MINIMAL NUMBER OF SINGULAR FIBERS IN A NONORIENTABLE LEFSCHETZ FIBRATION

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ABSTRACT. We show that there exists an admissible nonorientable genus g Lefschetz fibration with only one singular fiber over a closed orientable surface of genus h if and only if $g \geq 4$ and $h \geq 1$.

1. INTRODUCTION

Let S be a nonorientable closed surface and let t_α denote a Dehn twist about a two-sided simple closed curve α in S. We say that α is *trivial* if it bounds a disk or a Möbius band. Note that α is nontrivial if and only if t_α is nontrivial in the mapping class group $\mathcal{M}(S)$ of S, which is the group of isotopy classes of all self-diffeomorphisms of S.

Definition 1.1. Let X be a closed nonorientable 4-manifold, and Σ be a closed orientable 2-manifold. We say that a (nonorientable) Lefschetz fibration $f : X \to \Sigma$ is *admissible* if it has at least one singular fiber and all the vanishing cycles of f are nontrivial.

A Lefschetz fibration is said to be of genus g if the regular fiber is of genus g. In a nonorientable Lefschetz fibration, the regular fiber is a nonorientable surface, whose genus is defined to be the (maximal) number of projective planes in a connected sum decomposition. Note that the interactions between the relations in the mapping class groups of surfaces and Lefschetz fibrations listed in [9, Section 15.2] for the orientable case hold verbatim for the nonorientable case. In particular, for any closed nonorientable genus g surface S, if a product of m nontrivial Dehn twists is equal to a product of h commutators in $\mathcal{M}(S)$, then there is an admissible (nonorientable) genus g Lefschetz fibration with m singular fibers over a closed orientable surface of genus h.

For any $g \geq 1$ and $h \geq 0$, let $N(g,h)$ denote the minimal number of singular fibers in an admissible genus g Lefschetz fibration on a closed nonorientable 4-manifold over a closed orientable surface of genus h. According to [8, Proposition 1.11], any admissible genus one Lefschetz fibration over any closed orientable surface is a $\mathbb{P}^2$-bundle. Therefore, $N(1,h)$ is not defined for any $h \geq 0$, since there are no singular

fibers in this case. Moreover, $N(g,0) > 1$ for all $g \geq 2$, since a Dehn twist along a nontrivial curve is not isotopic to the identity, by definition.

On the other hand, any even power of any Dehn twist on a nonorientable surface S can be expressed as a single commutator in $\mathcal{M}(S)$ (see, for example, the Remark at the end of [10]). In particular, for any $g \geq 2$, there exists an admissible genus g Lefschetz fibration with *two* singular fibers over any closed orientable surface of genus $h \geq 0$, which implies that $N(g,h) \leq 2$ if $g \geq 2$ and $h \geq 0$. As a consequence of this discussion, a natural question arises: For which values of the pair (g,h), we have $N(g,h) = 1$?

Our main result is the following

Theorem 1.2. *$N(g,h) = 1$ iff $g \geq 4$ and $h \geq 1$.*

By our definitions in this paper, Theorem 1.2 is about nonorientable Lefschetz fibrations. An analogous result for the minimal number of singular fibers in *orientable* relatively minimal Lefschetz fibrations was obtained by Korkmaz and the second author [6], which says that $N(g,h) = 1$ iff $g \geq 3$ and $h \geq 2$.

Let $[G,G]$ denote the commutator subgroup of a group G. For any $x \in [G,G]$, the commutator length of x is defined to be the minimum number of factors needed to express x as a product of commutators. Our proof of Theorem 1.2 is based on the commutator lengths of Dehn twists on nonorientable surfaces. In [11], Szepietowski showed that the commutator length of any power of any Dehn twist on a closed nonorientable surface of genus at least 7 is equal to 1. Here, we prove the following results in the remaining low-genus cases.

Proposition 1.3. *Let S be a closed nonorientable surface of genus $4, 5$ or 6. Then for any nontrivial separating curve γ in S and for every $n \in \mathbb{Z}$, the commutator length of t_γ^n is equal to 1 in $\mathcal{M}(S)$.*

Proposition 1.4. *If S is a closed nonorientable surface of genus 2 or 3, then no Dehn twist along a nontrivial curve in S belongs to $[\mathcal{M}(S), \mathcal{M}(S)]$.*

In the rest of the paper, we assume that the genus of a closed nonorientable surface is at least 2, since the mapping class group of the real projective plane is trivial. We also assume that all curves are simple closed and two-sided.

2. Commutator lengths of Dehn twists on nonorientable surfaces

Let S be a closed nonorientable surface. When S is of genus at least 7, Szepietowski [11] proved that the commutator length of every power of any Dehn twist is equal to 1 in $\mathcal{M}(S)$. He also showed that if S is of genus 6, then the commutator length of every power of the Dehn twist along any nonseparating curve with

orientable complement is equal to 1 in $\mathcal{T}(S)$, the twist subgroup of $\mathcal{M}(S)$ which is generated by all Dehn twists. Note that, if the genus of S is at least 7, then we have

$$[\mathcal{M}(S), \mathcal{M}(S)] = [\mathcal{T}(S), \mathcal{T}(S)] = \mathcal{T}(S)$$

as shown in [4, Theorem 5.12], which is not true if the genus of S is at most 6, since in these cases $H_1(\mathcal{T}(S))$ is nontrivial [10]. We hope that this sheds some light on the genus assumptions in the aforementioned results of Szepietowski. In this note, we partially extend his results to cover the cases of genus $4, 5$ and 6, which is stated as Proposition 1.3 in the Introduction.

Proof. of *Proposition 1.3.* Let S be a closed nonorientable surface of genus 4 and let γ be a nontrivial separating curve in S. Note that at least one component of the complement of γ is a one-holed Klein bottle. In that component, we have $t_\gamma = y^2$, where y is the crosscap transposition. But we also know that y is conjugate to y^{-1} by a diffeomorphism f of the surface S. It follows that

$$t_\gamma^n = y^n y^n = y^n f y^{-n} f^{-1}.$$

Next, let S be a closed nonorientable surface of genus 5 or 6 and let γ be a nontrivial separating curve in S. When we cut the surface S along γ, we see that at least one of the components must be *nonorientable* of genus $2, 3$ or 4. If there is a nonorientable component of genus 2 in the complement of γ, then we can prove the desired result exactly as described in the first paragraph. Now suppose that one of the nonorientable components in the complement of γ is of genus 3. Recall that there is the following well-known relation

$$t_d t_e = (t_a t_b t_c)^4 \tag{2.1}$$

on the two-holed torus, where d and e are the boundary components. There is an orientation-reversing involution r of the two-holed torus such that $r(a) = c$, $r(b) = b$, and $r(c) = a$. Since r is orientation reversing, it conjugates right-handed Dehn twists to left-handed Dehn twists, and as a result we have

$$r(t_a t_b t_c)^{-2} r = (t_a t_b t_c)^2 \tag{2.2}$$

using that fact $r = r^{-1}$. By inserting the relation in (2.2) into (2.1), we obtain

$$t_d t_e = r(t_a t_b t_c)^{-2} r (t_a t_b t_c)^2. \tag{2.3}$$

Finally we embed the two-holed torus into our surface S such that the boundary component d is capped off with a Möbius band, and e is identified with γ. Then t_d becomes trivial in (2.3) and hence t_γ is a single commutator in $\mathcal{M}(S)$. Note that the involution r extends onto S. It follows that, for all $n \in \mathbb{Z}$, the commutator length of t_γ^n is also equal to 1 as well.

The only remaining case is that γ separates a closed nonorientable surface S of genus 6 into a one-holed torus and a nonorientable surface of genus 4. To take care of this case, we first make some preliminary observations. In a one-holed torus, let c denote a boundary parallel curve and let a, b denote the standard generators of the first homology group. The relation $(t_a t_b)^6 = t_c$ is well-known in the mapping class group of the one-holed torus. Using the braid relation $t_a t_b t_a = t_b t_a t_b$ we get

$$t_c = (t_b t_a^2 t_b t_a^2)^2. \tag{2.4}$$

There is an orientation-reversing involution r of the one-holed torus such that $r(a) = a$ and $r(b) = b$. Since r is orientation reversing, it conjugates right-handed Dehn twists to left-handed Dehn twists, and as a result we have

$$r(t_b t_a^2 t_b t_a^2)^{-1} r = t_a^2 t_b t_a^2 t_b. \tag{2.5}$$

Here we used that fact $r = r^{-1}$. By using the braid relation again we get

$$t_a^{-1}(t_a^2 t_b t_a^2 t_b) t_a = t_b t_a^2 t_b t_a^2. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$t_a^{-1} r (t_b t_a^2 t_b t_a^2)^{-1} r t_a = t_b t_a^2 t_b t_a^2. \tag{2.7}$$

Finally, we insert the relation in (2.7) into the relation in (2.4), to get

$$t_c = (t_b t_a^2 t_b t_a^2)^2 = t_a^{-1} r (t_b t_a^2 t_b t_a^2)^{-1} r t_a (t_b t_a^2 t_b t_a^2) = [t_b t_a^2 t_b t_a^2, t_a^{-1} r]. \tag{2.8}$$

Similarly, for any $n \in \mathbb{Z}$, we have

$$t_c^n = (t_b t_a^2 t_b t_a^2)^n (t_b t_a^2 t_b t_a^2)^n = t_a^{-1} r (t_b t_a^2 t_b t_a^2)^{-n} r t_a (t_b t_a^2 t_b t_a^2)^n = [(t_b t_a^2 t_b t_a^2)^n, t_a^{-1} r]. \tag{2.9}$$

There is an obvious embedding of the one holed torus into S so that the boundary c is identified with γ in S and the involution r of the one holed torus extends to an involution of S. This shows, combined with (2.9), that any power of the Dehn twist t_γ can be expressed as a single commutator in $\mathcal{M}(S)$. □

Next we show that one cannot possibly extend Szepietowski's results to cover the cases of genus 2 or 3, which is stated as Proposition 1.4 in the Introduction.

Proof. of *Proposition 1.4.* The mapping class group of the closed nonorientable surface S of genus 2 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [7]) and therefore its commutator subgroup is trivial. Hence the Dehn twist along the only nontrivial curve in S cannot be equal to a product of commutators.

Suppose that S is a closed nonorientable surface of genus 3 for the rest of the proof. It follows from the presentation of $\mathcal{M}(S)$ given in [1] that the homology

group $H_1(\mathcal{M}(S)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by the homology class $\bar{t}_a$ of a Dehn twist t_a and the homology class $\bar{z}$ of a crosscap slide z. In fact (cf. [3]) we have

$$H_1(\mathcal{M}(S)) = \langle \bar{t}_a, \bar{z} : \bar{t}_a^2 = \bar{z}^2 = 1, \bar{t}_a\bar{z} = \bar{z}\bar{t}_a \rangle.$$

Suppose that b is a nonseparating curve in S with nonorientable complement. Then t_b is conjugate to t_a. Since conjugate elements in a group G are homologous in $H_1(G)$, we have $\bar{t}_b = \bar{t}_a$. We conclude that $t_b \notin [\mathcal{M}(S), \mathcal{M}(S)]$, since otherwise, $\bar{t}_a = \bar{t}_b = 1$, which indeed contradicts to the fact that $\bar{t}_a$ is a generator of $H_1(\mathcal{M}(S))$. Note that there is no nonseparating curve with *orientable* complement in S. Moreover, since any separating curve in S is trivial, we deduce that no Dehn twist along a nontrivial curve in S is a commutator. □

Using similar techniques, one can prove the following result.

Lemma 2.1. *If S is a closed nonorientable surface of genus $4, 5$ or 6, then no Dehn twist along a nonseparating curve with nonorientable complement in S belongs to $[\mathcal{M}(S), \mathcal{M}(S)]$.*

Remark 2.2. Notice that there is no such curve if S is of genus 2 and the genus 3 case is already covered by Proposition 1.4. Moreover, if S is of genus 4, then it follows from [3] that the Dehn twist about a nonseparting curve with *orientable* complement represents a nontrivial element of $H_1(\mathcal{M}(S))$, and hence it does not belong to $[\mathcal{M}(S), \mathcal{M}(S)]$.

We now turn to the proof of our main Theorem 1.2.

Proof. Theorem 1.2. Szepietowski [11, Theorem 1.1] showed that if S is a closed nonorientable surface of genus at least 7, then for every nontrivial curve α in S, t_α is equal to a single commutator in $\mathcal{M}(S)$. Moreover, he showed (cf. [11, Theorem 1.2]) that there exists a nontrivial curve β in a closed nonorientable surface of genus 6, such that t_β is equal to a single commutator of elements of the twist subgroup $\mathcal{T}(S) \subset \mathcal{M}(S)$. It follows that for any $g \geq 6$ and $h \geq 1$, there is an admissible genus g Lefschetz fibration over a closed orientable surface of genus h, which has a unique singular fiber. The cases $g = 4, 5$ follow from Proposition 1.3. The cases $g = 2, 3$ follow from Proposition 1.4. □

3. A Lemma in the orientable case

In this final section, S denotes a closed *orientable* surface of positive genus. As shown by Korkmaz and the second author [6], if S is of genus at least 3, then the commutator length of any Dehn twist is equal to 2 in $\mathcal{M}(S)$. This result is optimum and there is no room for improvement. This is because, for a closed orientable surface S of genus 1 or 2, no Dehn twist belongs to $[\mathcal{M}(S), \mathcal{M}(S)]$ (see, for example, [6]) since $H_1(\mathcal{M}(S))$ is nontrivial in these cases. It is also known

that, for any closed orientable surface S, the commutator length of any power of a Dehn twist about a separating curve is at least 2 (see [2, Remark 3]) and the same is true for a sufficiently high power of a Dehn twist about a nonseparating curve [5, Corollary 2.4].

The extended mapping class group of an orientable surface S, denoted by $\mathcal{M}^\diamond(S)$, includes also the isotopy classes of orientation-reversing self-diffeomorphisms of S. In [11], Szepietowski showed that if S is of genus at least 3, then the commutator length of every power of any Dehn twist is equal to 1 in $\mathcal{M}^\diamond(S)$. It turns out that there is some room for improvement for his result.

Lemma 3.1. *Let γ be a nontrivial separating curve in a closed orientable surface S of genus 2. Then for every $n \in \mathbb{Z}$, the commutator length of t_γ^n is equal to 1 in $\mathcal{M}^\diamond(S)$.*

Remark 3.2. Note that for a closed orientable surface S of genus 1 or 2, no Dehn twist along a nonseparating curve belongs to $[\mathcal{M}^\diamond(S), \mathcal{M}^\diamond(S)]$. This follows from the fact that all nonseparating curves on a fixed surface S are equivalent and the homology class of the Dehn twist along some nonseparating curve is nontrivial in $H_1(\mathcal{M}^\diamond(S)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ ([3, Theorem 4.2]). We conclude that Lemma 3.1 cannot be improved any further.

Proof. of Lemma 3.1. Let γ be a nontrivial separating curve in a closed orientable surface S of genus 2. There is an obvious embedding of the one holed torus into S so that the boundary c is identified with γ in S and the involution r of the one holed torus extends to an involution of S. This shows, combined with (2.9), that any power of the Dehn twist t_γ can be expressed as a single commutator in $\mathcal{M}^\diamond(S)$. □

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