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Autoparallel Orbits in Kerr Brans-Dicke Spacetimes

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Abstract

The bounded orbital motion of a massive spinless test particle in the background of a Kerr Brans-Dicke geometry is analysed in terms of worldlines that are auto-parallels of different metric compatible space-time connections. In one case the connection is that of Levi-Civita with zero-torsion. In the second case the connection has torsion determined by the gradient of the Brans-Dicke background scalar field. The calculations permit one in principle to discriminate between these possibilities.

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1 Introduction

In general relativity it is commonly assumed that idealised massive spinless test *particles* have spacetime histories that coincide with time-like geodesics associated with the spacetime metric when acted on by gravitation alone. Such assumptions embody such notions as the "equivalence principle", the "equality of inertial and gravitational mass" and the tenets of "special relativity". Such a geometrical description of gravitational interactions is formulated in terms of Einstein's torsion-free metric-compatible (affine) connection. The particle histories then have self-parallel 4-velocities (tangent vectors) and may be termed "Levi-Civita auto-parallels". If a massive test particle moves in a bounded orbit in the geometry described by the exterior Schwarzshild metric one may use such hypotheses to calculate its orbital perihelion shift per revolution. Despite competing perturbations, such a perihelion shift of the planet Mercury in the gravitational field of the Sun is regarded as one of the classical tests of Einstein's theory of gravitation [1], [2]. Further evidence for these hypotheses is sought from observations of the pulse rate of the binary pulsar PSR 1913+16 that appears to be speeding up due to gravitational radiation [3]. A number of efforts have been made [4], [5], [6] to prove the *geodesic hypothesis* for test particles from a field theory approach but none are entirely convincing. However, (for spinless test particles) this hypothesis has almost become elevated to one of the natural laws of physics [7].

In 1961 Brans and Dicke [8] suggested a modification of Einsteinian gravitation by introducing a single additional scalar field with particular gravitational couplings to matter via the spacetime metric. They suggested that the experimental detection of such a new scalar component to gravitation might have been overlooked in traditional tests of gravitational theories. Their theory is arguably the simplest modification to Einstein's original description. Since then a number of astrophysical observations have indicated that gravitational scalar fields may indeed have relevance to the dynamics of matter. Furthermore on the theoretical side, modern "low energy effective string theories" are replete with scalar fields and most unified theories of the strong and electroweak interactions predict such fields with astrophysical implications [9].

Although the gravitational field equations were modified by Brans and

Dicke [8], they still assumed that the motion of test particles under the influence of gravitation was described by a Levi-Civita auto-parallel associated with the metric derived from the Brans-Dicke field equations. Dirac later suggested [10] that it was more natural to generate the motion of a test particle from a Weyl invariant action principle and that such a motion in general differed from a Brans-Dicke Levi-Civita auto-parallel. Although Dirac was concerned with the identification of electromagnetism with aspects of Weyl geometry, even for neutral test particles it turns out that such test particles would follow auto-parallels of a connection with torsion. In Ref. [11] we have shown that the theory of Brans and Dicke can be reformulated as a field theory on a spacetime with dynamic torsion T determined by the gradient of the Brans-Dicke scalar field Φ:

$$T = e^a \otimes \frac{d\Phi}{2\Phi} \otimes X_a - \frac{d\Phi}{2\Phi} \otimes e^a \otimes X_a \tag{1}$$

in any coframe $\{e^a\}$ with dual $\{X_a\}$. Although (in the absence of spinor fields) no new physics *of the fields* can arise from such a reformulation, the behaviour of spinless *massive test particles* in such a geometry with torsion is less clear cut. Two natural alternatives present themselves. One may assert that their histories are *either* time-like geodesics associated with auto-parallels of the Levi-Civita connection (as did Brans and Dicke) *or* the auto-parallels of the non-Riemannian connection with torsion. In [12, 13] we have shown that it is possible to theoretically compare these alternatives for the history of a mass in orbit about a spherically symmetric source of scalar-tensor gravity by regarding it as a spinless test particle. In principle such calculations could be confronted with observation in space experiments that measure the orbital parameters in binary systems. In practice there are many competing perturbations that may contribute significantly. In particular a rotating gravitational source will be expected to produce a modification whose magnitude will depend on both its mass and angular momentum as well as the scalar field. It is of interest therefore to compare the auto-parallel orbits derived from the two connections above for a spinless test particle in the metric of a spinning source.

In this note we perform such a calculation for a particle in the geometry derived from the exact Kerr Brans-Dicke metric and scalar field found in [14] (See also [15]). This solution to the Brans-Dicke equations is asymptotically flat and reduces to the Kerr metric solution to Einstein's equations when the

scalar field is replaced by a constant. The solution describes a stationary and axially symmetric metric and depends on parameters that may be identified with the scalar charge, mass and angular momentum of a localised source.

2 The motion of massive test particles

The Kerr Brans-Dicke solution found in [14] can be written in Boyer-Lindquist coordinates (t, r, θ, φ) as;

$$\begin{aligned} g &= \Phi_0^{-1} \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{-\frac{1}{2}A} \{ -\frac{\Sigma\Delta}{P} dt^2 + \frac{P \sin^2\theta}{\Sigma} (d\varphi - \frac{2Mlr}{P} dt)^2 \\ &\quad + \Sigma \left(\frac{(r-M)^2 - (M^2 - l^2)}{(r-M)^2 - (M^2 - l^2)\cos^2\theta} \right)^{2kA^2} (d\theta^2 + \frac{dr^2}{\Delta}) \} \end{aligned} \tag{2}$$

with the scalar field,

$$\Phi = \Phi_0 \left(\frac{r - (M + \sqrt{M^2 - l^2})}{r - (M - \sqrt{M^2 - l^2})} \right)^{\frac{A}{2}} \tag{3}$$

where $M > l$ denotes the source mass and l its angular momentum per unit mass. The constant A determines the strength of the scalar field and $k = \omega + 3/2$, in terms of the Brans-Dicke parameter ω. As usual

$$\begin{aligned} \Sigma &= r^2 + l^2 \cos^2\theta \\ \Delta &= r^2 + l^2 - 2Mr \\ P &= \Delta\Sigma + 2Mr(r^2 + l^2). \end{aligned} \tag{4}$$

We first examine test particle orbits C for massive spinless particles that follow Levi-Civita autoparallels. The equations of motion are

$$\hat{\nabla}_{\dot{C}} \dot{C} = 0$$

in terms of the torsion-free Levi-Civita connection $\hat{\nabla}$ and 4-velocity $\dot{C}$ normalised according to

$$\mathbf{g}(\dot{C}, \dot{C}) = -1, \tag{5}$$

(throughout units are adopted such that $G = 1$ and $c = 1$). If $C : \tau \mapsto x^\mu(\tau)$ in terms of proper time τ these equations yield

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \{^{\mu}_{\nu\lambda}\}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{6}$$

The metric above has two independent Killing vectors ∂_t and ∂_φ. These generate two constants of motion: the particle energy and orbital angular momentum. Since the orbits are planar we take $\theta = \frac{\pi}{2}$ and set

$$\bar{E} = m(\dot{t}g_{tt} + \dot{\varphi}g_{t\varphi}), \tag{7}$$

$$\bar{L} = m(\dot{t}g_{t\varphi} + \dot{\varphi}g_{\varphi\varphi}). \tag{8}$$

One can express $\dot{r}$ in (5) in terms of $\dot{t}$ and $\dot{\varphi}$ and metric components on the orbit. Eliminating $\dot{t}$ and $\dot{\varphi}$ from equations (7) and (8) and substituting into (5), the orbit equation for the particle equation may be written:

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{-\Delta\Phi^{-2}}{(g_{t\varphi}\tilde{E} - g_{tt}\tilde{L})^2 g_{rr}}\left\{\Phi^{-2}\Phi_0\Delta + 2g_{t\varphi}\tilde{E}\tilde{L} - g_{tt}\tilde{L}^2 - g_{\varphi\varphi}\tilde{E}^2\right\} \tag{9}$$

where $\tilde{L} = \frac{\bar{L}(\Phi_0)^{1/2}}{m}$ and $\tilde{E} = \frac{\bar{E}(\Phi_0)^{1/2}}{m}$. Now define,

$$A(r) = \frac{4M^2l^2 - \Delta r^2}{P_1} - 2Mlr \tag{10}$$

$$B(r) = \frac{-2Ml}{r} \tag{11}$$

$$C(r) = \frac{P_1}{r^2} \tag{12}$$

$$\Phi_1(r) = \left(\frac{r-(M+\sqrt{M^2-l^2})}{r-(M-\sqrt{M^2-l^2})}\right)^{-\frac{A}{2}} \tag{13}$$

$$P_1(r) = (r^2 + l^2)r^2 + 2Ml^2r \tag{14}$$

$$G(r) = \frac{r^2}{\Delta}\left(\frac{r^2+l^2-2Mr}{r^2+M^2-2Mr}\right)^{2kA^2} \tag{15}$$

and introduce $u = \frac{1}{r}$ so that:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{-u^4\Delta(1/u)}{\left(B(1/u)\tilde{E} - \tilde{L}A(1/u)\right)^2 G(1/u)}\{\Phi_1(1/u)\Delta(1/u) + 2B(1/u)\tilde{E}\tilde{L} - C(1/u)\tilde{E}^2 - \tilde{L}^2 A(1/u)\}. \quad (16)$$

To analyse this equation we employ the physically motivated approximations discussed in [13]. Thus if the radius of a (weak field) Newtonian orbit is much larger than the corresponding Schwarzschild radius of the source, one may expand this equation around $u = 0$ up to third order in order to compare its solutions with those in a Schwarzschild background:

$$\left(\frac{du}{d\varphi}\right)^2 \simeq S_0 + S_1 u + S_2 u^2 + S_3 u^3 \quad (17)$$

where the constants:

$$S_0 = \frac{1}{\tilde{L}^2}(\tilde{E}^2 - 1), \quad (18)$$

$$S_1 = \frac{1}{\tilde{L}^3} 4Ml\tilde{E}(\tilde{E}^2 - 1) + \frac{1}{\tilde{L}^2}(2M - \sqrt{M^2 - l^2}A), \quad (19)$$

$$\begin{aligned} S_2 &= -1 + \frac{1}{\tilde{L}^2}\{-\frac{1}{2}(M^2 - l^2)A^2 + 3l^2(\tilde{E}^2 - 1) \\ &\quad + M\sqrt{M^2 - l^2}A + 2k(M^2 - l^2)(\tilde{E}^2 - 1)A^2\} \\ &\quad + \frac{1}{\tilde{L}^3}\{8M^2\tilde{E}^3 l - 4Ml\tilde{E}\sqrt{M^2 - l^2}A\} \\ &\quad + \frac{12M^2 l^2 \tilde{E}^2}{\tilde{L}^4}(\tilde{E}^2 - 1), \end{aligned} \quad (20)$$

$$\begin{aligned} S_3 &= 2M + \frac{1}{\tilde{L}^2}\{-3l^2\sqrt{M^2 - l^2}A - \frac{1}{3}\sqrt{M^2 - l^2}(4M^2 - l^2)A \\ &\quad + 6Ml^2\tilde{E}^2 + 2M^2\sqrt{M^2 - l^2}A + 4k\tilde{E}^2 M(M^2 - l^2)A^2 \\ &\quad - (2k + \frac{1}{6})\left(M^2 - l^2\right)^{3/2} A^3\} + \frac{1}{\tilde{L}^3}\{-2Ml\tilde{E}(M^2 - l^2)A^2 \\ &\quad - 4M^2 l\tilde{E}\sqrt{M^2 - l^2}A + 16M^3 l\tilde{E}^3 \end{aligned}$$

$$+8k\tilde{E}Ml(\tilde{E}^2-1)(M^2-l^2)A^2+12M\tilde{E}l^3(\tilde{E}^2-1)\}$$
$$+\frac{12}{\tilde{L}^4}\{2M^3\tilde{E}^2l^2(2\tilde{E}^2-1)-M^2\tilde{E}^2l^2\sqrt{M^2-l^2}A\}$$
$$+\frac{1}{\tilde{L}^5}32l^3M^3\tilde{E}^3(\tilde{E}^2-1). \tag{21}$$

All the terms in S_2 except -1 and all the terms in S_3 give corrections to the Newtonian orbital equation.

By contrast we now compare this equation with the one obtained by assuming the worldline is a timelike auto-parallel of a particular connection with torsion:

$$\nabla_{\dot{C}}\dot{C}=0.$$

Here ∇ denotes the connection with the torsion (1) specified by the scalar field in terms of the 2-forms

$$T^a=e^a\wedge\frac{d\Phi}{2\Phi}$$

and the 4-velocity $\dot{C}$ is again normalized with

$$\mathbf{g}(\dot{C},\dot{C})=-1.$$

It is possible to re-write the worldline equation in terms of the Levi-Civita connection $\hat{\nabla}$ as

$$\widetilde{\hat{\nabla}_{\dot{C}}\dot{C}}=-\frac{1}{2\Phi}\iota_{\dot{C}}(d\Phi\wedge\tilde{\dot{C}})$$

where for any vector field V, $\tilde{V}=\mathbf{g}(V,-)$ is the metric related 1-form. This may be further simplified to

$$\hat{\nabla}_{\dot{C}}(\widetilde{\Phi^{1/2}\dot{C}})=-d\Phi^{1/2},$$

restricted to C, which in local coordinates gives:

$$\frac{d}{d\tau}\left(\Phi^{1/2}\frac{dx^\mu}{d\tau}\right)+\Phi^{1/2}\{^{\mu}_{\nu\lambda}\}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}=-g^{\mu\nu}\frac{\partial_\nu\Phi}{2\Phi^{1/2}}. \tag{22}$$

For any Killing vector K with $K\Phi=0$, the expression

$$\gamma_K=\Phi^{1/2}\mathbf{g}(K,\dot{C})$$

is constant along the worldline of the particle. As before the Killing vectors $K_t = \partial_t$ and $K_\varphi = \partial_\varphi$ generate, respectively, two constants of motion E and L, corresponding to energy and orbital angular momentum. Thus

$$E = m\left(\frac{\Phi}{\Phi_0}\right)^{1/2}(\dot{t}g_{tt} + \dot{\varphi}g_{t\varphi}), \tag{23}$$

$$L = m\left(\frac{\Phi}{\Phi_0}\right)^{1/2}(\dot{t}g_{t\varphi} + \dot{\varphi}g_{\varphi\varphi}) \tag{24}$$

in terms of metric functions evaluated on planar orbits. Eliminating $\dot{t}$ and $\dot{\varphi}$ from equations (23) and (24) and substituting in (5), one obtains the new auto-parallel orbit equation as

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{-\Delta\Phi^{-2}}{(g_{t\varphi}\hat{E} - g_{tt}\hat{L})^2 g_{rr}}\left\{\Phi^{-1}\Delta + 2g_{t\varphi}\hat{E}\hat{L} - g_{tt}\hat{L}^2 - g_{\varphi\varphi}\hat{E}^2\right\} \tag{25}$$

where $\hat{E} = \frac{E(\Phi_0)^{1/2}}{m}$ and $\hat{L} = \frac{L(\Phi_0)^{1/2}}{m}$. Expressed in terms of the variable $u = \frac{1}{r}$, it becomes

$$\begin{aligned}\left(\frac{du}{d\varphi}\right)^2 &= \frac{-u^4\Delta(1/u)}{\left(B(1/u)\hat{E} - \hat{L}A(1/u)\right)^2 G(1/u)}\{\Delta(1/u) + 2B(1/u)\hat{E}\hat{L} - \\ &\quad C(1/u)\hat{E}^2 - \hat{L}^2 A(1/u)\}.\end{aligned} \tag{26}$$

Expanding to third order in u as before, one finds

$$\left(\frac{du}{d\varphi}\right)^2 \simeq C_0 + C_1 u + C_2 u^2 + C_3 u^3 \tag{27}$$

in terms of the constants:

$$C_0 = \frac{1}{\hat{L}^2}(\hat{E}^2 - 1) \tag{28}$$

$$C_1 = 2\frac{M}{\hat{L}^2} + 4\frac{M\hat{E}l}{\hat{L}^3}(\hat{E}^2 - 1) \tag{29}$$

$$\begin{aligned}C_2 &= \frac{1}{\hat{L}^2}\{3l^2 + 2kA^2(M^2 - l^2)\}(\hat{E}^2 - 1) + \frac{1}{\hat{L}^3}8M^2 l\hat{E}^3 \\ &\quad + \frac{1}{\hat{L}^4}12M^2 l^2\hat{E}^2(\hat{E}^2 - 1) - 1\end{aligned} \tag{30}$$

$$
\begin{aligned}
C_3 &= 2M + \frac{1}{\hat{L}^2}\{(6M - 4MkA^2)l^2 + 4kM^3A^2\}\hat{E}^2 \\
&+ \frac{1}{\hat{L}^3}\{[(12M - 8kMA^2)l^3 + 8kM^3lA^2]\hat{E}(\hat{E}^2 - 1) + 16M^3\hat{E}^3 l\} \\
&+ \frac{1}{\hat{L}^4}\{24M^3l^2\hat{E}^2(2\hat{E}^2 - 1)\} + \frac{1}{\hat{L}^5}\{32M^3l^3\hat{E}^3(\hat{E}^2 - 1)\}. \qquad (31)
\end{aligned}
$$

The first three terms of C_2 and all terms in C_3 imply corrections to Newtonian orbits.

We note that both orbit equations have been written in the form:

$$\left(\frac{du}{d\varphi}\right)^2 \simeq g(u) = L_0 + L_1 u + L_2 u^2 + L_3 u^3. \qquad (32)$$

so their solutions can be analysed in terms of the corresponding constants according to the roots of the equation $g(u) = 0$. Suppose first that all three roots are real, i.e.

$$4L_1^3 L_3 + 4L_0 L_2^3 - L_1^2 L_2^2 + 27L_0^2 L_3^2 - 18L_0 L_1 L_2 L_3 < 0.$$

Suppose further that the roots are distinct and ordered to satisfy $u_1 < u_2 < u_3$. Then

$$u_1 + u_2 + u_3 = -\frac{L_2}{L_3}.$$

From the orbit equation, $g(u) \geq 0$ throughout the motion. Thus, $g(u)$ will have a local maximum between u_1 and u_2. Hence, for a bounded orbit, u_1 corresponds to aphelion and u_2 corresponds to perihelion. Consider the following cases [16]:

i. If $u_1 > 0$, one obtains bounded orbits of "elliptic "type. This requires that both L_0 and L_2 be negative provided that $L_3 > 0$. Then the particle is confined to the interval $u_1 < u < u_2$. (If $u_1 = u_2$, one obtains circular orbits.)

ii. If $u_1 = 0$, one obtains open orbits of "parabolic "type. This requires that $L_0 = 0$. This is possible for orbits associated with both Levi-Civita and torsional connections provided $E^2 = \Phi_0^{-1} m^2$.

iii. If $u_1 < 0$, one obtains open orbits of "hyperbolic"type. This requires that $E^2 > \Phi_0^{-1} m^2$ provided that $L_3 > 0$ where $L_3 = C_3$ if the orbit is associated with an autoparallel of the torsional connection and provided $L_3 = S_3$ if it is associated with the Levi-Civita connection.

3 The analysis of bounded orbits

We are interested here in bounded orbits [17] in which case the general solution of (32) can be expressed in terms of Jacobian elliptic functions. By introducing the variables

$$x = \frac{1}{2}\varphi\sqrt{L_3\,(u_3 - u_1)}, \qquad y = \sqrt{\frac{u - u_1}{u_2 - u_1}}$$

(32) becomes

$$\left(\frac{dy}{dx}\right)^2 = \left(1 - y^2\right)\left(1 - p^2 y^2\right) \tag{33}$$

with $p = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}$. Its general solution is

$$y = sn\,(x + \delta) \tag{34}$$

where δ is an arbitrary constant. Hence for both connections yielding orbits with perihelia,

$$u - u_1 = (u_2 - u_1)\,sn^2\left(\frac{1}{2}\varphi\sqrt{L_3(u_3 - u_1)} + \delta\right). \tag{35}$$

The periodicity of these solutions enables one to calculate a perihelion shift per revolution. The increase in φ between successive perihelia is given precisely by

$$\Delta\varphi = 2\int_{u_1}^{u_2} \frac{du}{\sqrt{L_3(u - u_1)(u - u_2)(u - u_3)}}. \tag{36}$$

With the transformation $y = \sqrt{\frac{u - u_1}{u_2 - u_1}}$, this becomes

$$\Delta\varphi = \frac{4K}{\sqrt{(u_3 - u_1)L_3}} \tag{37}$$

where

$$K = \int_0^1 \frac{dy}{\sqrt{(1 - y^2)(1 - p^2 y^2)}}.$$

Depending on circumstances one may be able to approximate this integral. This is possible if one is interested in non-relativistic bounded orbits in which the dimensionless quantity Mu remains small compared with unity and the orbital speed is small compared with the speed of light. This would, for example, arise for the motion of the planet Mercury regarded as a test particle in orbit about the Sun as a source. Even at the Sun's surface, where $R_\odot = 7 \times 10^8 m$, $M_\odot = 1.477 \times 10^3 m$, $Mu = 2.11 \times 10^{-6}$. At aphelion and perihelion, both $-\frac{L_3}{L_2}u_1$ and $-\frac{L_3}{L_2}u_2$ are small quantities [18]. Thus in the following we approximate $-\frac{L_3}{L_2}u_3 \simeq 1$. This means p^2 can be considered small so:

$$K \simeq \frac{1}{2}\pi\left(1 + \frac{1}{4}p^2\right)$$

and since the ratios $\frac{u_1}{u_3}$ and $\frac{u_2}{u_3}$ are small, we expand

$$p^2 \simeq \frac{(u_2 - u_1)}{u_3} \simeq -\frac{(u_2 - u_1)L_3}{L_2}.$$

We further approximate the term

$$\frac{1}{\sqrt{(u_3 - u_1)L_3}} = \frac{1}{\sqrt{|L_2|\left(1 + \frac{L_3}{L_2}(2u_1 + u_2)\right)}} \simeq \frac{1}{\sqrt{|L_2|}}\left(1 - \frac{L_3}{2L_2}(2u_1 + u_2)\right)$$

After a little algebra one finds that the increase in φ per revolution becomes

$$\Delta\varphi \simeq \frac{2\pi}{\sqrt{|L_2|}}\left(1 - \frac{3L_3}{4L_2}(u_1 + u_2)\right). \tag{38}$$

The advance of perihelion (perihelion shift) per revolution would be

$$\varepsilon = \Delta\varphi - 2\pi.$$

It is conventional to define e, the eccentricity of an elliptic orbit and r_0 its semi-axis major. Then

$$r_1 = (1 + e)r_0, \qquad r_2 = (1 - e)r_0$$

where $r_1 = \frac{1}{u_1}$ corresponds to aphelion distance and $r_2 = \frac{1}{u_2}$ corresponds to perihelion distance of an elliptic orbit. The perihelion shift may be expressed

in terms of these orbit parameters for the limiting Newtonian Kepler ellipse. Once a set of Kepler orbit parameters have been ascertained then these formulae permit one to match them to a relativistic orbit in terms of M, l, ω and A and the constants of motion [13].

Thus in summary, we have examined the bounded orbits of test particles in the presence of a particular solution to the Brans-Dicke theory of gravitation. Even with the same constants of motion and the same limiting Kepler orbits such orbits will in general differ. Those with histories determined by the Levi-Civita connection and satisfying the approximations above, have a perihelion shift per revolution

$$\varepsilon_1 = \frac{2\pi}{\sqrt{|S_2|}}\left(1 + \frac{3}{2(1-e^2)r_0}\frac{S_3}{|S_2|}\right) - 2\pi. \tag{39}$$

By contrast if the history is determined by the connection with torsion, the perihelion shift per revolution is different, being given by

$$\varepsilon_2 = \frac{2\pi}{\sqrt{|C_2|}}\left(1 + \frac{3}{2(1-e^2)r_0}\frac{C_3}{|C_2|}\right) - 2\pi. \tag{40}$$

Finally we note that, when $A = 0$ (i.e. the scalar field Φ is constant) both orbit equations describe geodesic motion given by the Levi-Civita connection in a background Kerr geometry. If one further sets $l = 0$, they both describe geodesic motion in a background Schwarzschild geometry. In this case the constants reduce to $C_3 = S_3 = 2M$ and $C_2 = S_2 = -1$ and the perihelion shift reduces to the classical value [18]

$$\varepsilon = \frac{6\pi M}{(1-e^2)r_0}. \tag{41}$$

4 Conclusion

An analysis of bounded orbital motion of a massive spinless test particle in the background of a Kerr Brans-Dicke geometry has been given. Such orbits have been discussed in terms of worldlines that are auto-parallels of different

metric compatible space-time connections. In one case the connection is torsion-free while in the other, the connection has a torsion that depends on the (spacetime) gradient of the scalar field. Such a connection differs from that of Levi-Civita by terms that contribute to the so called "improved stress-energy tensor" that enters into the original formulation of the Brans-Dicke theory. The variational derivation of this theory in terms of independent metric and connection variations naturally gives rise to a connection with such torsion.

It is notoriously difficult to discriminate between gravitational theories by observing planetary orbits in the solar system since many perturbations need to be taken into account. In the above calculations one has some information about the value of the constant ω but little guidance as to the value of the constants Φ_0 and A. Furthermore there is no unique Kerr Brans-Dicke geometry that one may, with confidence, ascribe to any particular spinning source. Despite these shortcomings, we feel that there may be astrophysical situations where considerations relevant to non-Riemannian geometries become relevant. The motion of bodies around spinning black holes, neutron or magnetic stars are examples. In order to obtain evidence for *any* gravitational interaction one needs to analyse observational data in terms of a particular theoretical framework. We have suggested that the detection of scalar-tensor gravitational interactions may benefit from a non-Riemmannian description. The calculations presented here offer such a framework in the context of the Brans-Dicke geometry with and without torsion. The effects of coupling other types of fields such as a Kalb-Ramond 2-form potential maybe worth investigating [19].

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References

[1] C M Will, **Theory and Experiment in Gravitational Physics** (Cambridge U P , 1993).

[2] C M Will, *The Confrontation between General Relativity and Experiment*, gr-qc/0103036 .

[3] A R Hulse, J H Taylor, Ap. J. Lett. **195** L51 (1975).

[4] A Einstein, L Infeld, B Hoffmann, Ann. Math. **39** 65 (1938).

[5] A Papapetrou, Proc. Roy. Soc. **A209** 248 (1951).

[6] W G Dixon, Proc. Roy. Soc. **A314** 499 (1970).

[7] S Weinberg, **Gravitation and Cosmology** (Wiley, 1972).

[8] C. H. Brans, R. Dicke, Phys. Rev. **124**(1961)925.

[9] C. H. Brans, *Gravity and ther tenacious scalar fields* (gr-qc/9705069) Contribution to Engelbert Schücking Festschrift.

[10] P A M Dirac, Proc. Roy. Soc. **A333** 403 (1973).

[11] T Dereli, R W Tucker, Phys. Letts. **B110** 206 (1982).

[12] T. Dereli, R. W. Tucker, *On the motion of matter in spacetime*, gr-qc/0107017.

[13] T. Dereli, R. W. Tucker, Mod. Phys. Lett. **A17** (2002)421 (gr-qc/0104050).

[14] C. B. G. McIntosh, Comm. Math. Physics **37**(1974) 335.

[15] H. Kim, Phys. Rev. **D60** (1999) 024001.

[16] W. Schmidt, Class. Q. Grav. **19** (2002) 2743.

[17] D. C. Wilkins, Phys. Rev. **D5** (1972) 814.

[18] J. L. Synge, **Relativity: The General Theory** (North-Holland Publishing, 1960).

[19] S. Kar, S. SenGupta, S. Sur, Phys. Rev. **D67** (2003) 044005.