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The Quadratic Symmetric Teleparallel Gravity in Two-Dimensions

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Abstract

A 2D symmetric teleparallel gravity model is given by a generic 4-parameter action that is quadratic in the non-metricity tensor. Variational field equations are derived. A class of conformally flat solutions are found. We also give static or cosmological solutions that need not be in this class.

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1 Introduction

In order to investigate the physical implications of General Relativity (GR) some simple limiting cases are often sought. The low energy, static limit is attained by assuming the existence of a time-like Killing vector. However, such configurations have no dynamics. Symmetric limits at arbitrary energy scales on the other hand are obtained by assuming one or more space-like Killing vectors. For example, the assumption of spherical symmetry reduces the gravitational action upon integrating over the angular variables to an effective 2D gravity model in (t, r) coordinates. Such 2D gravity models have recently been used much to discuss black hole dynamics, quantized gravity or numerical relativity [1]. It is well known that Einsteinean gravity in 2D has no dynamical degrees of freedom. One way of making this theory dynamical is to couple a dilaton scalar. It is also possible to introduce further dynamical degrees of freedom by going over to a non-Riemannian space-time by the inclusion of torsion and non-metricity. Generalized 2D gravity models with curvature and torsion or teleparallel theories with only torsion have been considered before (See the references in [2, 3]). On the other hand models including non-metricity received less attention [4, 5]. The symmetric teleparallel gravity (STPG) with vanishing curvature and torsion in four-dimensions has been introduced relatively recently [6, 7, 8]. Here we consider a 2D symmetric teleparallel gravity model given by the most general 4-parameter action that is quadratic in the non-metricity tensor. We give a class of conformally flat solutions to the variational field equations obtained by a re-scaling of the 2D Minkowski metric. We also discuss static and cosmological solutions that need not be in this class.

2 Mathematical Preliminaries

The triple $\{M, g, \nabla\}$ denotes the space-time where M is a 2-dimensional differentiable manifold, g is a non-degenerate metric and ∇ is a linear connection. g can be written as

$$g = \eta_{ab} e^a \otimes e^b \tag{1}$$

where $\{e^a\}$ are orthonormal co-frame $1-$forms and $\eta_{ab} = \mathrm{diag}(-1, +1)$. The orthonormal frame vectors $\{X_a\}$ dual to $\{e^a\}$ are determined by the relations $e^b(X_a) = \imath_a e^b = \delta^b_a$ where $\imath_a \equiv \imath_{X_a}$ denote the interior product operators with respect to the frame vectors X_a. We set the space-time orientation by the choice $\epsilon_{01} = +1$ or $*1 = e^0 \wedge e^1$ where $*$ is the Hodge map. Finally,

the connection ∇ is specified by a set of 1-forms $\{\Lambda^a{}_b\}$. Cartan structure equations below define the non-metricity, torsion and curvature forms:

$$\begin{aligned} D\eta_{ab} := -(\Lambda_{ab} + \Lambda_{ba}) &= -2Q_{ab} \ , & (2)\\ De^a := de^a + \Lambda^a{}_b \wedge e^b &= T^a \ , & (3)\\ D\Lambda^a{}_b := d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b &= R^a{}_b \ . & (4) \end{aligned}$$

$\wedge$, d and D denote the exterior product, exterior and covariant exterior derivatives, respectively. Bianchi identities follow as integrability conditions:

$$\begin{aligned} DQ_{ab} &= \frac{1}{2}(R_{ab} + R_{ba}) \ , & (5)\\ DT^a &= R^a{}_b \wedge e^b \ , & (6)\\ DR^a{}_b &= 0 \ . & (7) \end{aligned}$$

The connection 1-forms can be uniquely decomposed as

$$\Lambda^a_b = \omega^a_b + K^a_b + q^a_b + Q^a_b \tag{8}$$

where ω^a_b are the Levi-Civita connection 1-forms satisfying

$$\omega^a_b \wedge e^b = -de^a, \tag{9}$$

K^a_b are the contortion 1-forms satisfying

$$K^a_b \wedge e^b = T^a, \tag{10}$$

and

$$q_{ab} = -(\imath_a Q_{bc})e^c + (\imath_b Q_{ac})e^c \ . \tag{11}$$

We also make use of the following identities

$$\begin{aligned} D * e_a &= -Q \wedge *e_a + *(e_a \wedge e_b) \wedge T^b & (12)\\ D * (e_a \wedge e_b) &= -Q \wedge *(e_a \wedge e_b) & (13) \end{aligned}$$

where $Q := \Lambda^a{}_a = \eta^{ab}\Lambda_{ab}$ is the trace of the connection 1-forms.

We demand [11] that as the metric locally re-scales as

$$g \to e^{2\sigma} g \ , \tag{14}$$

connection should change according to

$$\Lambda^a_{\ b} \to \Lambda^a_{\ b} - \eta^a_{\ b} d\sigma \ . \tag{15}$$

Then we check through the structure equations the following scaling rules:

$$T^a \to e^\sigma T^a \quad , \quad R^a_{\ b} \to R^a_{\ b} \quad , \quad Q_{ab} \to Q_{ab} - \eta_{ab} d\sigma \ . \tag{16}$$

In writing down gravity models it is convenient to obtain first the irreducible decomposition of non-metricity tensor under the 2D Lorentz group. We write

$$Q_{ab} = \underbrace{\overline{Q}_{ab}}_{trace-free\ \ part} + \underbrace{\frac{1}{2}\eta_{ab} Q}_{trace\ \ part} \ . \tag{17}$$

Then

$$Q_{ab} = {}^{(1)}Q_{ab} + {}^{(2)}Q_{ab} + {}^{(3)}Q_{ab} \tag{18}$$

in terms of

$$\begin{aligned}
{}^{(1)}Q_{ab} &= \frac{1}{2}[(\imath_a \Lambda) e_b + (\imath_b \Lambda) e_a - \eta_{ab} \Lambda] \ , && (19)\\
{}^{(2)}Q_{ab} &= \frac{1}{2}\eta_{ab} Q \ , && (20)\\
{}^{(3)}Q_{ab} &= Q_{ab} - {}^{(1)}Q_{ab} - {}^{(2)}Q_{ab} && (21)
\end{aligned}$$

where $\Lambda := (\imath_b \overline{Q}^b_{\ a}) e^a$. These irreducible components satisfy

$$\eta_{ab}\, {}^{(1)}Q^{ab} = \eta_{ab}\, {}^{(3)}Q^{ab} = 0 \ , \ \imath_a\, {}^{(3)}Q^{ab} = 0 \ , \ e_a \wedge {}^{(3)}Q^{ab} = 0 \ . \tag{22}$$

Thus they are orthogonal in the following sense:

$${}^{(i)}Q^{ab} \wedge \ *^{(j)}Q_{ab} = \delta^{ij} N_{ij} \qquad (\text{no summation over } ij) \tag{23}$$

where δ^{ij} is the Kronecker symbol and N_{ij} are certain 2-forms. We calculate

$$\begin{aligned}
{}^{(1)}Q^{ab} \wedge \ *^{(1)}Q_{ab} &= (\imath^a Q_{ac})(\imath_b Q^{bc}) * 1 + \frac{1}{4} Q \wedge *Q - (\imath_a Q)(\imath_b Q^{ab}) * 1 \ , \\
{}^{(2)}Q^{ab} \wedge \ *^{(2)}Q_{ab} &= \frac{1}{2} Q \wedge *Q \ , && (24)\\
{}^{(3)}Q^{ab} \wedge \ *^{(3)}Q_{ab} &= Q^{ab} \wedge \ *Q_{ab} - {}^{(1)}Q^{ab} \wedge \ *^{(1)}Q_{ab} - {}^{(2)}Q^{ab} \wedge \ *^{(2)}Q_{ab} \ .
\end{aligned}$$

3 Symmetric Teleparallel Gravity

We formulate STPG by a variational principle from an action

$$I = \int_M \Big(L + R^a{}_b \rho_a{}^b + T^a \lambda_a \Big) \tag{25}$$

where L is a Lagrangian density 2-form quadratic in the non-metricity tensor, $\rho_a{}^b$ and λ_a are Lagrange multiplier 0-forms that impose the constraints

$$R^a{}_b = 0 \quad , \qquad T^a = 0 \, . \tag{26}$$

The gravitational field equations are derived from (25) by independent variations with respect to the connection $\{\Lambda^a{}_b\}$ and the orthonormal co-frame $\{e^a\}$ (and the Lagrange multipliers):

$$\lambda_a \wedge e^b + D\rho_a{}^b = -\Sigma_a{}^b \, , \tag{27}$$

$$D\lambda_a = -\tau_a \tag{28}$$

where

$$\Sigma_a{}^b = \frac{\partial L}{\partial \Lambda^a{}_b} \quad , \quad \tau_a = \frac{\partial L}{\partial e^a}.$$

In principle (27) is solved for the Lagrange multipliers λ_a and $\rho_a{}^b$ and substituted in (28) that governs the dynamics of the gravitational fields. It is important to notice that $D\lambda_a$ rather than the Lagrange multipliers themselves appear in (28). Thus we must be calculating $D\lambda_a$ directly and that we can do by taking the covariant exterior derivative of (27):

$$D\lambda_a \wedge e^b = -D\Sigma_a{}^b \, . \tag{29}$$

We used in the above derivation, the constraints

$$T^a = 0 \quad , \quad D^2 \rho_a{}^b = R^b{}_c \wedge \rho_a{}^c - R^c{}_a \wedge \rho_c{}^b = 0 \tag{30}$$

where the covariant exterior derivative of a $(1,1)$-type tensor is given by

$$D\rho_a{}^b = d\rho_a{}^b + \Lambda^b{}_c \wedge \rho_a{}^c - \Lambda^c{}_a \wedge \rho_c{}^b \, . \tag{31}$$

Finally we arrive at the gravitational field equations

$$D\Sigma_a{}^b - \tau_a \wedge e^b = 0 \, . \tag{32}$$

In order to use symmetry arguments we lower an index:

$$D\Sigma_{ab} + 2Q^c{}_b \wedge \Sigma_{ac} - \tau_a \wedge e_b = 0 \, . \tag{33}$$

The remaining set of field equations are derived via the covariant exterior derivative of (28):

$$D\tau_a = 0 \tag{34}$$

where we used $D^2\lambda_a = -R^b{}_a\lambda_b = 0$. Note that our system of equations (33) and (34) is complete in the sense that the total number of unknowns is $12 = 8$ (connection) $+ 4$ (co-frame), while the total number of equations is also $12 = 4$ (vanishing curvature)$+ 2$ (vanishing torsion) $+ 4$ (eqn.(33)) $+$ 2 (eqn.(34)).

We now write down the most general Lagrangian density 2-form which is quadratic in the non-metricity tensor [9]:

$$L = \sum_{I=1}^{3} k_I {}^{(I)}Q_{ab} \wedge *^{(I)}Q^{ab} + k_4 \left({}^{(1)}Q_{ab} \wedge e^b\right) \wedge * \left({}^{(2)}Q^{ac} \wedge e_c\right) \tag{35}$$

where k_1, k_2, k_3, k_4 are dimensionless coupling constants. Inserting (24) in (35), we simplify it to

$$L = c_1 Q_{ab} \wedge *Q^{ab} + c_2(\imath_a Q^{ac})(\imath^b Q_{bc}) * 1 + c_3 Q \wedge *Q + c_4(\imath_a Q)(\imath_b Q^{ab}) * 1 \tag{36}$$

in terms of new coupling constants

$$\begin{aligned} c_1 &= k_3, \\ c_2 &= k_1 - k_3 \ , \\ c_3 &= \frac{1}{4}k_1 + \frac{1}{2}k_2 - \frac{3}{4}k_3 + \frac{1}{4}k_4 \ , \\ c_4 &= -k_1 + k_3 - \frac{1}{2}k_4 \ . \end{aligned} \tag{37}$$

One should note that for the choice

$$c_1 = -1/2 \ , \ \ c_2 = 1 \ , \ \ c_3 = 1/2 \ , \ \ c_4 = -1, \tag{38}$$

the Lagrangian density (36) is an exact form. This is proven in the appendix. Furthermore the Lagrangian density (36) is scale invariant if and only if

$$c_1 + 2c_3 = 0 \ , \ c_2 = c_4 = 0 \ \ . \tag{39}$$

The model has no dynamics for the above two cases. We expect non-trivial solutions in all the other cases.

The variation of the action with (36) is long but straightforward. We list below the contributions to the variational field equations (33)-(34) coming from each term in (36):

$$\Sigma_a{}^b = \sum_{i=0}^{4} c_i \ {}^{(i)}\Sigma_a{}^b \quad , \qquad \tau_a = \sum_{i=0}^{4} c_i \ {}^{(i)}\tau_a \tag{40}$$

where

$$\begin{aligned}
{}^{(1)}\Sigma_{ab} &= 2 * Q_{ab} \ , \\
{}^{(2)}\Sigma_{ab} &= \imath^c Q_{ac} * e_b + \imath^c Q_{bc} * e_a \ , \\
{}^{(3)}\Sigma_{ab} &= 2\eta_{ab} * Q \ , \\
{}^{(4)}\Sigma_{ab} &= \frac{1}{2}(\imath_a Q) * e_b + \frac{1}{2}(\imath_b Q) * e_a + \eta_{ab}(\imath_c Q^{cd}) * e_d \ ,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
{}^{(1)}\tau_a &= -(\imath_a Q^{bc}) \wedge *Q_{bc} - Q^{bc} \wedge (\imath_a * Q_{bc}) \ , \\
{}^{(2)}\tau_a &= (\imath_b Q^{bc})(\imath^d Q_{cd}) * e_a - 2(\imath_a Q^{bd})(\imath^c Q_{cd}) * e_b \ , \\
{}^{(3)}\tau_a &= -(\imath_a Q) \wedge *Q - Q \wedge (\imath_a * Q) \ , \\
{}^{(4)}\tau_a &= (\imath_b Q)(\imath_c Q^{bc}) * e_a - (\imath_a Q)(\imath_c Q^{bc}) * e_b - (\imath_b Q)(\imath_a Q^{bc}) * e_c \ .
\end{aligned} \tag{42}$$

4 Solutions

The Minkowski metric with vanishing non-metricity;

$$g = -dT^2 + dR^2 \quad , \quad Q_{ab} = 0 \tag{43}$$

is a trivial solution of our system. We first consider a class of conformally flat solutions determined by metric

$$g = e^{2\psi(T,R)}(-dT^2 + dR^2) \tag{44}$$

and the non-metricity 1-forms

$$Q_{ab} = -\eta_{ab} d\psi(T,R) \tag{45}$$

that is generated by a conformal re-scaling of the flat solution with an arbitrary function $\psi(T,R)$. This configuration satisfies the constraints $R^a{}_b(\Lambda) = 0$ and $T^a(\Lambda) = 0$. The non-trivial equations come from the trace and the

symmetric parts of the field equation (33), and the zeroth and the first components of the field equation (34). They are,

$$\kappa(\psi_{TT} - \psi_{RR}) = 0 \tag{46}$$

$$\chi\psi_{TR} + \kappa\psi_T\psi_R = 0 \tag{47}$$

$$2\kappa\psi_T(\psi_{TT} - \psi_{RR}) = 0 \tag{48}$$

$$2\kappa\psi_R(\psi_{TT} - \psi_{RR}) = 0 \tag{49}$$

where

$$\kappa = 2c_1 + c_2 + 4c_3 + 2c_4 \quad , \quad \chi = c_2 + c_4 \ . \tag{50}$$

Here the lower indices T and R denote derivatives with respect to coordinates T and R, respectively. It is checked $\kappa = \chi = 0$ for the values (38) and (39) which means that any choice of $\psi(T, R)$ gives a solution. Otherwise, the general solution is given by

$$\begin{aligned}
\psi(T,R) &= \frac{\chi}{\kappa}\ln\left\{\cosh\left[\frac{a_1\sqrt{\kappa}}{\chi}(R+T+a_2)\right]\right\} \\
&\quad + \frac{\chi}{\kappa}\ln\left\{\cosh\left[\frac{a_1\sqrt{\kappa}}{\chi}(R-T+a_3)\right]\right\} + a_4 \\
&= \frac{\chi}{\kappa}\ln\left\{\cosh\left[\frac{a_1\sqrt{\kappa}}{\chi}(2R+a_2+a_3)\right]\right. \\
&\quad \left. + \cosh\left[\frac{a_1\sqrt{\kappa}}{\chi}(2T+a_2-a_3)\right]\right\} - \frac{\chi}{\kappa}\ln 2 + a_4
\end{aligned} \tag{51}$$

where a_1, a_2, a_3, a_4 are integration constants. We also give

$$\begin{aligned}
d\psi &= \frac{a_1}{\sqrt{\kappa}}(dR+dT)\tanh\left[\frac{a_1\sqrt{\kappa}}{\chi}(R+T+a_2)\right] \\
&\quad + \frac{a_1}{\sqrt{\kappa}}(dR-dT)\tanh\left[\frac{a_1\sqrt{\kappa}}{\chi}(R-T+a_3)\right] \ .
\end{aligned} \tag{52}$$

We wish now to discuss two other classes of solutions that may or may not be included in the class of conformally flat solutions above, depending on the parameter values.

4.1 Static solutions

Suppose $\psi = \psi(R)$ only. Then consider a coordinate transformation $(T, R) \to (t, r)$ such that

$$t = T \ , \ r = \int e^{2\psi(R)}dR$$

and let

$$\psi(R) = ln|f(r)| \, .$$

In the new coordinate chart our orthonormal co-frame reads

$$e^0 = f(r)dt \quad , \quad e^1 = dr/f(r) \quad . \tag{53}$$

We make the following ansatz independently for the non-metricity 1-forms

$$Q_{00} = f_r e^1 \quad , \quad Q_{11} = h(r)e^1 \quad , \quad Q_{01} = Q_{10} = 0 \tag{54}$$

where $h(r)$ is arbitrary. Here lower index r denotes derivative with respect to r. This configuration satisfies the constraints $R^a{}_b(\Lambda) = 0$ and $T^a(\Lambda) = 0$. The only non-trivial contribution to the field equation (33) comes from its trace: $d\Sigma^a{}_a = 0$, and the first component of the field equation (34): $D\tau_1 = 0$. The field equations reduce to

$$\alpha \left(fh\right)_r - \beta \left(ff_r\right)_r = 0 \tag{55}$$

$$\gamma(f_r)^2 h + \mu(h^3 - 2fhh_r) - \nu[(f_r)^3 + 2ff_r f_{rr}] \\ -\xi h^2 f_r + \eta(hff_{rr} + ff_r h_r) = 0 \tag{56}$$

where

$$\begin{aligned} \alpha &= 2c_1 + 2c_2 + 4c_3 + 3c_4 \\ \beta &= 2c_1 + 4c_3 + c_4 \\ \gamma &= c_1 + 3c_3 + c_4 \\ \mu &= c_1 + c_2 + c_3 + c_4 \\ \nu &= c_1 + c_3 \\ \xi &= c_1 + c_2 + 3c_3 + 2c_4 \\ \eta &= 2c_3 + c_4 \, . \end{aligned} \tag{57}$$

It is checked that all these constants are zero for the values (38) which means that any choice of $f(r)$ and $h(r)$ gives a solution. For a non-trivial solution we first solve for h from Eqn.(55):

$$h = \frac{\beta}{\alpha} f_r \, . \tag{58}$$

Substituting the above expression into (56) yields

$$Aff_{rr} + B(f_r)^2 = 0 \tag{59}$$

where

$$\begin{aligned} A &= 2\eta\beta\alpha^2 - 2\mu\alpha\beta^2 - 2\nu\alpha^3 \\ B &= \gamma\beta\alpha^2 + \mu\beta^3 - \nu\alpha^3 - \xi\alpha\beta^2 \ . \end{aligned} \tag{60}$$

Thus a solution is found as

$$f(r) = (a_1 + a_2 r)^{\frac{A}{A+B}} \tag{61}$$

where a_1, a_2 are integration constants. We note that if $\beta = -\alpha$, we can write the full connection 1-form as an exact form $\Lambda_{ab} = -\eta_{ab} d(\ln f(r))$.

We want to remark briefly on the singularity structure of the above solution. This is important for its physical interpretation. Within the simplest approach to singularities we examine scalars formed out of the main ingredients of space-time; e.g. $\mathcal{R}(\omega) = \imath_b \imath_a R^{ab}(\omega)$ in GR. In our case this amounts to looking at the quadratic non-metricity invariants such as $Q_{ab} \wedge *Q^{ab}$. Thus, from the Riemannian point of view we conclude that, since

$$\mathcal{R}(\omega) = \frac{2{a_2}^2 A(B-A)}{(A+B)^2} \frac{1}{(a_1 + a_2 r)^{\frac{2B}{A+B}}} \tag{62}$$

the surface $r = -a_1/a_2$ is an essential singularity (provided $\frac{2B}{A+B} > 0$). On the other hand from the symmetric teleparallel point of view we find

$$Q_{ab} \wedge *Q^{ab} = \left(1 + \frac{\beta^2}{\alpha^2}\right) \left(\frac{a_2 A}{A+B}\right)^2 \frac{1}{(a_1 + a_2 r)^{\frac{2B}{A+B}}} * 1 \ . \tag{63}$$

We see that the singularity structure of the above solution of STPG is basically similar to that of GR [7],[10].

4.2 Cosmological solutions

Suppose $\psi = \psi(T)$ only. Then consider a coordinate transformation $(T, R) \to (t, r)$ such that

$$t = \int e^{\psi(T)} dT \ , \ r = R \ .$$

and let

$$\psi(T) = ln|\mathcal{R}(t)| \ .$$

In the new coordinate chart our orthonormal co-frame reads

$$e^0 = dt \quad , \quad e^1 = \mathcal{R}(t) dr \quad . \tag{64}$$

We start with an independent ansatz for the non-metricity 1-forms

$$Q_{00} = \mathcal{S}(t)e^0 \quad , \quad Q_{11} = -\frac{\mathcal{R}_t}{\mathcal{R}}e^0 \quad , \quad Q_{01} = Q_{10} = 0 \tag{65}$$

where lower index t denotes derivative with respect to t. For this configuration, the only non-trivial part of (33) is again its trace; and the zeroth component of (34) which reads

$$\alpha(\mathcal{RS})_t + \beta\mathcal{R}_{tt} = 0 \tag{66}$$

$$\xi\mathcal{R}^2\mathcal{S}^2\mathcal{R}_t + \mu(\mathcal{R}^3\mathcal{S}^3 + 2\mathcal{R}^3\mathcal{S}\mathcal{S}_t) + \nu[2\mathcal{R}\mathcal{R}_t\mathcal{R}_{tt} - (\mathcal{R}_t)^3 + \mathcal{RS}(\mathcal{R}_t)^2] + \eta(\mathcal{SR}\mathcal{R}_{tt} + \mathcal{R}^2\mathcal{R}_t\mathcal{S}_t) = 0 \tag{67}$$

where all constants are defined in (57). The solution to these two equations is given by

$$\mathcal{S} = -\frac{\beta}{\alpha}\frac{\mathcal{R}_t}{\mathcal{R}} \quad , \quad \mathcal{R}(t) = (b_1 + b_2 t)^n \tag{68}$$

where b_1, b_2 are integration constants and

$$n = \frac{A}{A + C} \, . \tag{69}$$

Here A is given by (60) and

$$C = \mu\beta^3 + 2\mu\alpha\beta^2 + \nu\alpha^3 + \nu\beta\alpha^2 - \eta\beta\alpha^2 - \alpha\xi\beta^2 \, . \tag{70}$$

We determine a particular solution by setting $b_1 = 1$ and $b_2 = H/n$, and using the fact

$$\lim_{n\to\infty}\left(1 + \frac{H}{n}t\right)^n = e^{Ht} \, , \tag{71}$$

in the limit $C \to -A$. Then the metric takes a special form that satisfies the perfect cosmological principle:

$$g = -dt^2 + e^{2Ht}dr^2. \tag{72}$$

Now a co-ordinate transformation $\tan r = \rho$ yields the closed Robertson-Walker metric

$$g = -dt^2 + e^{2Ht}\frac{d\rho^2}{(1+\rho^2)^2} \, , \tag{73}$$

while another co-ordinate transformation $\tanh r = \rho$ yields the open Robertson-Walker metric

$$g = -dt^2 + e^{2Ht}\frac{d\rho^2}{(1-\rho^2)^2} \, . \tag{74}$$

We again note that if we set $\beta = -\alpha$, we can write the full connection 1-form an exact form $\Lambda_{ab} = -\eta_{ab}d(\ln|\mathcal{R}(t)|)$.

5 Conclusion

Theories of symmetric teleparallel gravity (STPG) in which the non-metricity tensor is dynamical are not easy to interpret [11]. However, they may play a role in certain astrophysical contexts. In addition to ordinary matter that couples gravitationally through its mass, we may think of dark matter in the universe that couples to a new kind of gravitational charge, called the Weyl charge [12], through the non-metricity tensor. The search for classical solutions is important to understand any observational effects that would be induced by such couplings. In this paper we investigated a symmetric teleparallel gravity model in two-dimensions. Firstly after writing scale transformation rules in non-Riemannian geometries, we discussed the decomposition of the full connection and orthogonal, irreducible decomposition of the non-metricity tensor under the 2D Lorentz group. Then we wrote the most general four-parameter Lagrangian density 2-form that is quadratic in the non-metricity tensor. For a specific set of values of the coupling constants the Lagrangian is an exact form and thus leads to a topological invariant. Besides, for some other special values of the coupling constants, the Lagrangian is scale invariant which means that the model does not have dynamics. We derived the variational field equations and found a class of conformally flat solutions. We also gave two other classes of solutions, one of which may be related to static, spherically symmetric solutions of 4D gravity while the other one takes the familiar form of an inflationary cosmological metric in 4D.

6 Appendix

We prove that for the following values of the coupling constants

$$c_1 = -1/2\ ,\ \ c_2 = 1\ ,\ \ c_3 = 1/2\ ,\ \ c_4 = -1, \tag{75}$$

Lagrangian (36) is an exact form:

$$L = d(\omega + q) \tag{76}$$

where $\omega_{ab} = \epsilon_{ab}\omega$ and $q_{ab} = \epsilon_{ab}q$. This means that the Lagrangian (36) with the values (75) do not yield dynamical field equations. We decompose the non-Riemannian curvature scalar by substituting the decomposition of the full connection (with $K^a{}_b = 0$), into the non-Riemannian curvature 2-form $R^a{}_b(\Lambda)$:

$$\begin{aligned} \mathcal{R}(\Lambda) * 1 &= \frac{1}{2} R^a{}_b(\Lambda) \wedge *e_a{}^b \\ &= \frac{1}{2}\epsilon_{ab} R^{ab}(\omega) + \frac{1}{2} d(\epsilon_{ab} q^{ab}) + \frac{1}{2}\epsilon_{ab} Q^a{}_c \wedge Q^{bc}\ . \end{aligned} \tag{77}$$

In STPG $R^a{}_b(\Lambda) = 0$ so that $\mathcal{R}(\Lambda) = 0$ and we have $R^{ab}(\omega) = d\omega^{ab}$ in two-dimensions. Therefore

$$\epsilon_a{}^b Q^{ac} \wedge Q_{cb} = -\epsilon_{ab}(d\omega^{ab} + dq^{ab})\ . \tag{78}$$

Here we write the left hand side in terms of the volume form as

$$\epsilon_a{}^b Q^{ac} \wedge Q_{cb} = (\imath_a Q^{ac})(\imath^b Q_{bc}) * 1 - (\imath^a Q^{bc})(\imath_c Q_{ab}) * 1 \tag{79}$$

and calculate the second term of the right hand side of this expression as follows:

$$\begin{aligned} 0 &= \epsilon_a{}^b q^{ac} \wedge q_{cb} \\ &= Q_{ab} \wedge *Q^{ab} - (\imath_a Q^{ac})(\imath^b Q_{bc}) * 1 - Q \wedge *Q \\ &\quad +2(\imath_a Q)(\imath_b Q^{ab}) * 1 - (\imath^a Q^{bc})(\imath_c Q_{ab}) * 1\ . \end{aligned} \tag{80}$$

Finally we insert these results into (78) together with $\omega_{ab} = \epsilon_{ab} d\omega$ and $q_{ab} = \epsilon_{ab} q$ to get

$$\begin{aligned} d(\omega + q) = -\frac{1}{2} Q_{ab} \wedge *Q^{ab} + (\imath_a Q^{ac})(\imath^b Q_{bc}) * 1 + \frac{1}{2} Q \wedge *Q \\ -(\imath_a Q)(\imath_b Q^{ab}) * 1 \end{aligned} \tag{81}$$

which leads to (75) and (76).

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