaces with disjoint boundaries in ∂M are also disjoint, this is no longer true for minimal surfaces. In Theorem 4.4, we proved that for any strictly mean convex 3-manifold M, there are disjoint simple closed curves in ∂M, bounding intersecting stable minimal surfaces.

It might be interesting to study the space of curves bounding nonembedded minimal surfaces as described in Section 4. In other words, we showed the existence of such extreme curves for any strictly mean convex 3-manifold M in Theorem 4.8. However, a condition on the curve to guarantee the embeddedness of all minimal surfaces it bounds would be very interesting. In our construction for the curves bounding nonembedded minimal surfaces, we used two intersecting minimal surfaces in M which have disjoint boundaries in ∂M. Then, we used a bridge between them to get a nonembedded minimal surface. This implies that for the final boundary curve Γ and near the bridge, the ratio between the extrinsic distance (in ∂M) and

the intrinsic distance (in ∂M) gets very large. Maybe, a bound (of course, depending on M) on this ratio might be a good condition to guarantee the embeddedness of all minimal surfaces which Γ bounds in M.

Furthermore, for any $g > 0$, Almgren and Thurston showed existence of unknotted simple closed curves Γ_g in $\mathbb{R}^3$ with the property that the absolutely area minimizing surface Γ_g bounds has genus at least g, see [2]. Indeed, they showed that for any $g > 0$, there are curves Γ_g such that any embedded surface they bound in their convex hull of the curve must have genus at least g. Since any minimal surface with boundary Γ must be in the convex hull of Γ, and any absolutely area minimizing surface is embedded, this automatically implies that the absolutely area minimizing surface Γ_g bound must have genus at least g. In this paper's terminology, this means Almgren and Thurston showed that for any $g > 0$, $\mathcal{A}_g \neq \emptyset$. However, their examples are not extreme curves. Here, one of the corollaries of Theorem 3.22 is that for any convex body Ω in $\mathbb{R}^3$ and for any $g > 0$, there is a curve $\Gamma_g \in \partial\Omega$ such that the absolutely area minimizing surface, which Γ_g bounds, has genus at least g.

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